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Fractional neutral evolution equations with nonlocal conditions

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Abstract
In the present paper, we deal with the fractional neutral differential equations involving nonlocal initial conditions. The existence of mild solutions are established. The results are obtained by using the fractional power of operators and the Sadovskii's fixed point theorem. An application to a fractional partial differential equation with nonlocal initial condition is also considered.
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1 Introduction

The nonlocal condition, which is a generalization of the classical condition, was motivated by physical problems. The pioneering work on nonlocal conditions is due to Byszewski (see [1–3]). Existence results for semilinear evolution equations with nonlocal conditions were investigated in [4–6]. Neutral differential equations arises in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for neutral functional differential equations is the Hale book [7].

Fractional differential equations describe many practical dynamical phenomena arising in engineering, physics, economy and science. In particular, we can find numerous applications in viscoelasticity, electrochemistry, control, electromagnetic, seepage flow in porous media and in fluid dynamic traffic models (see [8–10]). The result obtained is a generalization and a continuation of some results reported in [11–15].

The main purpose of this paper is to study the existence of mild solutions of semilinear neutral fractional differential equations with nonlocal conditions in the following form

$$\begin{aligned} &{}^cD^\alpha\bigl[x(t)+F\bigl(t,x(t),x\bigl(b_1(t)\bigr),\dots,x\bigl(b_m(t)\bigr)\bigr)\bigr]+Ax(t)\\ &\quad=G\bigl(t,x(t),x\bigl(a_1(t)\bigr),\dots,x\bigl(a_n(t)\bigr)\bigr),\quad t\in J=[0,b],\\ &x(0)+g(x)=x_0, \end{aligned}\tag{1.1}$$

where $-A$ is the infinitesimal generator of an analytic semigroup and the functions F, G and g are given functions to be defined later. The fractional derivative ${}^cD^\alpha$, $0<\alpha<1$ is understood in the Caputo sense.

2 Preliminaries

Throughout this paper, X will be a Banach space with the norm $\|\cdot\|$ and $-A : D(A) \to X$ is the infinitesimal generator of an analytic compact semigroup of uniformly bounded linear operators $\{S(t), t \geq 0\}$. This means that there exists a $M \geq 1$ such that $\| S(t) \| \leq M$. We assume without loss of generality that $0 \in \rho(A)$. This allows us to define the fractional power A^γ, for $0 < \gamma \leq 1$, as a closed linear operator on its domain $D(A^\gamma)$ with inverse $A^{-\gamma}$.

We will introduce the following basic properties of A^γ.

Theorem 2.1 (see [16])

(1) *$X_\gamma = D(A^\gamma)$ is a Banach space with the norm $\| x \|_\gamma = \| A^\gamma x \|$, $x \in X_\gamma$.*
(2) *$S(t) : X \to X_\gamma$ for each $t > 0$ and $A^\gamma S(t)x = S(t)A^\gamma x$ for each $x \in X_\gamma$ and $t \geq 0$.*
(3) *For every $t > 0$, $A^\gamma S(t)$ is bounded on X and there exists a positive constant C_γ such that*

$$\left\|A^\gamma S(t)\right\| \leq \frac{C_\gamma}{t^\gamma}. \tag{2.1}$$

(4) *If $0 < \beta < \gamma \leq 1$, then $D(A^\gamma) \hookrightarrow D(A^\beta)$ and the embedding is compact whenever the resolvent operator of A is compact.*

Let us recall the following known definitions.

Definition 2.1 (see [17–19]) The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}\, ds, \quad t > 0$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where $\Gamma(\cdot)$ is the Gamma function.

Definition 2.2 (see [17–19]) The Caputo derivative of order α with the lower limit zero for a function f can be written as

$${}^cD^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}\, ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, 0 \leq n-1 < \alpha < n.$$

If f is an abstract function with values in X, then the integrals appearing in the above definitions are taken in Bochner's sense.

We list the following basic assumptions of this paper.

