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Generalized Meixner-Pollaczek polynomials

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Abstract

We consider the generalized Meixner-Pollaczek (GMP) polynomials $P_n^\lambda(x;\theta,\psi)$ of a variable $x \in \mathbb{R}$ and parameters $\lambda > 0$, $\theta \in (0,\pi)$, $\psi \in \mathbb{R}$, defined *via* the generating function

$$G^\lambda(x;\theta,\psi;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1-ze^{i\psi})^{\lambda+ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x;\theta,\psi)z^n, \quad |z| < 1.$$

We find the three-term recurrence relation, the explicité formula, the hypergeometric representation, the difference equation and the orthogonality relation for GMP polynomials $P_n^\lambda(x;\theta,\psi)$. Moreover, we study the special case of $P_n^\lambda(x;\theta,\psi)$ corresponding to the choice $\psi = \pi + \theta$ and $\psi = \pi - \theta$, which leads to some interesting families of polynomials. The limiting case ($\lambda \to 0$) of the sequences of polynomials $P_n^\lambda(x;\theta,\pi+\theta)$ is obtained, and the orthogonality relation in the strip $S = \{z \in \mathbb{C} : |\Im(z)| < 1\}$ is shown.

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1 Introduction

The classical Koebe function is a function holomorphic in $\mathbb{D} = \{z : |z| < 1\}$ and given by the formula

$$k_2(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left\{\left(\frac{1+z}{1-z}\right)^2 - 1\right\} = z + 2z^2 + 3z^3 + \cdots, \quad z \in \mathbb{D}.$$

The importance of k_2 follows from the extremality for the famous Bieberbach conjecture. The Koebe function is univalent and starlike in $\mathbb{D}$ and maps the unit disk $\mathbb{D}$ onto the complex plane minus a slit $(-\infty, -\frac{1}{4}]$.

Several generalizations of k_2 appeared in the literature. Robertson [1] proved that $k_{2(1-\alpha)}(z) = \frac{z}{(1-z)^{2(1-\alpha)}}$ $(0 \le \alpha < 1)$ is the extremal function for the class of functions starlike of order α. The function

$$k_\alpha(z) = \frac{1}{2\alpha}\left\{\left(\frac{1+z}{1-z}\right)^\alpha - 1\right\}, \quad \alpha \in \mathbb{R} \setminus \{0\}, z \in \mathbb{D},$$

was extensively studied by Pommerenke [2], who investigated a universal invariant family U_α.

The definition of k_α was extended for a nonzero complex number α by Yamashita [3]. The classical result of Hille [4] ascertains that k_α is univalent in $\mathbb{D}$ if and only if $\alpha \neq 0$ is in the union A of the closed disks $\{|z+1| \leq 1\}$ and $\{|z-1| \leq 1\}$. Making use of geometric properties, Yamashita [3] described how k_α tends to be univalent in the whole $\mathbb{D}$ as α tends to each boundary point of A from outside.

The properties of $\log k_c'$, where

$$k_c(z) = \frac{1}{2c}\left[\left(\frac{1+z}{1-z}\right)^c - 1\right], \quad c \in \mathbb{C} \setminus \{0\}, \qquad k_0(z) = \frac{1}{2}\log\frac{1+z}{1-z}, \quad z \in \mathbb{D}, \tag{1.1}$$

were studied in [5] by Campbell and Pfaltzgraff. Pommerenke [2] examined the special case of (1.1), *i.e.*,

$$k_{i\gamma}(z) = \frac{1}{2i\gamma}\left[\left(\frac{1+z}{1-z}\right)^{i\gamma} - 1\right], \quad \gamma > 0, z \in \mathbb{D},$$

for which

$$k_{i\gamma}'(z) = \frac{1}{(1+z)^{1-i\gamma}(1-z)^{1-i\gamma}}.$$

An evident and important extension of (1.1) was given by the following formulas ($\theta \in \mathbb{R}, \psi \in \mathbb{R}$):

$$k_c(\theta, \psi; z) = \frac{1}{(e^{i\psi} - e^{i\theta})c}\left[\left(\frac{1 - ze^{i\theta}}{1 - ze^{i\psi}}\right)^c - 1\right], \quad c \in \mathbb{C} \setminus \{0\}, e^{i\psi} \neq e^{i\theta}, z \in \mathbb{D},$$

and for the case when $c = 0$,

$$k_0(\theta, \psi; z) = \frac{1}{(e^{i\psi} - e^{i\phi})}\log\frac{1 - ze^{i\theta}}{1 - ze^{i\psi}}, \quad e^{i\psi} \neq e^{i\theta}, z \in \mathbb{D}.$$

