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Controllability and observability of complex [r]-matrix time-varying impulsive systems

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Abstract

Since many equations of practical systems such as Schrödinger equation, Ginzburg-Landau equation and Orr-Sommerfeld equation are defined in complex number fields, in this paper, the issue of controllability and observability for an [r]-matrix time-varying impulsive system defined in complex fields is addressed. Several sufficient and necessary conditions for state controllability and observability of such a system are established. Meanwhile, corresponding controllability and observability criteria for the [r]-matrix time-invariant impulsive system are also obtained.

MSC: 34H05; 34H99

Keywords: complex-valued systems; impulsive systems; matrix differential equation; controllability; observability

1 Introduction

Since many evolution processes, optimal control models in economics, stimulated neural networks, frequency-modulated systems and some motions of missiles or aircrafts are characterized by impulsive dynamical behavior, the study of impulsive systems is of great importance. Nowadays, there has been an increasing interest in the analysis and synthesis of impulsive systems, or impulsive control systems, due to their theoretical and practical significance; for example, [1–12] and the references therein.

As the fundamental issues of modern control theory, the controllability and observability have been studied extensively in the context of finite-dimensional linear systems, nonlinear systems, infinite-dimensional systems, n-dimensional systems and hybrid systems using different kinds of approaches [13–17]. In particular, many efforts have been focused on the problem of controllability and observability for various kinds of impulsive systems using different approaches. The geometric analysis of reachability, controllability and observability for impulsive systems in terms of invariant subspaces were presented in [16, 18]. By proposing the rank condition, Guan *et al.* [17], Zhao *et al.* [10] and Shi *et al.* [15] proposed the sufficient and necessary conditions for state controllability and observability of different kinds of linear time-varying impulsive systems, respectively.

However, the common setting adopted in the above-mentioned works except [3, 10, 19] is always in real number fields. In fact, many classical systems such as Schrödinger equation, Ginzburg-Landau equation, Riccati equation and Orr-Sommerfeld equation are considered in complex number fields. But there have been few reports about the analysis and synthesis of complex dynamical systems; for example [20–25] and references therein.

More abstract than real system, the control theory of complex-valued dynamical systems has many potential applications in science and engineering. For example, recently research on the control theory of quantum systems has attracted considerable attention [26–29]. Quantum systems are a class of complex dynamical systems which take values in a Banach space in a complex field.

Matrix differential equations are relevant to the description of many phenomena in physics and engineering, ranging from such diverse applications as control theory to game theory [30]. The motivation for considering $[r]$-matrix differential systems arises from the demand for a level of generality sufficient to deal with the increasingly important matrix linear systems of control theory such as those associated with matrix Riccati differential equations and matrix bilinear control systems [24, 31–35]. In particular, recently research on the control theory of multidimentional systems has attracted attention of quite a few scientists [36, 37]. Multidimentional systems are a class of matrix differential systems which have extensive application in image processing. Due to these reasons, it is important and necessary to study the control theory of complex matrix impulsive systems.

To the best of our knowledge, there is no result so far about the control theory of complex matrix impulsive systems. Inspired by [10, 22], in this paper, we consider the fundamental concepts of controllability and observability of complex $[r]$-matrix time-varying impulsive systems by an algebraic approach. The main difficulty is to investigate the conditions for controllability and observability of complex $[r]$-matrix impulsive systems in the context of complex matrices. Explicit characterization for controllability and observability of this kind of a system in terms of the rank conditions is presented by use of the matrix differential theory in a complex field. These questions are meaningful and challenging.

The paper is organized as follows. In Section 2, the complex matrix time-varying impulsive systems to be dealt with are formulated and several new results about the variation of parameters for such systems are presented. Several sufficient and necessary conditions for state controllability and state observability of complex matrix time-varying impulsive systems and corresponding complex matrix time-invariant impulsive systems are established in Sections 3 and 4, respectively. An example is given to explain those results in Section 5. Finally, some conclusions are drawn in Section 6.

2 Notations and preliminaries

In order to make precise the concept of a complex $[r]$-matrix time-varying impulsive system, we use the terminologies in [22] and [24]. Let $M^{p\times q}(\mathbb{C}^{r\times r})$ be the set of all block matrices with $p \geq 1$ rows and $q \geq 1$ columns over the ring $\mathbb{C}^{r\times r}$ of all $r \times r$ complex matrices. Fix an open interval $I \subset \mathbb{R}$ and let the symbols $L^1_{\mathrm{loc}}(I, \mathbb{C})$, $L^\infty_{\mathrm{loc}}(I, \mathbb{C})$ and $AC_{\mathrm{loc}}(I, \mathbb{C})$ denote, respectively, the spaces of complex-valued Lebesgue measurable functions on I which are locally integrable, locally bounded and locally absolutely continuous on I. Here 'local' implies a property holding on all compact subintervals of I. In the same way, denote by $L^1_{\mathrm{loc}}(I, M^{p\times q}(\mathbb{C}^{r\times r}))$ all $p \times q$ $[r]$-matrices whose entries are locally integrable on I, and similar notations hold for $L^\infty_{\mathrm{loc}}(I, M^{p\times q}(\mathbb{C}^{r\times r}))$ and other relevant classes of matrix functions on I.