(H1) $F : J \times X^{m+1} \to X$ is a continuous function, and there exists a constant $\beta \in (0,1)$ and $M_1, M_2 > 0$ such that the function $A^\beta F$ satisfies the Lipschitz condition:

$$\left\|A^\beta F(s_1, x_0, x_1, \ldots, x_m) - A^\beta F(s_2, y_0, y_1, \ldots, y_m)\right\| \leq M_1 \Bigl(|s_1 - s_2| + \max_{i=0,1,\ldots,m} \|x_i - y_i\|\Bigr),$$

for $0 \leq s_1, s_2 \leq b$, $x_i, y_i \in X$, $i = 0, 1, \ldots, m$ and the inequality

$$\left\|A^\beta F(t, x_0, x_1, \ldots, x_m)\right\| \leq M_2 \Bigl(\max_{i=0,1,\ldots,m} \|x_i\| + 1\Bigr), \tag{2.2}$$

holds for $(t, x_0, x_1, \ldots, x_m) \in J \times X^{m+1}$.

(H2) The function $G : J \times X^{n+1} \to X$ satisfies the following conditions:

(i) for each $t \in J$, the function $G(t,\cdot) : X^{n+1} \to X$ is continuous and for each $(x_0, x_1, \ldots, x_n) \in X^{n+1}$ the function $G(\cdot, x_0, x_1, \ldots, x_n) : J \to X$ is strongly measurable;

(ii) for each positive number $q \in N$, there is a positive function $h_q(\cdot) : [0,b] \to R^+$ such that

$$\sup_{\|x_0\|,\ldots,\|x_n\| \le q} \left\| G(t, x_0, x_1, \ldots, x_n) \right\| \le h_q(t),$$

the function $s \to (t-s)^{1-\alpha} h_q(s) \in L^1([0,t], R^+)$ and there exists a $\Lambda > 0$ such that

$$\lim_{q\to\infty} \inf \frac{\int_0^t (t-s)^{1-\alpha} h_q(s)\,ds}{q} = \Lambda < \infty, \quad t \in [0,b],$$

(H3) $a_i, b_j \in C(J,J)$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. $g \in C(E,X)$, here and hereafter $E = C(J,X)$, and g satisfies that:

(i) There exist positive constants M_3 and M_4 such that $\|g(x)\| \le M_3\|x\| + M_4$ for all $x \in E$;

(ii) g is a completely continuous map.

At the end of this section, we recall the fixed-point theorem of Sadoviskii [20], which is used to establish the existence of the mild solution of the nonlocal Cauchy problem (1.1).

Theorem 2.2 (Sadovskii's fixed-point theorem) *Let Φ be a condensing operator on a Banach space X, that is, Φ is continuous and takes bounded sets into bounded sets, and $\mu(\Phi(B)) \le \mu(B)$ for every bounded set B of X with $\mu(B) > 0$. If $\Phi(\Upsilon) \subset \Upsilon$ for a convex, closed and bounded set Υ of X, then Φ has a fixed point in X (where $\mu(\cdot)$ denotes Kuratowski's measure of noncompactness).*

3 Main result

In this section, we study the existence of mild solutions for the neutral fractional differential equations with nonlocal conditions (1.1), so we introduce the concept of a mild solution.

Definition 3.1 (see [21, 22]) A continuous function $x(\cdot) : J \to X$ is said to be a mild solution of the nonlocal Cauchy problem (1.1) if the function $(t-s)^{\alpha-1} A T_\alpha(t-s) F(s, x(s), x(b_1(s)), \ldots, x(b_m(s)))$, $s \in [0,b)$ is integrable on $[0,b)$ and the following integral equation is verified:

$$\begin{aligned} x(t) &= S_\alpha(t)\bigl[x_0 + F\bigl(0, x(0), x\bigl(b_1(0)\bigr), \ldots, x\bigl(b_m(0)\bigr)\bigr) - g(x)\bigr] \\ &\quad - F\bigl(t, x(t), x\bigl(b_1(t)\bigr), \ldots, x\bigl(b_m(t)\bigr)\bigr) \\ &\quad - \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) F\bigl(s, x(s), x\bigl(b_1(s)\bigr), \ldots, x\bigl(b_m(s)\bigr)\bigr)\,ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) G\bigl(s, x(s), x\bigl(a_1(s)\bigr), \ldots, x\bigl(a_n(s)\bigr)\bigr)\,ds, \quad 0 \le t \le b, \end{aligned} \tag{3.1}$$

where

$$S_\alpha(t)x = \int_0^\infty \eta_\alpha(\theta) S\bigl(t^\alpha\theta\bigr)x\,d\theta, \qquad T_\alpha(t)x = \alpha\int_0^\infty \theta\eta_\alpha(\theta) S\bigl(t^\alpha\theta\bigr)x\,d\theta$$

with η_α is a probability density function defined on $(0,\infty)$, that is $\eta_\alpha(\theta) \geq 0$, $\theta \in (0,\infty)$ and $\int_0^\infty \eta_\alpha(\theta)\,d\theta = 1$.