We have

$$k_c'(\theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{1-c}(1 - ze^{i\psi})^{1+c}}, \quad c \in \mathbb{C}.$$

Comparing $k_{i\gamma}'(\theta, \psi; z) = \frac{1}{(1-ze^{i\theta})^{1-i\gamma}(1-ze^{i\psi})^{1+i\gamma}}$ with the generating function for Meixner-Pollaczek polynomials $P_n^\lambda(x; \theta)$ [6],

$$G^\lambda(x; \theta, -\theta; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix}(1 - ze^{-i\theta})^{\lambda + ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta) z^n, \quad z \in \mathbb{D},$$

where $\lambda > 0$, $\theta \in (0, \pi)$, $x \in \mathbb{R}$, we were motivated to introduce the generalized Meixner-Pollaczek polynomials (GMP) [7] $P_n^\lambda(x; \theta, \psi)$ of a variable $x \in \mathbb{R}$ and parameters $\lambda > 0$, $\theta \in (0, \pi)$, $\psi \in \mathbb{R}$ *via* the generating function

$$G^\lambda(x; \theta, \psi; z) = \frac{1}{(1 - ze^{i\theta})^{\lambda - ix}(1 - ze^{i\psi})^{\lambda + ix}} = \sum_{n=0}^{\infty} P_n^\lambda(x; \theta, \psi) z^n, \quad z \in \mathbb{D}. \tag{1.2}$$

Obviously, we have $P_n^\lambda(x; \theta, -\theta) = P_n^\lambda(x; \theta)$.

2 Orthogonal polynomials

Let $\mathfrak{L}$ denote the *moment functional* that is a linear map $\mathbb{C}[x] \to \mathbb{C}$. A sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$ is an *orthogonal polynomials sequence* (OPS) with respect to $\mathfrak{L}$ if $P_n(x)$ has degree n, $\mathfrak{L}[P_m(x)P_n(x)] = 0$ for $m \neq n$ and $\mathfrak{L}[P_n^2(x)] \neq 0$ for all n.

In this paper we consider orthogonal polynomial systems defined recursively. Every monic OPS $P_n(x)_{n=0}^{\infty}$ may be described by a recurrence formula of the form

$$P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \quad n = 1, 2, 3, \ldots, \tag{2.1}$$

where $P_{-1}(x) = 0$, $P_0(x) = 1$, the numbers c_n and λ_n are constants, $\lambda_n \neq 0$ for $n > 1$ and λ_1 is arbitrary (see [8, Ch. I, Theorem 4.1]). The sequences of orthogonal polynomials are *symmetric* if $P_n(x) = (-1)^n P_n(-x)$ for all n (see [8, Ch. I, Theorem 4.3]) or that c_n in (2.1) are all zero.

Polynomials with exponential generating functions are among the most often studied polynomials. One of them is the Meixner-Pollaczek polynomials. The Meixner-Pollaczek polynomials were first invented by Meixner [9]. The same polynomials were also considered independently by Pollaczek [10]. These polynomials are classified in the Askey-scheme of orthogonal polynomials [6, 11].

Some of the main properties of these polynomials are presented in Erdélyi *et al.* [12], Chihara [8], Askey and Wilson [11] and in the report by Koekoek and Swarttouw [6]. Detailed analyses with applications of these polynomials are also made by several authors. Among others, the works of Rahman [13], Atakishiyev and Suslov [14], Bender *et al.* [15], Koornwinder [16] and the extensive work of Li and Wong [17] may be included.

This paper is mainly concerned about the generalized Meixner-Pollaczek (GMP) polynomials. We also study the special cases of $P_n^{\lambda}(x;\theta,\psi)$, corresponding to the choice $\psi = \pi + \theta$ and $\psi = \pi - \theta$, which lead to some interesting families of polynomials.

For complex numbers a, b and c ($c \neq 0, -1, -2, \ldots$), the *Gaussian hypergeometric function* ${}_2F_1(a, b, c; z)$ is defined by

$${}_2F_1(a, b, c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \quad z \in \mathbb{D}, \tag{2.2}$$

where $(a)_n$ is the *Pochhammer symbol* described by

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad n \in \mathbb{N}, \qquad (a)_0 = 1.$$

Notice that ${}_2F_1(a, b, c; z)$ is symmetric in a and b, and the series terminates if either a or b is zero or a negative integer. In general, the series ${}_2F_1(z)$ is absolutely convergent in $\mathbb{D}$. If $\Re(c - a - b) > 0$, it is also convergent on $\partial\mathbb{D}$, and it is known that

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad Re(c-a-b) > 0; c \neq 0, -1, -2, \ldots. \tag{2.3}$$

3 Generalized Meixner-Pollaczek polynomials

In this section we find the three-term recurrence relation, the explicité formula, the hypergeometric representation, the difference equation and the orthogonality relation for (GMP) polynomials $P_n^{\lambda}(x;\theta,\psi)$.