We are now able to define an $[r]$-matrix linear time-varying impulsive system on the interval $I = [t_0, +\infty)$

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)u(t), & t \neq t_k, \\ \triangle x = E_k x(t_k) + F_k u_k, & t = t_k, \\ y(t) = C(t)x(t) + D(t)u(t), \\ x(t_0{}^+) = x_0, \end{cases} \tag{1}$$

where $k = 1, 2, \ldots$, $A(t) \in L^1_{\mathtt{loc}}(I, M^{n\times n}(\mathbb{C}^{r\times r}))$, $B(t) \in L^1_{\mathtt{loc}}(I, M^{n\times m}(\mathbb{C}^{r\times r}))$, $C(t) \in L^1_{\mathtt{loc}}(I, M^{p\times n}(\mathbb{C}^{r\times r}))$, $D(t) \in L^1_{\mathtt{loc}}(I, M^{p\times m}(\mathbb{C}^{r\times r}))$, $x \in L^1_{\mathtt{loc}}(I, M^{n\times 1}(\mathbb{C}^{r\times r}))$ is the state vector, $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$ is the initial state, $u(t) \in L^\infty_{\mathtt{loc}}(I, M^{m\times 1}(\mathbb{C}^{r\times r})) \cap L^1_{\mathtt{loc}}(I, M^{m\times 1}(\mathbb{C}^{r\times r}))$ is the control input, $u_k = u(t_k)$, $E_k \in M^{n\times n}(\mathbb{C}^{r\times r})$, $F_k \in M^{n\times m}(\mathbb{C}^{r\times r})$, $y \in L^1_{\mathtt{loc}}(I, M^{p\times 1}(\mathbb{C}^{r\times r}))$ is the output, $\triangle x(t_k) = x(t_k^+) - x(t_k^-)$, where $x(t_k^+) = \lim_{h\to 0^+} x(t_k + h)$, $x(t_k) = x(t_k^-) = \lim_{h\to 0^+} x(t_k - h)$, and the discontinuity points $t_0 < t_1 < t_2 < \cdots < t_k \cdots$, $\lim_{k\to\infty} t_k = +\infty$, which implies that the solution of (1) is left-continuous at t_k.

Definition 1 The $[r]$-matrix time-varying impulsive system (1) is called state controllable on $[t_0, t_f]$ $(t_f > t_0)$ if for any given initial state $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$, there exists a piecewise locally bounded controller $u(t) : [t_0, t_f] \to M^{m\times 1}(\mathbb{C}^{r\times r})$ such that the corresponding solution of (1) satisfies $x(t_f) = 0$.

Definition 2 System (1) is said to be observable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$ if any initial state $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$ can be uniquely determined by the corresponding system input $u(t)$ and output $y(t)$ for $t \in [t_0, t_f]$.

To study system (1), we first consider the following $[r]$-matrix time-varying system:

$$\dot{x}(t) = A(t)x(t) + B(t)u(t). \tag{2}$$

See Everitt and Markus [22], a solution $x(t)$ of (2) is a column $[r]$-vector in $AC_{\mathtt{loc}}(I_1, M^{n\times 1}(\mathbb{C}^{r\times r}))$, which is determined by a controller $u(t)$ on $I_1 = [t_0, t_1] \subset I$, and an initial state $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$, according to the classical Lagrange formula of variations of parameters

$$x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)B(s)u(s)\,ds. \tag{3}$$

Here $X(t, s) = X(t)X^{-1}(s)$ is the transition matrix and $X(t)$ is the $n \times n$ $[r]$-matrix fundamental solution characterized by

$$\dot{X}(t) = A(t)X(t).$$

Let A^* be the conjugated transpose of a complex matrix or a complex block matrix A, and let $\prod_{i=k}^{1} A_i$ stand for a matrix product $A_k A_{k-1} \cdots A_1$. Next we will present the solution of system (1).

Lemma 1 *For any $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots$, the solution of system* (1) *is*

$$x(t) = X(t, t_k)x\bigl(t_k^+\bigr) + \int_{t_k}^{t} X(t, s)B(s)u(s)\,ds, \tag{4}$$

where

$$
\begin{aligned}
x(t_k^+) = {} & \prod_{j=k}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
& + \sum_{i=1}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
& + \sum_{i=2}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_k u_k.
\end{aligned}
$$

Proof According to (3), we have

$$
x(t) = X(t, t_0)x_0 + \int_{t_0}^{t} X(t, s)B(s)u(s)\,ds, \quad t \in [t_0, t_1],
$$

so

$$
x(t_1) = X(t_1, t_0)x_0 + \int_{t_0}^{t_1} X(t_1, s)B(s)u(s)\,ds.
$$

Since $\triangle x(t_k) = x(t_k^+) - x(t_k) = E_k x(t_k) + F_k u_k$, we have

$$
\begin{aligned}
x(t_1^+) &= (I + E_1)x(t_1) + F_1 u_1 \\
&= (I + E_1)X(t_1, t_0)\left[x_0 + \int_{t_0}^{t_1} X(t_0, s)B(s)u(s)\,ds\right] + F_1 u_1.
\end{aligned}
\tag{5}
$$

For $t \in (t_1, t_2]$,

$$
x(t) = X(t, t_1)x(t_1^+) + \int_{t_1}^{t} X(t, s)B(s)u(s)\,ds,
$$

where $x(t_1^+)$ is given by (5). This implies that Lemma 1 holds for $k = 1$.