Remark $\int_0^\infty \theta\eta_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}$.

Lemma 3.1 (see [22]) *The operators $S_\alpha(t)$ and $T_\alpha(t)$ have the following properties*:

(I) *for any fixed $x \in X$, $\|S_\alpha(t)x\| \leq M\|x\|$, $\|T_\alpha(t)x\| \leq \frac{\alpha M}{\Gamma(\alpha+1)}\|x\|$*;

(II) *$\{S_\alpha(t), t \geq 0\}$ and $\{T_\alpha(t), t \geq 0\}$ are strongly continuous*;

(III) *for every $t > 0$, $S_\alpha(t)$ and $T_\alpha(t)$ are also compact operators*;

(IV) *for any $x \in X$, $\beta \in (0,1)$ and $\delta \in (0,1)$, we have $AT_\alpha(t)x = A^{1-\beta}T_\alpha(t)A^\beta x$ and $\|A^\delta T_\alpha(t)\| \leq \frac{\alpha C_\delta \Gamma(2-\delta)}{t^{\alpha\delta}\Gamma(1+\alpha(1-\delta))}$, $t \in (0,b]$.*

Theorem 3.1 *If the assumptions* (H1)-(H3) *are satisfied and $x_0 \in X$, then the nonlocal Cauchy problem* (1.1) *has a mild solution provided that*

$$L_0 = M_1\left[(M+1)M_0 + \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right] < 1 \tag{3.2}$$

and

$$M\left[M_0M_2 + M_3 + \frac{\alpha\Lambda}{\Gamma(\alpha+1)}\right] + M_0M_2 + \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}M_2}{\beta\Gamma(1+\alpha\beta)} < 1, \tag{3.3}$$

where $M_0 = \|A^{-\beta}\|$.

Proof For the sake of brevity, we rewrite that

$$\begin{aligned}
&\bigl(t, x(t), x\bigl(b_1(t)\bigr), \ldots, x\bigl(b_m(t)\bigr)\bigr) = \bigl(t, v(t)\bigr) \quad \text{and} \\
&\bigl(t, x(t), x\bigl(a_1(t)\bigr), \ldots, x\bigl(a_n(t)\bigr)\bigr) = \bigl(t, u(t)\bigr).
\end{aligned}$$

Define the operator Φ on E by

$$\begin{aligned}
(\Phi x)(t) = {} & S_\alpha(t)\bigl[x_0 + F\bigl(0, v(0)\bigr) - g(x)\bigr] - F\bigl(t, v(t)\bigr) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F\bigl(s, v(s)\bigr)\,ds \\
& + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)G\bigl(s, u(s)\bigr)\,ds, \quad t \in J.
\end{aligned}$$

For each positive integer q, let $B_q = \{x \in E : \|x(t)\| \leq q, 0 \leq t \leq b\}$.

Then for each q, B_q is clearly a bounded closed convex set in E.

From Lemma 3.1 and (2.2) yield

$$\begin{aligned}
&\left\|\int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F\bigl(s, v(s)\bigr)\,ds\right\| \\
&\quad \leq \int_0^t \bigl\|(t-s)^{\alpha-1}A^{1-\beta}T_\alpha(t-s)A^\beta F\bigl(s, v(s)\bigr)\bigr\|\,ds \\
&\quad \leq \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}\int_0^t (t-s)^{\alpha\beta-1}\bigl\|A^\beta F\bigl(s, v(s)\bigr)\bigr\|\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}M_2\Bigl(\max_{i=1,2,\dots,m}\|x_i\|+1\Bigr) \\
&\le \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}M_2(q+1) \qquad (3.4)
\end{aligned}
$$

