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Generalized q-Bessel function and its properties

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Abstract

In this paper, the generalized q-Bessel function, which is a generalization of the known q-Bessel functions of kinds 1, 2, 3, and the new q-analogy of the modified Bessel function presented in (Mansour and Al-Shomarani in J. Comput. Anal. Appl. 15(4):655-664, 2013) is introduced. We deduced its generating function, recurrence relations and q-difference equation, which gives us the differential equation of each of the Bessel function and the modified Bessel function when q tends to 1. Finally, the quantum algebra $E_2(q)$ and its representations presented an algebraic derivation for the generating function of the generalized q-Bessel function.

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Keywords: q-Bessel functions; generating function; q-difference equation; quantum algebra; irreducible representation

1 Introduction

The q-shifted factorials are defined by [1]

$$\begin{aligned}
&(a;q)_0 = 1,\\
&(a;q)_k = \prod_{i=o}^{k-1}(1-aq^k),\\
&(a_1,\dots,a_r;q)_k = \prod_{j=1}^{r}(a_j;q)_k; \quad k=0,1,2,\dots,\\
&(a;q)_\infty = \prod_{i=0}^{\infty}(1-aq^i), \quad \text{where } a, a_i\text{'s}, q \in \mathbb{R} \text{ such that } 0<q<1.
\end{aligned}$$

The one-parameter family of q-exponential functions

$$E_q^{(\alpha)}(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{\alpha n^2}{2}}}{(q;q)_n} z^n$$

with $\alpha \in \mathbb{R}$ has been considered in [2]. Consequently, in the limit when $q \to 1$, we have $\lim_{q\to 1} E_q^{(\alpha)}((1-q)z) = e^z$. Exton [3] presented the following q-exponential functions:

$$E(\mu, z; q) = \sum_{n=0}^{\infty} \frac{q^{\mu n(n-1)}}{[n]_q!} z^n,$$

where $[n]_q! = \frac{(q;q)_n}{(1-q)^n}$. The relation between these two notations is given by

$$E(\lambda, z; q) = E_q^{(2\lambda)}\big(q^{-\lambda}(1-q)z\big).$$

In Exton's formula, if we replace z by $\frac{x}{1-q}$ and μ by $2a$, we get the following q-exponential function:

$$E_q(x,a) = \sum_{n=0}^{\infty} \frac{q^{a\binom{n}{2}}}{(q;q)_n} x^n, \tag{1}$$

which satisfies the functional relation [4]

$$E_q(x,a) - E_q(qx,a) = xE_q\big(q^a x, a\big),$$

which can be rewritten by the formula

$$D_q E_q(x,a) = \frac{1}{1-q} E_q\big(q^a x, a\big), \tag{2}$$

where the Jackson q-difference operator D_q is defined by [5]

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x} \tag{3}$$

and satisfies the product rule

$$D_q\big(f(x)g(x)\big) = f(qx)D_q g(x) + g(x)D_q f(x).$$

There are two important special cases of the function $E_q(x,a)$

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q,q)_n}, \quad |x| < 1 \tag{4}$$

and

$$E_q(x) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} x^n}{(q,q)_n}. \tag{5}$$

The q-Bessel functions of kinds 1, 2 and 3 are defined by [6]

$$J_n^{(1)}(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^n {}_2\varphi_1\left(\begin{matrix} 0,0 \\ q^{n+1} \end{matrix} \,\middle|\, q; -\frac{x^2}{4}\right), \tag{6}$$

$$J_n^{(2)}(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^n {}_0\varphi_1\left(\begin{matrix} - \\ q^{n+1} \end{matrix} \,\middle|\, q; -\frac{q^{n+1}x^2}{4}\right), \tag{7}$$

$$J_n^{(3)}(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^n {}_1\varphi_1\left(\begin{matrix} 0 \\ q^{n+1} \end{matrix} \,\middle|\, q; -\frac{q^{\frac{n+1}{2}}x^2}{4}\right), \tag{8}$$

where ${}_r\varphi_s$ is the basic hypergeometric function [1]

$$ {}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix} \middle| q; z \right) = \sum_{k=0}^{\infty} \frac{(a_1, \dots, a_r; q)_k}{(b_1, \dots, b_s; q)_k} \left((-1)^k q^{\frac{k}{2}(k-1)} \right)^{1+s-r} \frac{z^k}{(q;q)_k}. \tag{9} $$

The functions $J_n^{(i)}(x;q)$, $i = 1, 2$, are q-analogues of the Bessel function, and the function $J_n^{(3)}(x;q)$ is a q-analogue of the modified Bessel function.