Now, assume that Lemma 1 holds when $k = m$, namely, for $t \in (t_m, t_{m+1}]$,

$$
x(t) = X(t, t_m)x(t_m^+) + \int_{t_m}^{t} X(t, s)B(s)u(s)\,ds,
\tag{6}
$$

where

$$
\begin{aligned}
x(t_m^+) = {} & \prod_{j=m}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
& + \sum_{i=1}^{m}\prod_{j=m}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
& + \sum_{i=2}^{m}\prod_{j=m}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_m u_m.
\end{aligned}
$$

Then, (6) leads to

$$x(t_{m+1}) = X(t_{m+1}, t_m)x(t_m^+) + \int_{t_m}^{t_{m+1}} X(t_{m+1}, s)B(s)u(s)\,ds,$$

and hence we have

$$\begin{aligned}
x(t_{m+1}^+) &= (I + E_{m+1})x(t_{m+1}) + F_{m+1}u_{m+1} \\
&= (I + E_{m+1})\Bigg[X(t_{m+1}, t_m)x(t_m^+) \\
&\quad + \int_{t_m}^{t_{m+1}} X(t_{m+1}, s)B(s)u(s)\,ds\Bigg] + F_{m+1}u_{m+1} \\
&= (I + E_{m+1})X(t_{m+1}, t_m)x(t_m^+) + (I + E_{m+1})X(t_{m+1}, t_m) \\
&\quad \times \int_{t_m}^{t_{m+1}} X(t_m, s)B(s)u(s)\,ds + F_{m+1}u_{m+1} \\
&= (I + E_{m+1})X(t_{m+1}, t_m)\Bigg[\prod_{j=m}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
&\quad + \sum_{i=1}^{m}\prod_{j=m}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
&\quad + \sum_{i=2}^{m}\prod_{j=m}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_m u_m\Bigg] \\
&\quad + (I + E_{m+1})X(t_{m+1}, t_m)\int_{t_m}^{t_{m+1}} X(t_m, s)B(s)u(s)\,ds + F_{m+1}u_{m+1} \\
&= \prod_{j=m+1}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
&\quad + \sum_{i=1}^{m+1}\prod_{j=m+1}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
&\quad + \sum_{i=2}^{m+1}\prod_{j=m+1}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_{m+1}u_{m+1}.
\end{aligned}$$

Thus, when $t \in (t_{m+1}, t_{m+2}]$,

$$x(t) = X(t, t_{m+1})x(t_{m+1}^+) + \int_{t_{m+1}}^{t} X(t, s)B(s)u(s)\,ds,$$

which implies that Lemma 1 is true when $k = m + 1$. According to the mathematical induction, we can immediately conclude that Lemma 1 is true. This completes the proof. □

3 Controllability

In this subsequent section, we discuss the controllability criteria of complex-valued $[r]$-matrix impulsive system (1) using the algebraic method.

For $t_f \in (t_k, t_{k+1}]$, assume that $I + E_j$ ($j = 1, \ldots, k$) are invertible and denote the following $n \times n$ block matrices:

$$\begin{aligned}
&\Phi_0 := I(\text{identity matrix}), \qquad \Phi_i := \prod_{j=1}^{i} X(t_{j-1}, t_j)(I + E_j)^{-1}, \\
&W_i := W(t_{i-1}, t_i) = \int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)B^*(s)X(t_{i-1}, s)^* \, ds, \\
&W_{k+1} := W(t_k, t_f) = \int_{t_k}^{t_f} X(t_k, s)B(s)B^*(s)X(t_{i-1}, s)^* \, ds, \\
&W(\Phi_{i-1}, t_{i-1}, t_i) := \int_{t_{i-1}}^{t_i} \Phi_{i-1}X(t_{i-1}, s)B(s)B^*(s)X(t_{i-1}, s)^* \Phi_{i-1}^* \, ds, \\
&W(\Phi_k, t_k, t_f) := \int_{t_k}^{t_f} \Phi_k X(t_k, s)B(s)B^*(s)X(t_k, s)^* \Phi_k^* \, ds, \\
&V_i := \Phi_i F_i, \quad i = 1, 2, \ldots, k.
\end{aligned} \tag{7}$$

Now we present a sufficient condition and a necessary condition for the controllability of $[r]$-matrix time-varying impulsive system (1).

Theorem 1 *System* (1) *is controllable on* $[t_0, t_f]$ ($t_f \in (t_k, t_{k+1}]$) *if there exist an* $l \in \{1, 2, \ldots, k\}$ *and a matrix* $G \in M^{m\times n}(\mathbb{C}^{r\times r})$ *such that* W_l *is invertible or* $GF_l = I$ (*identity matrix*).