Theorem 1 *Let us set $P_{-1}^{\lambda} = 0$. The polynomials $P_n^{\lambda} = P_n^{\lambda}(x;\theta,\psi)$ have the following properties*:

(a) *P_n^{λ} satisfy the three-term recurrence relation*

$$\begin{aligned} &P_0^{\lambda} = 1, \\ &nP_n^{\lambda} = \bigl[(\lambda - ix)e^{i\theta} + (\lambda + ix)e^{i\psi} + (n-1)\bigl(e^{i\theta} + e^{i\psi}\bigr)\bigr]P_{n-1}^{\lambda} \\ &\qquad - (2\lambda + n - 2)e^{i(\theta+\psi)}P_{n-2}^{\lambda}, \quad n \geq 1. \end{aligned}$$

(b) *P_n^{λ} are given by the formula*

$$P_n^{\lambda}(x;\theta,\psi) = e^{in\theta}\sum_{j=0}^{n}\frac{(\lambda + ix)_j(\lambda - ix)_{n-j}}{j!(n-j)!}e^{ij(\psi-\theta)}, \quad n \in \mathbb{N}\cup\{0\}. \tag{3.1}$$

(c) *P_n^{λ} have the hypergeometric representation*

$$n!P_n^{\lambda}(x;\theta,\psi) = (2\lambda)_n e^{in\theta}{}_2F_1\bigl(-n, \lambda + ix, 2\lambda; 1 - e^{i(\psi-\theta)}\bigr). \tag{3.2}$$

(d) *Let $y(x) = P_n^{\lambda}(x;\theta,\psi)$. The function $y(x)$ satisfies the following difference equation*:

$$\begin{aligned} &e^{i\theta}(\lambda - ix)y(x+i) + \bigl[ix\bigl(e^{i\theta} + e^{i\psi}\bigr) - (n+\lambda)\bigl(e^{i\theta} - e^{i\psi}\bigr)\bigr]y(x) \\ &\quad - e^{i\psi}(\lambda + ix)y(x-i) = 0. \end{aligned} \tag{3.3}$$

Proof

(a) We differentiate the formula (1.2) with respect to z, and after multiplication by $(1 - ze^{i\theta})(1 - ze^{i\psi})$, we compare the leading coefficients of z^{n-1}.

(b) The Cauchy product of the power series

$$\bigl(1 - ze^{i\theta}\bigr)^{-(\lambda-ix)} = \sum_{n=0}^{\infty}\frac{(\lambda - ix)_n e^{in\theta}}{n!}z^n$$

and

$$\bigl(1 - ze^{i\psi}\bigr)^{-(\lambda+ix)} = \sum_{n=0}^{\infty}\frac{(\lambda + ix)_n e^{in\psi}}{n!}z^n$$

gives (3.1).

(c) Applying the formula from [12, vol.1, p.82],

$$(1-s)^{a-c}(1 - s + sz)^{-a} = \sum_{n=0}^{\infty}\frac{(c)_n}{n!}{}_2F_1(-n, a; c; z)s^n, \quad |s| < 1, \bigl|s(1-z)\bigr| < 1,$$

with $s = ze^{i\theta}$, $a = \lambda + ix$, $c = 2\lambda$, $z = 1 - e^{i(\psi-\theta)}$, one obtains

$$\bigl(1 - ze^{i\theta}\bigr)^{-(\lambda-ix)}\bigl(1 - ze^{i\psi}\bigr)^{-(\lambda+ix)} = \sum_{n=0}^{\infty}\frac{e^{in\theta}(2\lambda)_n}{n!}{}_2F_1\bigl(-n, \lambda + ix, 2\lambda; 1 - e^{i(\psi-\theta)}\bigr)z^n.$$

Comparing the coefficients of the power series, we get (3.2).

(d) Inserting $(x+i)$ and $(x-i)$ instead of x into the generating function (1.2), we find that

$$y(x+i) = \sum_{k=0}^{n-1} P_k^{\lambda}(x;\theta,\psi)\bigl[e^{i(n-k)\theta} - e^{i[(n-k-1)\theta+\psi]}\bigr] + P_n^{\lambda},$$

$$y(x-i) = \sum_{k=0}^{n-1} P_k^{\lambda}(x;\theta,\psi)\bigl[e^{i(n-k)\psi} - e^{i[(n-k-1)\psi+\theta]}\bigr] + P_n^{\lambda},$$

which implies that

$$\begin{aligned} &e^{i\theta}(\lambda - ix)y(x+i) - e^{i\psi}(\lambda+ix)y(x-i) \\ &\quad = \bigl(e^{i\theta} - e^{i\psi}\bigr)\sum_{k=0}^{n-1} P_k^{\lambda}(x;\theta,\psi)\bigl[(\lambda - ix)e^{i(n-k)\theta} + (\lambda+ix)e^{i(n-k)\psi}\bigr] \\ &\qquad + \bigl[e^{i\theta}(\lambda - ix) - e^{i\psi}(\lambda+ix)\bigr]P_n^{\lambda}. \end{aligned} \tag{3.4}$$