Rogov [7, 8] introduced generalized modified q-Bessel functions, similarly to the classical case [9], as

$$ I_n^i(x;q) = \frac{(q^{n+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2} \right)^n {}_\delta\varphi_1 \left(\begin{matrix} 0, 0, \dots, 0 \\ q^{n+1} \end{matrix} \middle| q; \frac{q^{\frac{(n+1)(2-\delta)}{2}} x^2}{4} \right), $$

where

$$ \delta = \begin{cases} 2 & \text{for } i = 1, \\ 0 & \text{for } i = 2, \\ 1 & \text{for } i = 3. \end{cases} $$

Recently, Mansour and *et al.* [10] studied the following q-Bessel function:

$$ J_n^{(4)}(x;q) = \frac{(x/2)^n}{(q;q)_n} {}_0\varphi_2 \left(\begin{matrix} - \\ 0, q^{n+1} \end{matrix} \middle| q; \frac{-q^{\frac{3(n+1)}{2}} x^2}{4} \right), $$

which is a q-analogy of the modified Bessel function.

In this paper, we define the generalized q-Bessel function and study some of its properties. Also, in analogy with the ordinary Lie theory [11, 12], we derive algebraically the generating function of the generalized q-Bessel function.

2 The generalized q-Bessel function and its generating function

Definition 2.1 The generalized q-Bessel function is defined by

$$ J_n(x,a;q) = \frac{(x/2)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak}{2}(k+n)}}{(q^{n+1};q)_k} \frac{(x^2/4)^k}{(q;q)_k}, \tag{10} $$

which converges absolutely for all x when $a \in \mathbb{Z}^+$ and for $|x| < 2$ if $a = 0$.

As special cases of $J_n(x,a;q)$, we get

$$ \begin{aligned} J_n^{(1)}(x;q) &= J_n(x,0;q), \\ J_n^{(2)}(x;q) &= J_n(x,2;q), \\ J_n^{(3)}(x;q) &= J_n(x,1;q), \\ J_n^{(4)}(x;q) &= J_n(x,3;q). \end{aligned} $$

Lemma 2.2 *The function $J_n(x,a;q)$ is a q-analogy of each of the Bessel function and the modified Bessel function.*

Proof

$$\begin{aligned}\lim_{q\to 1} J_n\big((1-q)x,a;q\big) &= \lim_{q\to 1}\left\{\frac{(1-q)^n}{(q,q)_n}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty}(-1)^{k(a+1)} q^{\frac{ak(k+n)}{2}} \frac{(1-q)^{2k}}{(q^{n+1};q)_k(q;q)_k}\left(\frac{x}{2}\right)^{2k}\right\} \\ &= \frac{1}{(1)_n}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)}}{(n+1)_k(1)_k}\left(\frac{x}{2}\right)^{2k} \\ &= \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)}}{\Gamma(n+k+1)\Gamma(k+1)}\left(\frac{x}{2}\right)^{2k}.\end{aligned}$$

Hence, we get

$$\lim_{q\to 1} J_n\big((1-q)x,a;q\big) = J_n(x); \quad a = 0,2,4,\ldots \tag{11}$$

and

$$\lim_{q\to 1} J_n\big((1-q)x,a;q\big) = I_n(x); \quad a = 1,3,5,\ldots, \tag{12}$$

where $J_n(x)$ is the Bessel function and $I_n(x)$ is the modified Bessel function. □

Lemma 2.3 *The function $J_n(x,a;q)$ satisfies*

$$J_{-n}(x,a;q) = (-1)^{n(a+1)} J_n(x,a;q), \quad n \in \mathbb{Z}. \tag{13}$$

Proof Using the definition (10), we get

$$J_{-n}(x,a;q) = \sum_{k=n}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak(k-n)}{2}} (q^{-n+k+1};q)_\infty}{(q;q)_\infty (q;q)_k}\left(\frac{x}{2}\right)^{2k-n}.$$