Proof (i) Without loss of generality, suppose that there exists an $l \in \{1, 2, \ldots, k\}$ such that the complex matrix W_l is invertible. For an initial state x_0, choose

$$u(t) = \begin{cases} -B(t)^* X(t_{l-1}, t)^* W_l^{-1} \prod_{j=l-1}^{1} (I + E_j)X(t_j, t_{j-1})x_0, & t \in (t_{l-1}, t_l), \\ 0, & t \in [t_0, t_f] \setminus (t_{l-1}, t_l), \end{cases} \tag{8}$$

which implies also that the control is piecewise locally bounded on $[t_0, t_f]$. Thus applying (8) into (4) yields that

$$\begin{aligned}
x(t_f) &= X(t_f, t_k)\Bigg[\prod_{j=k}^{1} (I + E_j)X(t_j, t_{j-1})x_0 - \prod_{j=k}^{l} (I + E_j)X(t_j, t_{j-1}) \\
&\quad \times \int_{t_{l-1}}^{t_l} X(t_{l-1}, s)B(s)B(s)^* X(t_{l-1}, s)^* \, ds W^{-1}(t_{l-1}, t_l) \\
&\quad \times \prod_{j=l-1}^{1} (I + E_j)X(t_j, t_{j-1})x_0\Bigg] \\
&= X(t_f, t_k)\Bigg[\prod_{j=k}^{1} (I + E_j)X(t_j, t_{j-1}) - \prod_{j=k}^{1} (I + E_j)X(t_j, t_{j-1})\Bigg]x_0 = 0.
\end{aligned}$$

It follows that system (1) is controllable on $[t_0, t_f]$.

(ii) Without loss of generality, suppose that there exist an $l \in \{1, 2, \ldots, k\}$ and a complex matrix $G \in M^{m\times n}(\mathbb{C}^{r\times r})$ satisfying $GF_l = I$. Then, given an initial state $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$, we design the following control law:

$$u(t) = \begin{cases} -G\prod_{j=l}^{1}(I+E_j)X(t_j,t_{j-1})x_0, & t = t_l, \\ 0, & t \in [t_0,t_f] \setminus t_l, \end{cases} \tag{9}$$

which implies that the control $u(t)$ is piecewise locally bounded on $[t_0, t_f]$. By Lemma 1 and (9), the corresponding solution of (1) yields

$$\begin{aligned} x(t_f) &= X(t_f,t_k)\Bigg\{\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})x_0 - \prod_{j=k}^{l+1}(I+E_j)X(t_j,t_{j-1})F_lG \\ &\quad \times \prod_{j=l}^{1}(I+E_j)X(t_j,t_{j-1})x_0\Bigg\} \\ &= X(t_f,t_k)\Bigg[\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1}) - \prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\Bigg]x_0 = 0. \end{aligned}$$

Hence, by the definition of controllability, system (1) is controllable on $[t_0, t_f]$. This completes the proof. □

Theorem 2 *If system* (1) *is controllable on* $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$, *then*

$$\operatorname{rank}\big\{W(\Phi_0,t_0,t_1),\ldots,W(\Phi_k,t_k,t_f),V_1,\ldots,V_k\big\} = nr, \tag{10}$$

where $W(\cdot,\cdot,\cdot)$, Φ_i *and* V_i *are defined in* (7).

Proof Suppose that system (1) is controllable on $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$, while

$$\operatorname{rank}\big\{W(\Phi_0,t_0,t_1),\ldots,W(\Phi_k,t_k,t_f),V_1,\ldots,V_k\big\} < nr.$$

Then there exists a nonzero $nr \times 1$ complex vector x_1 such that

$$\begin{aligned} 0 &= x_1^* V_i = x_1^*\Phi_i F_i, \quad i = 1,2,\ldots,k, \\ 0 &= x_1^* W(\Phi_{i-1},t_{i-1},t_i)x_1 \\ &= x_1^*\Phi_{i-1}\int_{t_{i-1}}^{t_i} X(t_{i-1},s)B(s)B^*(s)X^*(t_{i-1},s)\,ds\,\Phi_{i-1}^* x_1, \\ 0 &= x_1^* W(\Phi_k,t_k,t_f)x_1 = x_1^*\Phi_k\int_{t_k}^{t_f} X(t_k,s)B(s)B^*(s)X^*(t_k,s)\,ds\,\Phi_k^* x_1. \end{aligned} \tag{11}$$

The integrands of (11) are nonnegative $\|x_1^*\Phi_{i-1}X(t_{i-1},s)B(s)\|^2$, $i = 1,2,\ldots,k$, and it follows that

$$\begin{cases} x_1^*\Phi_{i-1}X(t_{i-1},t)B(t) = 0, & t \in (t_{i-1},t_i), i = 1,2,\ldots,k, \\ x_1^*\Phi_k X(t_k,t)B(t) = 0, & t \in (t_k,t_f]. \end{cases} \tag{12}$$

Since system (1) is controllable on $[t_0, t_f]$, then for the initial state $x_0 \in M^{n\times 1}(\mathbb{C}^{r\times r})$, whose first column is x_1, there exists a piecewise locally bounded controller $u(t)$ such that

$$\begin{aligned}
x(t_f) = X(t_f, t_k)\Bigg[&\prod_{j=k}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
&+ \sum_{i=1}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
&+ \sum_{i=2}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_k u_k\Bigg] \\
&+ \int_{t_k}^{t_f} X(t_f, s)B(s)u(s)\,ds = 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
&-X(t_f, t_k)\prod_{j=k}^{1}(I + E_j)X(t_j, t_{j-1})x_0 \\
&\quad= X(t_f, t_k)\Bigg[\sum_{i=1}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds \\
&\qquad+ \sum_{i=2}^{k}\prod_{j=k}^{i}(I + E_j)X(t_j, t_{j-1})F_{i-1}u_{i-1} + F_k u_k\Bigg] + \int_{t_k}^{t_f} X(t_f, s)B(s)u(s)\,ds.
\end{aligned}$$