it follows that $(t-s)^{\alpha-1}AT_\alpha(t-s)F(s,v(s))$ is integrable on J, by Bochner's theorem [23] so Φ is well defined on B_q. Similarly, from (H2)(ii), we obtain

$$
\begin{aligned}
\left\|\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)G\bigl(s,u(s)\bigr)\,ds\right\| &\le \int_0^t \bigl\|(t-s)^{\alpha-1}T_\alpha(t-s)G\bigl(s,u(s)\bigr)\bigr\|\,ds \\
&\le \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\bigl\|G\bigl(s,u(s)\bigr)\bigr\|\,ds \\
&\le \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}h_q(s)\,ds. \qquad (3.5)
\end{aligned}
$$

We claim that there exists a positive number q such that $\Phi B_q \subseteq B_q$. If it is not true, then for each positive number q, there is a function $x_q(\cdot) \in B_q$, but $\Phi x_q \notin B_q$, but $\|\Phi x_q(t)\| > q$ for some $t(q) \in J$, where $t(q)$ denotes that t is dependent of q. However, from equations (2.2), (3.4) and (3.5) and (H3)(i), we have

$$
\begin{aligned}
q &\le \bigl\|(\Phi x_q)(t)\bigr\| \\
&\le M\bigl[\|x_0\| + M_0M_2(q+1) + (M_3q + M_4)\bigr] + M_0M_2(q+1) \\
&\quad + \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}M_2(q+1) + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}h_q(s)\,ds. \qquad (3.6)
\end{aligned}
$$

Dividing both sides of (3.6) by q and taking the lower limit as $q \to +\infty$, we get

$$
M\left[M_0M_2 + M_3 + \frac{\alpha\Lambda}{\Gamma(\alpha+1)}\right] + M_0M_2 + \frac{C_{1-\beta}\Gamma(1+\beta)b^{\alpha\beta}M_2}{\beta\Gamma(1+\alpha\beta)} \ge 1.
$$

This contradicts (3.3). Hence, for positive q, $\Phi B_q \subseteq B_q$.

Next, we will show that the operator Φ has a fixed point on B_q, which implies that equation (1.1) has a mild solution. We decompose Φ as $\Phi = \Phi_1 + \Phi_2$, where the operators Φ_1 and Φ_2 are defined on B_q, respectively, by

$$
(\Phi_1 x)(t) = S_\alpha(t)F\bigl(0,v(0)\bigr) - F\bigl(t,v(t)\bigr) - \int_0^t (t-s)^{\alpha-1}AT_\alpha(t-s)F\bigl(s,v(s)\bigr)\,ds
$$

and

$$
(\Phi_2 x)(t) = S_\alpha(t)\bigl[x_0 - g(x)\bigr] + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)G\bigl(s,u(s)\bigr)\,ds,
$$

for $t \in J$. We will show that Φ_1 verifies a contraction condition while Φ_2 is a compact operator.

To prove that Φ_1 satisfies a contraction condition, we take $x_1, x_2 \in B_q$, then for each $t \in J$ and by condition (H1) and (3.2), we have

$$\begin{aligned}
&\left\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\right\| \\
&\quad \le \left\|S_\alpha(t)\left[F\left(0, v_1(0)\right) - F\left(0, v_2(0)\right)\right]\right\| + \left\|F\left(t, v_1(t)\right) - F\left(t, v_2(t)\right)\right\| \\
&\qquad + \left\|\int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s)\left[F\left(s, v_1(s)\right) - F\left(s, v_2(s)\right)\right] ds\right\| \\
&\quad \le (M+1) M_0 M_1 \sup_{0 \le s \le b} \left\|x_1(s) - x_2(s)\right\| + \frac{C_{1-\beta}\Gamma(1+\beta) M_1 b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)} \sup_{0 \le s \le b} \left\|x_1(s) - x_2(s)\right\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\right\| &\le M_1\left[(M+1) M_0 + \frac{C_{1-\beta}\Gamma(1+\beta) b^{\alpha\beta}}{\beta\Gamma(1+\alpha\beta)}\right] \sup_{0 \le s \le b} \left\|x_1(s) - x_2(s)\right\| \\
&= L_0 \sup_{0 \le s \le b} \left\|x_1(s) - x_2(s)\right\|.
\end{aligned}$$

Thus,

$$\left\|(\Phi_1 x_1)(t) - (\Phi_1 x_2)(t)\right\| \le L_0 \sup_{0 \le s \le b} \left\|x_1(s) - x_2(s)\right\|,$$

and by assumption $0 < L_0 < 1$, we see that Φ_1 is a contraction.