For $s = k - n$, we obtain

$$J_{-n}(x,a;q) = \sum_{s=0}^{\infty} \frac{(-1)^{(s+n)(a+1)} q^{\frac{as(s+n)}{2}} (q^{s+1};q)_\infty}{(q;q)_\infty (q;q)_{s+n}}\left(\frac{x}{2}\right)^{2s+n},$$

and using the relations [1]

$$\begin{aligned}&\big(q^{s+1};q\big)_\infty = \big(q^{n+s+1};q\big)_\infty \big(q^{s+1};q\big)_n, \\ &(q;q)_{s+n} = (q;q)_s \big(q^{s+1};q\big)_n,\end{aligned}$$

we obtain

$$\begin{aligned}J_{-n}(x,a;q) &= (-1)^{n(a+1)} \sum_{s=0}^{\infty} \frac{(-1)^{s(a+1)} q^{\frac{as(s+n)}{2}} (q^{n+s+1};q)_\infty}{(q;q)_\infty (q;q)_s}\left(\frac{x}{2}\right)^{2s+n} \\ &= (-1)^{n(a+1)} J_n(x,a;q).\end{aligned}$$

□

Lemma 2.4 *The function $J_n(x,a;q)$ satisfies the relation*

$$J_n(-x,a;q) = (-1)^n J_n(x,a;q), \quad n \in \mathbb{Z}, \tag{14}$$

and hence it is even (*or odd*) *function if the integer n is even* (*or odd*).

Now we will deduce the generating function of the generalized q-Bessel function $J_n(x,a;q)$.

Theorem 1 *The generating function $g(x,t,a;q)$ of the function $J_n(x,a;q)$ is given by*

$$g(x,t,a;q) = E_q\left(\frac{q^{\frac{a}{4}}xt}{2}, \frac{a}{2}\right) E_q\left(\frac{(-1)^{a+1}q^{\frac{a}{4}}x}{2t}, \frac{a}{2}\right) = \sum_{n=-\infty}^{\infty} q^{\frac{an^2}{4}} J_n(x,a;q)t^n. \tag{15}$$

Proof Let

$$g(x,t,a;q) = E_q\left(\frac{q^{\frac{a}{4}}xt}{2}, \frac{a}{2}\right) E_q\left(\frac{(-1)^{a+1}q^{\frac{a}{4}}x}{2t}, \frac{a}{2}\right),$$

then

$$g(x,t,a;q) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{(-1)^{s(a+1)}q^{\frac{a}{2}[\binom{r}{2}+\binom{s}{2}]+\frac{a}{4}(s+r)}}{(q;q)_r(q;q)_s} \left(\frac{x}{2}\right)^{s+r} t^{r-s}.$$

For $s = r - n$, we get

$$g(x,t,a;q) = \sum_{n=-\infty}^{\infty}\sum_{r=0}^{\infty} \frac{(-1)^{(r-n)(a+1)}q^{\frac{a}{2}[\binom{r}{2}+\binom{r-n}{2}]+\frac{a}{4}(2r-n)}}{(q;q)_r(q;q)_{r-n}} \left(\frac{x}{2}\right)^{2r-n} t^{n}.$$

Hence, for $n \leq 0$, the coefficient of t^n is given by

$$c_n = (-1)^{-n(a+1)} q^{\frac{an^2}{4}} \frac{(x/2)^{-n}}{(q;q)_{-n}} \sum_{r=0}^{\infty} \frac{(-1)^{r(a+1)}q^{\frac{ar}{2}(r-n)}}{(q;q)_r(q^{-n+1};q)_r} \left(\frac{x}{2}\right)^{2r},$$

where $(q,q)_{r-n} = (q;q)_{-n}(q^{-n+1};q)_r$ for $n \leq 0$. Then

$$c_n = (-1)^{-n(a+1)} q^{\frac{an^2}{4}} J_{-n}(x,a;q) = q^{\frac{an^2}{4}} J_n(x,a;q).$$