Multiplying both sides of the above equation by $\prod_{j=1}^{k}[X(t_{j-1}, t_j)(I + E_j)^{-1}]X(t_k, t_f)$ from left yields that

$$\begin{aligned}
-x_0 = \Bigg[&\sum_{i=1}^{k}\Phi_{i-1}\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u(s)\,ds + \sum_{i=2}^{k}\Phi_{i-1}F_{i-1}u_{i-1} + \Phi_k F_k u_k\Bigg] \\
&+ \Phi_k\int_{t_k}^{t_f} X(t_k, s)B(s)u(s)\,ds,
\end{aligned}$$

so

$$\begin{aligned}
-x_1 = \Bigg[&\sum_{i=1}^{k}\Phi_{i-1}\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u_1(s)\,ds + \sum_{i=2}^{k}\Phi_{i-1}F_{i-1}u_1(t_{i-1}) + \Phi_k F_k u_1(t_k)\Bigg] \\
&+ \Phi_k\int_{t_k}^{t_f} X(t_k, s)B(s)u_1(s)\,ds,
\end{aligned}$$

where $u_1(t)$ denotes the first column of $u(t)$. Moreover, multiplying x_1^* to both sides of the above equality from left, from (11) and (12), we have

$$\begin{aligned}
-x_1^* x_1 = \Bigg[&\sum_{i=1}^{k}x_1^*\Phi_{i-1}\int_{t_{i-1}}^{t_i} X(t_{i-1}, s)B(s)u_1(s)\,ds + \sum_{i=1}^{k}x_1^*\Phi_i F_i u_1(t_i)\Bigg] \\
&+ x_1^*\Phi_k\int_{t_k}^{t_f} X(t_k, s)B(s)u_1(s)\,ds = 0.
\end{aligned}$$

This contradicts the assumption that $x_1 \neq 0$, and we conclude that (10) holds. This completes the proof. □

Remark For system (1) with $r = 1$ and continuous $A(t)$, $B(t)$, $C(t)$ and $D(t)$, the sufficient controllability criteria and necessary controllability criteria obtained in Theorem 1 and Theorem 2 are the existing results in [10]. Furthermore, when system (1) is defined in the real number fields, the sufficient controllability criteria and necessary controllability criteria obtained in Theorem 1 and Theorem 2 are the existing results in [17]. However, since the controllability criteria in Theorem 1 are sufficient conditions, there exists some conservatism in Theorem 1, further research is needed for the controllability of system (1).

4 Observability

In this section, our objective is to explicitly characterize the observability criteria of complex-valued $[r]$-matrix impulsive system (1). First, a claim is presented [38]: For $n \times n$ complex matrix A, there exist scalar functions $\beta_0(t), \beta_1(t), \dots, \beta_{n-1}(t)$ such that

$$e^{At} = \sum_{j=0}^{n-1} \beta_j(t) A^j. \tag{13}$$

From system (1) and Lemma 1, we can get the output

$$y(t) = C(t)X(t,t_0)x_0 + C(t)\int_{t_0}^{t} X(t,s)B(s)u(s)\,ds + D(t)u(t), \quad t \in [t_0, t_1], \tag{14}$$

and

$$\begin{aligned} y(t) &= C(t)x(t) + D(t)u(t) \\ &= C(t)X(t,t_l)\Bigg\{\prod_{j=l}^{1}(I+E_j)X(t_j,t_{j-1})x_0 \\ &\quad + \sum_{i=1}^{l}\prod_{j=l}^{i}(I+E_j)X(t_j,t_{j-1})\int_{t_{i-1}}^{t_i} X(t_{i-1},s)B(s)u(s)\,ds \\ &\quad + \sum_{i=2}^{l}\prod_{j=l}^{i}(I+E_j)X(t_j,t_{j-1})F_{i-1}u_{i-1} + F_l u_l\Bigg\} \\ &\quad + C(t)\int_{t_l}^{t} X(t,s)B(s)u(s)\,ds + D(t)u(t), \quad t \in (t_l, t_{l+1}], \end{aligned} \tag{15}$$

where $l = 1, 2, \dots, k$. Rewrite (14) and (15)

$$\tilde{y}(t) = \begin{cases} C(t)X(t,t_0)x_0, & t \in (t_0, t_1], \\ C(t)X(t,t_l)\prod_{j=l}^{1}(I+E_j)X(t_j,t_{j-1})x_0, & t \in (t_l, t_{l+1}], \end{cases} \tag{16}$$

where

$$\tilde{y}(t) = y(t) - C(t)\int_{t_0}^{t} X(t,s)B(s)u(s)\,ds - D(t)u(t), \quad t \in [t_0, t_1],$$

and

$$\begin{aligned}\tilde{y}(t) &= y(t) - \sum_{i=1}^{l}\prod_{j=l}^{i}(I+E_j)X(t_j,t_{j-1})\int_{t_{i-1}}^{t_i}X(t_{i-1},s)B(s)u(s)\,ds \\ &\quad - \sum_{i=2}^{l}\prod_{j=l}^{i}(I+E_j)X(t_j,t_{j-1})F_{i-1}u_{i-1} - F_l u_l \\ &\quad - C(t)\int_{t_l}^{t}X(t,s)B(s)u(s)\,ds - D(t)u(t), \quad t\in(t_l,t_{l+1}].\end{aligned}$$