To prove that Φ_2 is compact, firstly we prove that Φ_2 is continuous on B_q.

Let $\{x_n\} \subseteq B_q$ with $x_n \to x$ in B_q, then for each $s \in J$, $u_n(s) \to u(s)$, and by (H2)(i), we have $G(s, u_n(s)) \to G(s, u(s))$, as $n \to \infty$.

By the dominated convergence theorem, we have

$$\begin{aligned}
&\|\Phi_2 x_n - \Phi_2 x\| \\
&\quad = \sup_{0 \le t \le b} \Bigl\| S_\alpha(t)\left[g(x) - g(x_n)\right] \\
&\qquad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s)\left[G\left(s, u_n(s)\right) - G\left(s, u(s)\right)\right] ds \Bigr\| \to 0,
\end{aligned}$$

as $n \to \infty$, that is continuous.

Next, we prove that the family $\{\Phi_2 x : x \in B_q\}$ is a family equicontinuous functions. To do this, let $\epsilon > 0$ small, $0 < t_1 < t_2$, then

$$\begin{aligned}
&\left\|(\Phi_2 x)(t_2) - (\Phi_2 x)(t_1)\right\| \\
&\quad \le \left\|S_\alpha(t_2) - S_\alpha(t_1)\right\| \left\|x_0 - g(x)\right\| \\
&\qquad + \int_0^{t_1-\epsilon} \left\|(t_2-s)^{\alpha-1} T_\alpha(t_2-s) - (t_1-s)^{\alpha-1} T_\alpha(t_1-s)\right\| \left\|G\left(s, u(s)\right)\right\| ds \\
&\qquad + \int_{t_1-\epsilon}^{t_1} \left\|(t_2-s)^{\alpha-1} T_\alpha(t_2-s) - (t_1-s)^{\alpha-1} T_\alpha(t_1-s)\right\| \left\|G\left(s, u(s)\right)\right\| ds \\
&\qquad + \int_{t_1}^{t_2} \left\|(t_2-s)^{\alpha-1} T_\alpha(t_2-s)\right\| \left\|G\left(s, u(s)\right)\right\| ds.
\end{aligned}$$

We see that $\|(\Phi_2 x)(t_2) - (\Phi_2 x)(t_1)\|$ tends to zero independently of $x \in B_q$ as $t_2 \to t_1$, with ϵ sufficiently small since the compactness of $S_\alpha(t)$ for $t > 0$ (see [16]) implies the continuity of $S_\alpha(t)$ for $t > 0$ in t in the uniform operator topology. Similarly, using the compactness of the set $g(B_q)$ we can prove that the function $\Phi_2 x$, $x \in B_q$ are equicontinuous at $t = 0$. Hence, Φ_2 maps B_q into a family of equicontinuous functions.

It remains to prove that $V(t) = \{(\Phi_2 x)(t) : x \in B_q\}$ is relatively compact in X. Obviously, by condition (H3), $V(0)$ is relatively compact in X.

Let $0 < t \leq b$ be fixed, $0 < \epsilon < t$, arbitrary $\delta > 0$, for $x \in B_q$, we define

$$\begin{aligned}
\left(\Phi_2^{\epsilon,\delta} x\right)(t) &= \int_\delta^\infty \eta_\alpha(\theta) S\left(t^\alpha \theta\right)\left[x_0 - g(x)\right] d\theta \\
&\quad + \alpha \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \\
&= S\left(\epsilon^\alpha \delta\right) \int_\delta^\infty \eta_\alpha(\theta) S\left(t^\alpha \theta - \epsilon^\alpha \delta\right)\left[x_0 - g(x)\right] d\theta \\
&\quad + \alpha S\left(\epsilon^\alpha \delta\right) \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta - \epsilon^\alpha \delta\right) G\left(s, u(s)\right) d\theta\, ds.
\end{aligned}$$

Since $S(\epsilon^\alpha \delta)$, $\epsilon^\alpha \delta > 0$ is a compact operator, then the set $V^{\epsilon,\delta}(t) = \{(\Phi_2^{\epsilon,\delta} x)(t) : x \in B_q\}$ is relatively compact in X for every ϵ, $0 < \epsilon < t$ and for all $\delta > 0$.