Similarly, for $n \geq 0$. □

As special cases of $g_n(x,t,a;q)$, we obtain

$$g_n(x,t,0;q) = E_q\left(\frac{xt}{2},0\right) E_q\left(\frac{-x}{2t},0\right) = e_q\left(\frac{xt}{2}\right) e_q\left(\frac{-x}{2t}\right), \tag{16}$$

which is a generating function of the q-Bessel function $J_n^{(1)}(x;q)$ [13],

$$g_n\left(x, \frac{t}{\sqrt{q}}, 2; q\right) = E_q\left(\frac{xt}{2},1\right) E_q\left(\frac{-qx}{2t},1\right) = E_q\left(\frac{xt}{2}\right) E_q\left(\frac{-qx}{2t}\right), \tag{17}$$

which is a generating function of the q-Bessel function $J_n^{(2)}(x;q)$ [13],

$$g_n(x,t,1;q) = E_q\left(\frac{\sqrt[4]{q}xt}{2},\frac{1}{2}\right)E_q\left(\frac{\sqrt[4]{q}x}{2t},\frac{1}{2}\right), \tag{18}$$

which is a generating function of the q-Bessel function $J_n^{(3)}(x;q)$ and

$$g_n(x,t,3;q) = E_q\left(\frac{q^{3/4}xt}{2},\frac{3}{2}\right)E_q\left(\frac{q^{3/4}x}{2t},\frac{3}{2}\right), \tag{19}$$

which is a generating function of the q-Bessel function $J_n^{(4)}(x;q)$ [10].

3 The q-difference equation of the function $J_n(x,a;q)$

Now the generating function method [13] will be used to deduce the q-difference equation of the generalized q-Bessel function. Using equation (15), we have

$$E_q\left(\frac{q^{\frac{a}{4}}xth}{2},\frac{a}{2}\right)E_q\left(\frac{(-1)^{a+1}q^{\frac{a}{4}}xh}{2t},\frac{a}{2}\right) = \sum_{n=-\infty}^{\infty} q^{\frac{an^2}{4}} J_n(xh,a;q)t^n; \quad h\in\mathbb{R}-\{0\}. \tag{20}$$

By applying the operator D_q, we get

$$\begin{aligned}&\frac{(-1)^{a+1}q^{\frac{a}{4}}h}{2(1-q)t}E_q\left(\frac{q^{\frac{a+4}{4}}xth}{2},\frac{a}{2}\right)E_q\left(\frac{(-1)^{a+1}q^{\frac{3a}{4}}xh}{2t},\frac{a}{2}\right)\\&\quad+\frac{q^{\frac{a}{4}}th}{2(1-q)}E_q\left(\frac{q^{\frac{3a}{4}}xth}{2},\frac{a}{2}\right)E_q\left(\frac{(-1)^{a+1}q^{\frac{a}{4}}xh}{2t},\frac{a}{2}\right)\\&=\sum_{n=-\infty}^{\infty} q^{\frac{an^2}{4}} D_qJ_n(xh,a;q)t^n.\end{aligned}$$

Using equation (20), we obtain

$$\begin{aligned}&\frac{(-1)^{a+1}q^{\frac{a}{4}+\frac{a}{2}\binom{n+1}{2}+\frac{n+1}{2}}}{2(1-q)}J_{n+1}\left(q^{\frac{a+2}{4}}xh,a;q\right)+\frac{q^{\frac{a}{4}+\frac{an(n-1)}{4}}}{2(1-q)}J_{n-1}\left(q^{\frac{a}{4}}xh,a;q\right)\\&=\frac{q^{\frac{an^2}{4}}}{h}D_qJ_n(xh,a;q).\end{aligned}$$

Hence

$$\begin{aligned}&D_qJ_n(xh,a;q)\\&=\frac{q^{\frac{a}{4}}h}{2(1-q)}\left\{(-1)^{a+1}q^{\frac{n(a+2)+2}{4}}J_{n+1}\left(q^{\frac{a+2}{4}}xh,a;q\right)+q^{\frac{-an}{4}}J_{n-1}\left(q^{\frac{a}{4}}xh,a;q\right)\right\}.\end{aligned} \tag{21}$$

Similarly, we can prove the following relation:

$$\begin{aligned}&D_qJ_n(xh,a;q)\\&=\frac{q^{\frac{a}{4}}h}{2(1-q)}\left\{(-1)^{a+1}q^{\frac{na}{4}}J_{n+1}\left(q^{\frac{a}{4}}xh,a;q\right)+q^{\frac{(-an-2n+2)}{4}}J_{n-1}\left(q^{\frac{a+2}{4}}xh,a;q\right)\right\}.\end{aligned} \tag{22}$$