It is easy to see, from Definition 2, that the observability of system (1) is equivalent to the estimation of x_0 from $\tilde{y}(t)$. We denote the $nr \times nr$ block matrices $M(t_0,t_f)$ as follows:

$$\begin{aligned}M(t_0,t_f) &= \sum_{i=1}^{k}M(t_{i-1},t_i) + M(t_k,t_f), \\ M(t_0,t_1) &= \int_{t_0}^{t_1}X^*(s,t_0)C(s)^*C(s)X(s,t_0)\,ds, \\ M(t_{i-1},t_i) &= \int_{t_{i-1}}^{t_i}\left[\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_{i-1})C(s)^*C(s) \\ &\quad \times X(s,t_{i-1})\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\,ds, \quad i=2,\ldots,k+1, \\ M(t_k,t_f) &= \int_{t_k}^{t_f}\left[\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_k)C(s)^*C(s)X(s,t_k) \\ &\quad \times \prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\,ds.\end{aligned} \tag{17}$$

Now we present the sufficient and necessary condition for the observability of system (1).

Theorem 3 *System* (1) *is observable on* $[t_0,t_f]$ $(t_f\in(t_k,t_{k+1}])$ *if and only if* $nr \times nr$ *complex block matrix* $M(t_0,t_f)$ *defined in* (17) *is invertible.*

Proof Multiplying both sides of (16), respectively, by $X^*(t,t_0)C(t)^*$ and

$$\left[\prod_{j=l}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(t,t_{i-1})C(t)^*$$

from left and integrating with respect to t from t_0 to t_f yield that

$$\begin{aligned}&\int_{t_0}^{t_1}X^*(s,t_0)C(s)^*\tilde{y}(s)\,ds + \sum_{i=2}^{k}\int_{t_{i-1}}^{t_i}\left[\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_{i-1}) \\ &\quad \times C(s)^*\tilde{y}(s)\,ds + \int_{t_k}^{t_f}\left[\prod_{j=k-1}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_k)C(s)^*\tilde{y}(s)\,ds\end{aligned}$$

$$
\begin{aligned}
&= \int_{t_0}^{t_1} X^*(s,t_0)C(s)^*C(s)X(s,t_0)\,ds + \sum_{i=2}^{k}\int_{t_{i-1}}^{t_i}\left[\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* \\
&\quad \times X^*(s,t_{i-1})C(s)^*C(s)X(s,t_{i-1})\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\,ds \\
&\quad + \int_{t_k}^{t_f}\left[\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_k)C(s)^*C(s)X(s,t_k)\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\,ds \\
&= \left[\sum_{i=1}^{k}M(t_{i-1},t_i)+M(t_k,t_f)\right]x_0 = M(t_0,t_f)x_0.
\end{aligned}
\tag{18}
$$

It is easy to see that the left-hand side of (18) depends on $\tilde{y}(t)$, $t\in[t_0,t_f]$. So, if $M(t_0,t_f)$ is invertible, then the initial state $x(t_0)=x_0$ is uniquely determined by the corresponding complex system output $y(t)$ and input $u(t)$ for $t\in[t_0,t_f]$.

Next we consider the necessary part. If the complex matrix $M(t_0,t_f)$ is not invertible, then there exists a nonzero $nr\times 1$ vector x_α such that $x_\alpha^* M(t_0,t_f)x_\alpha = 0$. Since $M(t_{i-1},t_i)$ $(i=1,\dots,k)$ and $M(t_k,t_f)$ are positive semidefinite matrices, we have

$$
x_\alpha^* M(t_{i-1},t_i)x_\alpha = 0,\quad i=1,2,\dots,k,\qquad x_\alpha^* M(t_k,t_f)x_\alpha = 0.
$$

If let initial state $x(t_0)=x_0=(x_\alpha,x_\alpha,\dots,x_\alpha)$, namely, each column of x_0 is x_α, then

$$
x_0^* M(t_{i-1},t_i)x_0 = 0,\quad i=1,2,\dots,k,\qquad x_0^* M(t_k,t_f)x_0 = 0,
\tag{19}
$$

it follows from (16) and (17) that

$$
\begin{aligned}
&\int_{t_0}^{t_f}\tilde{y}(s)^*\tilde{y}(s)\,ds \\
&\quad = \sum_{i=1}^{k}\int_{t_{i-1}}^{t_i}\tilde{y}(s)^*\tilde{y}(s)\,ds + \int_{t_k}^{t_f}\tilde{y}(s)^*\tilde{y}(s)\,ds \\
&\quad = \int_{t_0}^{t_1} x_0^* X^*(s,t_0)C^*(s)C(s)X(s,t_0)x_0\,ds + \sum_{i=2}^{k}\int_{t_{i-1}}^{t_i} x_0^*\left[\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* \\
&\qquad \times X^*(s,t_{i-1})C(s)^*C(s)X(s,t_{i-1})\prod_{j=i-1}^{1}(I+E_j)X(t_j,t_{j-1})x_0\,ds \\
&\qquad + \int_{t_k}^{t_f} x_0^*\left[\prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})\right]^* X^*(s,t_k)C(s)^*C(s)X(s,t_k) \\
&\qquad \times \prod_{j=k}^{1}(I+E_j)X(t_j,t_{j-1})x_0\,ds \\
&\quad = x_0^*\left[\sum_{i=1}^{k}M(t_{i-1},t_i)+M(t_k,t_f)\right]x_0 = 0,
\end{aligned}
$$