Moreover, for every $x \in B_q$, we have

$$\begin{aligned}
\left\|(\Phi_2 x)(t) - \left(\Phi_2^{\epsilon,\delta} x\right)(t)\right\| &\leq \left\| \int_0^\delta \eta_\alpha(\theta) S\left(t^\alpha \theta\right)\left[x_0 - g(x)\right] d\theta \right\| \\
&\quad + \alpha \left\| \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \right\| \\
&\quad + \alpha \left\| \int_0^t \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \right. \\
&\quad \left. - \int_0^{t-\epsilon} \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \right\| \\
&\leq \left\| \int_0^\delta \eta_\alpha(\theta) S\left(t^\alpha \theta\right)\left[x_0 - g(x)\right] d\theta \right\| \\
&\quad + \alpha \left\| \int_0^t \int_0^\delta \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \right\| \\
&\quad + \alpha \left\| \int_{t-\epsilon}^t \int_\delta^\infty \theta (t-s)^{\alpha-1} \eta_\alpha(\theta) S\left((t-s)^\alpha \theta\right) G\left(s, u(s)\right) d\theta\, ds \right\| \\
&\leq M\left[\|x_0\| + M_3 \|x\| + M_4\right] \int_0^\delta \eta_\alpha(\theta)\, d\theta \\
&\quad + \alpha M \left(\int_0^t (t-s)^{\alpha-1} h_q(s)\, ds \right) \int_0^\delta \theta \eta_\alpha(\theta)\, d\theta \\
&\quad + \alpha M \left(\int_{t-\epsilon}^t (t-s)^{\alpha-1} h_q(s)\, ds \right) \int_0^\infty \theta \eta_\alpha(\theta)\, d\theta \\
&\leq M\left[\|x_0\| + M_3 \|x\| + M_4\right] \int_0^\delta \eta_\alpha(\theta)\, d\theta
\end{aligned}$$

$$+ \alpha M\left(\int_0^t (t-s)^{\alpha-1} h_q(s)\,ds\right)\int_0^\delta \theta\eta_\alpha(\theta)\,d\theta$$
$$+ \frac{\alpha M}{\Gamma(1+\alpha)}\int_{t-\epsilon}^t (t-s)^{\alpha-1} h_q(s)\,ds.$$

Therefore, there are relative compact sets arbitrary close to the set $V(t)$, $t > 0$. Hence, the set $V(t)$, $t > 0$ is also relatively compact in X.

Thus, by Arzela-Ascoli theorem Φ_2 is a compact operator. Those arguments enable us to conclude that $\Phi = \Phi_1 + \Phi_2$ is a condensing map on B_q, and by the fixed-point theorem of Sadovskii there exists a fixed point $x(\cdot)$ for Φ on B_q. Therefore, the nonlocal Cauchy problem (1.1) has a mild solution, and the proof is completed. □

4 Example

Let $X = L^2([0,\pi], R)$, we consider the following fractional neutral partial differential equations

$$^c\partial_t^\alpha\left[u(t,z) + \int_0^\pi a(z,y)u(t,y)\,dy\right] = \partial_z^2 u(t,z) + \partial_z h\bigl(t,u(t,z)\bigr),\quad 0 \le t \le b, 0 \le z \le \pi,$$

$u(t,0) = u(t,\pi) = 0$, $0 \le t \le b$,

$$u(0,z) + \sum_{i=1}^p \int_0^\pi k(z,y)u(t_i,y)\,dy = u_0(z),\quad 0 \le z \le \pi, \tag{4.1}$$

where $^c\partial_t^\alpha$ is a Caputo fractional partial derivative of order $0 < \alpha < 1$, $b > 0$, $z \in [0,\pi]$, p is a positive integer, $0 < t_0 < t_1 < \cdots < t_p < b$.