By using equations (21) and (22), we obtain

$$\begin{aligned}&(-1)^{a+1}q^{\frac{n(a+1)+1}{2}}J_{n+1}(\sqrt{q}xh,a;q)+J_{n-1}(xh,a;q)\\&\quad=(-1)^{(a+1)}q^{\frac{an}{2}}J_{n+1}(xh,a;q)+q^{\frac{1-n}{2}}J_{n-1}(\sqrt{q}xh,a;q).\end{aligned}\tag{23}$$

But

$$\begin{aligned}J_{n-1}(\sqrt{q}xh,a;q)&=\frac{(\frac{\sqrt{q}xh}{2})^{n-1}}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k(a+1)}q^{\frac{ak}{2}(k+n-1)}(q^{n+k};q)_\infty}{(q;q)_k}\left(\frac{\sqrt{q}xh}{2}\right)^{2k}\\&=\frac{q^{\frac{n-1}{2}}(\frac{xh}{2})^{n-1}}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^{k(a+1)}q^{\frac{ak}{2}(k+n-1)}(q^{n+k};q)_\infty}{(q;q)_k}(q^k-1+1)\left(\frac{xh}{2}\right)^{2k}\\&=q^{\frac{n-1}{2}}J_{n-1}(xh,a;q)-\frac{q^{\frac{n-1}{2}}(\frac{xh}{2})^{n-1}}{(q;q)_\infty}\sum_{k=0}^{\infty}\frac{(-1)^{(k+1)(a+1)}q^{\frac{a(k+1)(k+n)}{2}}}{(q;q)_k}\\&\quad\times\left(q^{n+k+1};q\right)_\infty\left(\frac{xh}{2}\right)^{2k+2}\\&=q^{\frac{n-1}{2}}\left\{J_{n-1}(xh,a;q)+\frac{(-1)^a q^{\frac{an}{2}}xh}{2}J_n\left(q^{\frac{a}{4}}xh,a;q\right)\right\}.\end{aligned}$$

Then

$$q^{\frac{1-n}{2}}J_{n-1}(\sqrt{q}xh,a;q)-J_{n-1}(xh,a;q)=\frac{(-1)^a q^{\frac{an}{2}}xh}{2}J_n\left(q^{\frac{a}{4}}xh,a;q\right).\tag{24}$$

Equations (23) and (24) give the relation

$$q^{\frac{1+n}{2}}J_{n+1}(\sqrt{q}xh,a;q)-J_{n+1}(xh,a;q)=\frac{-xh}{2}J_n\left(q^{\frac{a}{4}}xh,a;q\right).\tag{25}$$

Equations (21) and (25) give the relation

$$\left\{D_q+\frac{q^{\frac{a(1-n)}{4}}}{(1-q)x}\left[q^{\frac{n}{2}}\delta_q-1\right]\right\}J_n(xh,a;q)=\frac{(-1)^{a+1}hq^{\frac{(a+2)(n+1)}{4}}}{2(1-q)}J_{n+1}\left(q^{\frac{a+2}{4}}xh,a;q\right),\tag{26}$$

where the operator δ_q is given by $\delta_q f(x)=f(\sqrt{q}x)$.

Now consider the following operator:

$$M_{n,q}=\left\{D_q+\frac{q^{\frac{a(1-n)}{4}}}{(1-q)x}\left[q^{\frac{n}{2}}\delta_q-1\right]\right\},\tag{27}$$

then we can rewrite equation (26) by the formula

$$M_{n,q}J_n(xh,a;q)=\frac{(-1)^{a+1}hq^{\frac{(a+2)(n+1)}{4}}}{2(1-q)}J_{n+1}\left(q^{\frac{a+2}{4}}xh,a;q\right).\tag{28}$$

Also, equations (22) and (24) give the relation

$$\left\{D_q+\frac{q^{\frac{-a(1+n)}{4}}}{(1-q)x}\left[q^{\frac{-n}{2}}\delta_q-1\right]\right\}J_n(xh,a;q)=\frac{hq^{\frac{(a+2)(1-n)}{4}}}{2(1-q)}J_{n-1}\left(q^{\frac{a+2}{4}}xh,a;q\right).\tag{29}$$