which implies that $\int_{t_0}^{t_f} \operatorname{trac}(\tilde{y}(s)^* \tilde{y}(s))\, ds = \int_{t_0}^{t_f} \|\tilde{y}(s)\|_F^2\, ds = 0$. Thus by (16),

$$0 = \tilde{y}(t) = \begin{cases} C(t)X(t,t_0)x_0, & t \in (t_0, t_1], \\ C(t)X(t,t_{i-1}) \prod_{j=i-1}^{1} (I + E_j) X(t_j, t_{j-1}) x_0, & t \in (t_{i-1}, t_i], \\ C(t)X(t,t_k) \prod_{j=k}^{1} (I + E_j) X(t_j, t_{j-1}) x_0, & t \in (t_k, t_f]. \end{cases}$$

From Definition 2, system (1) is not observable on $[t_0, t_f]$ ($t_f \in (t_k, t_{k+1}]$). This contradicts the assumption of observability. This completes the proof. □

For system (1), when $A(t) = A$, $B(t) = B$, $C(t) = C$, $D(t) = D$, the complex impulsive system becomes a complex linear time-invariant impulsive system. We have a more concise result than Theorem 3. Denote

$$S := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{nr-1} \end{bmatrix}, \qquad \tilde{S} := \begin{bmatrix} S \\ S\hat{E}_1 \\ \vdots \\ S\hat{E}_k \end{bmatrix}, \tag{20}$$

where $\hat{E}_i = \prod_{j=i}^{1} (I + E_j)$, $i = 1, 2, \ldots, k$.

Theorem 4 *If complex* $[r]$*-matrix impulsive system* (1) *has complex constant coefficient matrices* A, B, C, D, *then the following conclusions hold.*

(i) *If* $\operatorname{rank}(S) = nr$, *then complex* $[r]$*-matrix linear impulsive system* (1) *is observable on* $[t_0, t_f]$ $(t_f \in (t_k, t_{k+1}])$.
(ii) *Assume that* $AE_i = E_iA$, $i = 1, 2, \ldots, k$. *If complex* $[r]$*-matrix system* (1) *is observable, then* $\operatorname{rank}(\tilde{S}) = nr$.

Proof (i) If $\operatorname{rank}(S) = nr$ while complex $[r]$-matrix system (1) is not observable, then by Theorem 3 the matrix $M(t_0, t_f)$ is not invertible, which implies that there exists a nonzero $nr \times 1$ vector x_α satisfying $x_\alpha^* M(t_0, t_f) x_\alpha = 0$. Since the matrices $M(t_{i-1}, t_i)$ are non-negative definite, we obtain

$$x_\alpha^* M(t_0, t_1) x_\alpha = \int_{t_0}^{t_1} \left[C e^{A(s-t_0)} x_\alpha \right]^* \left[C e^{A(s-t_0)} x_\alpha \right] ds = 0.$$

This shows that

$$C e^{A(t-t_0)} x_\alpha = 0, \quad t \in (t_0, t_1]. \tag{21}$$

Clearly, when $t = t_0$, we have $Cx_\alpha = 0$. Differentiating (21) j times and evaluating the results at $t = t_0$ yield that

$$CA^j x_\alpha = 0, \quad j = 0, 1, \ldots, nr - 1. \tag{22}$$

Hence we deduce that $Sx_\alpha = 0$ for $x_\alpha \neq 0$. It follows that $\operatorname{rank}(S) < nr$, which leads to a contradiction with the assumption that $\operatorname{rank}(S) = nr$. The proof of part (i) is completed.

(ii) If otherwise, assume that complex impulsive system (1) is observable while $\operatorname{rank}(\tilde{S}) < nr$, then there exists a vector $x_\alpha \neq 0$ satisfying $\tilde{S}x_\alpha = 0$ which reduces from (20) to

$$CA^l x_\alpha = 0, \qquad CA^l \prod_{j=i}^{1}(I+E_j)x_\alpha = 0, \quad l = 0,1,\ldots,nr-1, \tag{23}$$

where $i = 1,\ldots,k$. From (23), (13) and the fact that $AE_i = E_iA$, we obtain

$$\begin{aligned}
M(t_0,t_1)x_\alpha &= \int_{t_0}^{t_1} \left[e^{A(s-t_0)}\right]^* C^* \sum_{l=0}^{nr-1} \beta_l(s-t_0) CA^l x_\alpha\, ds = 0, \\
M(t_{i-1},t_i)x_\alpha &= \int_{t_{i-1}}^{t_i} \left[\prod_{j=i-1}^{1}(I+E_j)e^{A(t_{j-1}-t_j)}\right]^* \left[e^{A(s-t_{i-1})}\right]^* C^* \\
&\quad \times \sum_{l=0}^{nr-1} \beta_l(s-t_{i-1}) CA^l \prod_{j=i-1}^{1}(I+E_j)x_\alpha\, ds = 0, \quad i = 2,\ldots,k, \\
M(t_k,t_f)x_\alpha &= \int_{t_k}^{t_f} \left[\prod_{j=k}^{1}(I+E_j)e^{A(t_{j-1}-t_j)}\right]^* \left[e^{A(s-t_k)}\right]^* C^* \\
&\quad \times \sum_{l=0}^{nr-1} \beta_l(s-t_k) CA^l \prod_{j=k}^{1}(I+E_j)x_\alpha\, ds = 0.
\end{aligned}$$