$$u_0(z) \in X = L^2\bigl([0,\pi], R\bigr),\qquad k(z,y) \in L^2\bigl([0,\pi]\times[0,\pi], R\bigr).$$

We define an operator A by $Af = -f''$ with the domain

$$D(A) = \bigl\{f(\cdot) \in X : f, f' \text{absolutely continuous}, f'' \in X, f(0) = f(\pi) = 0\bigr\}.$$

Then $-A$ generates a strongly continuous semigroup $\{S(t)\}_{t\ge 0}$ which is compact, analytic, and self-adjoint. Furthermore, $-A$ has a discrete spectrum, the eigenvalues are $-n^2$, $n \in N$, with the corresponding normalized eigenvectors $u_n(z) = (2/\pi)^{1/2}\sin(nz)$. We also use the following properties:

(a) If $f \in D(A)$, then $Af = \sum_{n=1}^\infty n^2\langle f, u_n\rangle u_n$.
(b) For each $f \in X$, $A^{-1/2}f = \sum_{n=1}^\infty \frac{1}{n}\langle f, u_n\rangle u_n$. In particular, $\|A^{-1/2}\| = 1$.
(c) The operator $A^{1/2}$ is given by

$$A^{1/2}f = \sum_{n=1}^\infty n\langle f, u_n\rangle u_n$$

on the space $D(A^{1/2}) = \{f(\cdot) \in X, \sum_{n=1}^\infty n\langle f, u_n\rangle u_n \in X\}$.

The system (4.1) can be reformulated as the following nonlocal Cauchy problem in X:

$$\begin{aligned}
&{}^{c}D^{\alpha}\bigl[x(t) + F\bigl(t, x(t), x\bigl(b_1(t)\bigr), \ldots, x\bigl(b_m(t)\bigr)\bigr)\bigr] + Ax(t) \\
&\quad = G\bigl(t, x(t), x\bigl(a_1(t)\bigr), \ldots, x\bigl(a_n(t)\bigr)\bigr), \quad t \in J = [0, b], \\
&x(0) + g(x) = x_0,
\end{aligned}$$

where $x(t) = u(t, \cdot)$ that is $(x(t))(z) = u(t, z)$, $t \in [0, b]$, $z \in [0, \pi]$.

The function $F : [0, b] \times X \to X$ is given by

$$\bigl(F(t, \varphi)\bigr)(z) = \int_0^{\pi} a(z, y)\varphi(y)\, dy$$

holds for $(\varphi, t) \in [0, b] \times X \to X$ and $z \in [0, \pi]$.

The function $G : [0, b] \times X \to X$ is given by

$$\bigl(G(t, \varphi)\bigr)(z) = \partial_z h\bigl(t, u(t, z)\bigr)$$

holds for $(\varphi, t) \in [0, b] \times X \to X$ and $z \in [0, \pi]$, and the function $g : E \to X$ is given by

$$g(x) = \sum_{i=0}^{p} K_g x(t_i),$$

where $K_g(u)(z) = \int_0^{\pi} k(z, y)u(y)\, dy$, for $z \in [0, \pi]$.

We can take $\alpha = \frac{1}{2}$ and $G(t, x) = \frac{1}{t^{1/3}} \sin x$, then (H2) is satisfied. Furthermore, assume that $M_3 = M_4 = (p + 1)[\int_0^{\pi} \int_0^{\pi} k^2(z, y)\, dy\, dz]^{1/2}$. Then (H3) is satisfied (noting that $K_g : X \to X$ is completely continuous).

Moreover, we assume the following conditions hold:

(i) The function $a(z, y)$, $z, y \in [0, \pi]$ is measurable and

$$\int_0^{\pi} \int_0^{\pi} a^2(z, y)\, dy\, dz < \infty.$$

(ii) The function $\partial_z a(z, y)$ is measurable, $a(0, y) = a(\pi, y) = 0$, and let

$$N_1 = \biggl[\int_0^{\pi} \int_0^{\pi} \bigl(\partial_z a(z, y)\bigr)^2\, dy\, dz\biggr]^{1/2} < \infty.$$

Therefore, the conditions (H1)-(H3) are all satisfied. Hence, according to Theorem 3.1, system (4.1) has a mild solution provided that (3.2) and (3.3) hold.

Competing interests

The author declares that he has no competing interests.

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