If we consider the operator

$$N_{n,q} = \left\{ D_q + \frac{q^{\frac{-a(1+n)}{4}}}{(1-q)x} \left[q^{\frac{-n}{2}} \delta_q - 1 \right] \right\}, \tag{30}$$

then we can rewrite equation (29) by the formula

$$N_{n,q} J_n(xh, a; q) = \frac{hq^{\frac{(a+2)(1-n)}{4}}}{2(1-q)} J_{n-1}\left(q^{\frac{a+2}{4}} xh, a; q\right). \tag{31}$$

Hence, the q-difference equation of the function $J_n(x, a; q)$ takes the formula

$$M_{n-1,q} N_{n,q} J_n(xh, a; q) = \frac{(-1)^{a+1} q^{\frac{a+2}{4}} h^2}{4(1-q)^2} J_n\left(q^{\frac{a+2}{2}} xh, a; q\right). \tag{32}$$

If we replace h by $1 - q$ and consider the limit as q tends to 1, then we obtain

$$\left(\frac{1}{2} \frac{d}{dx} - \frac{n-1}{2x} \right) \left(\frac{1}{2} \frac{d}{dx} + \frac{n}{2x} \right) y(x) = \frac{(-1)^{a+1}}{4} y(x), \tag{33}$$

or

$$x^2 y''(x) + xy' - \left(n^2 + (-1)^{a+1} x^2 \right) y = 0. \tag{34}$$

The differential equation (34) gives the Bessel function at $a = 0, 2, 4, \ldots$ and the modified Bessel function at $a = 1, 3, 5, \ldots$, which proves again that $J_n(x, a; q)$ is a q-analogy of each of them.

4 The recurrence relations of the function $J_n(x, a; q)$

Lemma 4.1

$$J_n(x, a; q) = \frac{2}{x} \left(1 - q^{n+1}\right) J_{n+1}\left(q^{-a/4} x, a; q\right) + (-1)^{a+1} q^{\frac{(a+2)(n+1)}{2}} J_{n+2}(x, a; q). \tag{35}$$

Proof

$$\begin{aligned}
J_n(x, a; q) &= \frac{(x/2)^n}{(q;q)_n} \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak}{2}(k+n)}}{(q;q)_k (q^{n+1};q)_{k+1}} \left(1 - q^{n+1} + q^{n+1} - q^{n+k-1}\right) \left(\frac{x^2}{4} \right)^k \\
&= \frac{2}{x} \left(1 - q^{n+1}\right) \frac{(x/2)^{n+1}}{(q;q)_{n+1}} \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak}{2}(k+n+1)}}{(q;q)_k (q^{n+2};q)_k} \left(\frac{(q^{-\frac{a}{4}} x)^2}{4} \right)^k \\
&\quad + q^{n+1} \frac{(x/2)^{n+2}}{(q;q)_{n+2}} \sum_{k=1}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak}{2}(k+n)}}{(q;q)_{k-1} (q^{n+3};q)_{k-1}} \left(\frac{x^2}{4} \right)^{k-1} \\
&= \frac{2}{x} \left(1 - q^{n+1}\right) J_{n+1}\left(q^{-a/4} x, a; q\right) \\
&\quad + (-1)^{a+1} q^{\frac{(a+2)(n+1)}{2}} \frac{(x/2)^{n+2}}{(q;q)_{n+2}} \sum_{k=0}^{\infty} \frac{(-1)^{k(a+1)} q^{\frac{ak}{2}(k+n+2)}}{(q;q)_k (q^{n+3};q)_k} \left(\frac{x^2}{4} \right)^k \\
&= \frac{2}{x} \left(1 - q^{n+1}\right) J_{n+1}\left(q^{-a/4} x, a; q\right) + (-1)^{a+1} q^{\frac{(a+2)(n+1)}{2}} J_{n+2}(x, a; q).
\end{aligned}$$

□

Similarly, if we write $(1-q^k+q^k-q^{n+k-1})$ instead of $(1-q^{n+1}+q^{n+1}-q^{n+k-1})$, we can prove the following lemma.