So $M(t_0,t_f)x_\alpha = 0$. Because $x_\alpha \neq 0$, the matrix $M(t_0,t_f)$ is not invertible. Hence complex $[r]$-matrix impulsive system (1) is not observable from Theorem 3, and it contradicts the assumption of observability. This completes the proof. □

5 Example

In system (1), let $E_k = kI$, $t_k = k$, $t_0 = 0$, $t_f = t_2 = 2$, $m = n = r = 2$,

$$A(t) = \begin{pmatrix} i & 1 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{pmatrix},$$

where $i = \sqrt{-1}$, and for the convenience of calculation, we take $B(t)$, $C(t)$ such that $B(t)B^*(t) = I$, $C(t)C^*(t) = I$.

Next we consider the controllability and observability of system (1) on $[0,2]$. It is easy to compute

$$X(t) = \begin{pmatrix} e^{it}J(t) & 0 \\ 0 & e^{\frac{1}{2}t^2}I \end{pmatrix}, \qquad X(t,s) = \begin{pmatrix} e^{-i(t-s)}J(t-s) & 0 \\ 0 & e^{\frac{1}{2}(t^2-s^2)}I \end{pmatrix},$$

where $J(t) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, and from (7)

$$W_2 = \int_1^2 X(1,s)X^*(1,s)\, ds = \int_1^2 \begin{pmatrix} J(2-2s) & 0 \\ 0 & e^{1-s^2}I \end{pmatrix} ds = \begin{pmatrix} J(1) & 0 \\ 0 & \int_1^2 e^{1-s^2}\, dsI \end{pmatrix}.$$

Obviously, W_2 is invertible, according to Theorem 1, system (1) is controllable. At the same time, by Theorem 2, we should have

$$\operatorname{rank}\bigl\{W(\Phi_0,0,1), W(\Phi_1,1,2), W(\Phi_2,2,2), V_1, V_2\bigr\} = 4. \tag{24}$$

In fact, it is easy to calculate

$$\begin{aligned}
W(\Phi_0,0,1) &= \int_0^1 \Phi_0 X(0,s)B(s)B^*(s)X^*(0,s)\Phi_0^*\,ds \\
&= \int_0^1 \begin{pmatrix} e^{is}J(-s) & 0 \\ 0 & e^{-\frac{s^2}{2}}I \end{pmatrix} ds \\
&= \begin{pmatrix} i(1-e^i) & 1+(i-1)e^i & 0 & 0 \\ 0 & i(1-e^i) & 0 & 0 \\ 0 & 0 & \int_0^1 e^{-\frac{1}{2}s^2}\,ds & 0 \\ 0 & 0 & 0 & \int_0^1 e^{-\frac{1}{2}s^2}\,ds \end{pmatrix},
\end{aligned}$$

and $\operatorname{rank}(W(\Phi_0,0,1)) = 4$, so (24) is right.

For the observability, from (17), we have

$$\begin{aligned}
M(t_0,t_1) = M(0,1) &= \int_0^1 X^*(s,0)X(s,0)\,ds = \begin{pmatrix} J(1) & 0 \\ 0 & \int_0^1 e^{s^2} I \end{pmatrix}, \\
M(t_1,t_2) = M(1,2) &= \int_1^2 \bigl[I + E_1X(1,0)\bigr]^* X^*(s,0)X(s,0)(I+E_1)X(1,0)\,ds \\
&= 4\int_1^2 X^*(1,0)X^*(s,0)X(s,0)X(1,0)\,ds \\
&= \begin{pmatrix} 4J(5) & 0 \\ 0 & \int_1^2 e^{1+s^2} I \end{pmatrix}, \\
M(t_0,t_f) = M(0,1) + M(1,2) &= \begin{pmatrix} J(1)+4J(5) & 0 \\ 0 & (\int_0^1 e^{s^2} + \int_1^2 e^{1+s^2})I \end{pmatrix},
\end{aligned}$$

$M(t_0,t_f)$ is invertible, according to Theorem 3, system (1) is observable on $[0,2]$. This completes the example.

6 Conclusion

In this paper, the issue of the controllability and observability criteria for a class of complex $[r]$-matrix time-varying impulsive systems has been addressed for the first time. Taking advantage of the matrix differential equation theory in complex fields, several sufficient and necessary conditions for state controllability and observability of such systems have been established respectively without imposing extra conditions. Moreover, the corresponding criteria for controllability and observability of complex $[r]$-matrix time-invariant systems have also been derived.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
TF completed the proof and wrote the initial draft. JS provided the problem and gave some suggestions of amendment. TF then finalized the manuscript. Correspondence was mainly done by JS. All authors read and approved the final manuscript.

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