Lemma 4.2

$$J_n(x,a;q)=\frac{2}{x}\bigl(1-q^{n+1}\bigr)J_{n+1}\bigl(q^{\frac{2-a}{4}}x,a;q\bigr)+(-1)^{a+1}q^{\frac{a(n+1)}{2}}J_{n+2}(x,a;q). \tag{36}$$

Now, if we replace a by $a+2$ in the recurrence relation (36), we get the recurrence relation (35). Then we have the following lemma.

Lemma 4.3 *The two functions $J_n(x,a;q)$ and $J_n(x,a+2;q)$ have the same recurrence relation.*

Then we have two cases of the recurrence relation.

Case (1): The function $J_n(x,a;q)$ has the recurrence relation

$$J_n(x,a;q)=\frac{2}{x}\bigl(1-q^{n+1}\bigr)J_{n+1}(x,a;q)-q^{n+1}J_{n+2}(x,a;q);\quad \forall a=0,2,4,\ldots, \tag{37}$$

which is the recurrence relation of each of $J_n^{(1)}(x;q)$ and $J_n^{(2)}(x;q)$.

Case (2): The function $J_n(x,a;q)$ has the recurrence relation

$$\begin{aligned}J_n(x,a;q)&=q^{\frac{n}{4}}\left[\frac{2}{x}\bigl(1-q^{n+1}\bigr)-q^{\frac{n}{2}}\frac{x}{2}\right]J_{n+1}(x,a;q)\\&\quad+q^{n+1/2}J_{n+2}(x,a;q);\quad \forall a=1,3,5,\ldots,\end{aligned} \tag{38}$$

which is the recurrence relation of each of $J_n^{(3)}(x;q)$ and $J_n^{(4)}(x;q)$.

5 The quantum algebra approach to $J_n(x,a;q)$

The quantum algebra $E_q(2)$ is determined by generators H, E_+ and E_- with the commutation relations

$$[H,E_+]=E_+,\qquad [H,E_-]=-E_-,\qquad [E_-,E_+]=0. \tag{39}$$

By considering the irreducible representations (ω) of $E_q(2)$ characterized by $\omega\in\mathbb{C}$, then the spectrum of the operator H will be the set of integers $\mathbb{Z}$, and the basis vectors f_m, $m\in\mathbb{Z}$, satisfy

$$E_\pm f_m=\omega f_{m\pm1},\qquad Hf_m=mf_m,\qquad E_+E_-f_m=\omega^2 f_m, \tag{40}$$

where $C=E_+E_-$ is the Casimir operator which commutes with the generators H, E_+ and E_-. The following differential operators presented a simple realization of (ω)

$$H=z\frac{d}{dz},\qquad E_+=\omega z,\qquad E_-=\frac{\omega}{z} \tag{41}$$

acting on the space of all linear combinations of the functions z^m, z a complex variable, $m\in\mathbb{Z}$, with basis vectors $f_m(z)=z^m$.

In the ordinary Lie theory, matrix elements T_{sm} of the complex motion group in the representation (ω) are typically defined by the expansions [7–9]

$$e^{\alpha E_+}e^{\beta E_-}e^{\tau H}f_m = \sum_{s=-\infty}^{\infty} T_{sm}(\alpha,\beta,\tau)f_s. \tag{42}$$

If we replace the mapping e^x by the mapping $E_q(x,a/2)$ from the Lie algebra to the Lie group with putting $\tau = 0$ in equation (42), we can use the model (41) to find the following q-analog of matrix elements of (ω):

$$E_q(\alpha E_+, a/2)E_q(\beta E_-, a/2)f_m = \sum_{s=-\infty}^{\infty} T_{sm}(\alpha,\beta)f_s, \tag{43}$$

and hence

$$E_q\left(\alpha\omega z, \frac{a}{2}\right)E_q\left(\beta\frac{\omega}{z}, \frac{a}{2}\right)z^m = \sum_{r,t=0}^{\infty} \frac{q^{\frac{a}{2}[\binom{r}{2}+\binom{t}{2}]}}{(q;q)_r(q;q)_t}\omega^{r+t}\alpha^r\beta^t z^{m-t+r}.$$

Now replace s by $m-t+r$ to get

$$E_q\left(\alpha\omega z, \frac{a}{2}\right)E_q\left(\beta\frac{\omega}{z}, \frac{a}{2}\right)z^m = \sum_{s=-\infty}^{\infty}\sum_{r=0}^{\infty} \frac{q^{\frac{a}{2}[\binom{r}{2}+\binom{m+r-s}{2}]}}{(q;q)_r(q;q)_{m+r-s}}\omega^{m+2r-s}\alpha^r\beta^{m+r-s}z^s$$

and by equating the coefficient of z^s for $m \ge s$ on both sides, we get

$$\begin{aligned}
T_{sm}(\alpha,\beta) &= \frac{q^{\frac{a}{2}\binom{m-s}{2}}}{(q;q)_{m-s}}(\omega\beta)^{m-s}\sum_{r=0}^{\infty}\frac{q^{\frac{a}{2}r(r+m-s-1)}}{(q;q)_r(q^{m-s+1};q)_r}\left(\omega^2\alpha\beta\right)^r, \quad m \ge s \\
&= q^{\frac{a}{4}(m-s)^2}\left((-1)^{-(a+1)}\frac{\beta}{\alpha}\right)^{\frac{m-s}{2}}(-1)^{(m-s)(a+1)}J_{m-s}\left(2q^{\frac{-a}{4}}\omega\sqrt{(-1)^{a+1}\alpha\beta}, a; q\right), \\
&\quad (-1)^{a+1}\alpha\beta > 0; m \ge s \\
&= q^{\frac{a}{4}(s-m)^2}\left((-1)^{(a+1)}\frac{\alpha}{\beta}\right)^{\frac{s-m}{2}}J_{s-m}\left(2q^{\frac{-a}{4}}\omega\sqrt{(-1)^{a+1}\alpha\beta}, a; q\right), \\
&\quad (-1)^{a+1}\alpha\beta > 0; m \ge s,
\end{aligned}$$

where $J_{-n}(x,a;q) = (-1)^{n(a+1)}J_n(x,a;q)$.

Similarly,

$$T_{sm}(\alpha,\beta) = q^{\frac{a}{4}(s-m)^2}\left((-1)^{(a+1)}\frac{\alpha}{\beta}\right)^{\frac{s-m}{2}}J_{s-m}\left(2q^{\frac{-a}{4}}\omega\sqrt{(-1)^{a+1}\alpha\beta}, a; q\right),$$
$$(-1)^{a+1}\alpha\beta > 0; s \ge m.$$

The combination between the two cases gives us the following expression:

$$T_{sm}(\alpha,\beta) = q^{\frac{a}{4}(s-m)^2}\left((-1)^{(a+1)}\frac{\alpha}{\beta}\right)^{\frac{s-m}{2}}J_{s-m}\left(2q^{\frac{-a}{4}}\omega\sqrt{(-1)^{a+1}\alpha\beta}, a; q\right),$$
$$(-1)^{a+1}\alpha\beta > 0, \tag{44}$$

which is valid for all $m, s \in \mathbb{Z}$. Then we get the following result.

Lemma 5.1

$$E_q\left(\alpha\omega z, \frac{a}{2}\right)E_q\left(\beta\frac{\omega}{z}, \frac{a}{2}\right) = \sum_{s=-\infty}^{\infty} q^{\frac{a}{4}s^2}\left((-1)^{(a+1)}\frac{\alpha}{\beta}\right)^{\frac{s}{2}} J_s\left(2q^{\frac{-a}{4}}\omega\sqrt{(-1)^{a+1}\alpha\beta}, a; q\right)z^s, \tag{45}$$

where $(-1)^{a+1}\alpha\beta > 0$.

As special cases:
Considering (45) with $a = 0$, $\alpha = -\beta = 1$, $z = t$ and $\omega = \frac{x}{2}$, we obtain the relation (16).
Considering (45) with $a = 2$, $\alpha = -\beta = 1$, $z = \frac{t}{\sqrt{q}}$ and $\omega = \frac{\sqrt{q}x}{2}$, we obtain the relation (17).
Considering (45) with $a = 1$, $\alpha = \beta = 1$, $z = t$ and $\omega = \frac{\sqrt[4]{q}x}{2}$, we obtain the relation (18).
Considering (45) with $a = 3$, $\alpha = \beta = 1$, $z = t$ and $\omega = \frac{q^{3/4}x}{2}$, we obtain the relation (19).

Competing interests
The author declares that he has no competing interests.

Author's contributions
The author read and approved the final manuscript.

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