Differentiation of the generating function (1.2) with respect to z and equating the leading coefficient of z^{n-1} yields

$$nP_n^{\lambda}(x;\theta,\psi) = \sum_{k=0}^{n-1} P_k^{\lambda}(x;\theta,\psi)\bigl[(\lambda - ix)e^{i(n-k)\theta} + (\lambda+ix)e^{i(n-k)\psi}\bigr],$$

which together with (3.4) gives (3.3).

□

Theorem 2 *The polynomials $P_n^{\lambda}(x;\theta,\psi)$ are orthogonal on $(-\infty,+\infty)$ with the weight $w_{\theta,\psi}^{\lambda}(x) = \frac{1}{2\pi}e^{(\theta-\psi+\pi)x}|\Gamma(\lambda+ix)|^2$, for $\lambda > 0$, $\theta \in (-\frac{\pi}{2},\frac{\pi}{2})$, and*

$$\int_{-\infty}^{+\infty} w_{\theta,\psi}^{\lambda}(x)P_n^{\lambda}(x;\theta,\psi)\overline{P_m^{\lambda}(x;\theta,\psi)}\,dx = \delta_{nm}\frac{\Gamma(n+2\lambda)}{n!(2\cos(\theta-\psi+\pi)/2)^{2\lambda}}.$$

Proof Let $F(s)$ and $G(s)$ be the Mellin transforms of $f(x)$ and $g(x)$, *i.e.*,

$$\{\mathcal{M}f\}(s) = F(s) = \int_0^{\infty} f(x)x^{s-1}\,dx, \qquad \{\mathcal{M}g\}(s) = G(s) = \int_0^{\infty} g(x)x^{s-1}\,dx.$$

Then the following formula (Parseval's identity) holds [18]:

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)G(1-s)\,ds = \int_0^{\infty} f(x)g(x)\,dx \tag{3.5}$$

and [12]

$$\int_0^{+\infty} u^{\alpha-1}e^{-pu}e^{-iqu}\,du = \Gamma(\alpha)\bigl(p^2+q^2\bigr)^{-\frac{\alpha}{2}}e^{-i\alpha\arctan(p/q)}. \tag{3.6}$$

For $f(x) = 2x^{2(\lambda+j)}e^{-x^2}$ and $g(x) = 2x^{2(\lambda+k)-1}e^{-x^2}$, we have $F(s) = \Gamma(\lambda+j+\frac{s}{2})$, $G(s) = \Gamma(\lambda+k+\frac{s-1}{2})$. By the well-known property

$$\{\mathcal{M}f\}\bigl(e^{i\theta}x\bigr) = e^{-i\theta s}F(s),$$

we have

$$\{\mathcal{M}f\}\big(e^{i(\theta-\psi+\pi)/2}x\big) = e^{-is(\theta-\psi+\pi)/2}F(s).$$

Consecutively, applying first the formula $(a)_j = \frac{\Gamma(a+j)}{\Gamma(a)}$ $(j = 1, 2, \ldots)$ and (3.5), and then setting $\alpha = 2\lambda + k + j$, $p = \cos(\theta - \psi + \pi) + 1$, $q = \sin(\theta - \psi + \pi)$ in (3.6), we have

$$\begin{aligned}
&\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(\theta-\psi+\pi)x}(\lambda+ix)_j(\lambda-ix)_k\big|\Gamma(\lambda+ix)\big|^2\,dx \\
&\quad= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{(\theta-\psi+\pi)x}\Gamma(\lambda+j+ix)\Gamma(\lambda+k-ix)\,dx \\
&\quad= \frac{1}{4\pi i}\int_{-\infty}^{+\infty} e^{-ix(\theta-\psi+\pi)/2}\Gamma\left(\lambda+j+\frac{x}{2}\right)\Gamma\left(\lambda+k-\frac{x}{2}\right)dx \\
&\quad= 2e^{i(\theta-\psi+\pi)(\lambda+j)}\int_0^{+\infty} x^{2(2\lambda+k+j)-1}\exp\big(-\big(e^{(\theta-\psi+\pi)i}+1\big)x^2\big)\,dx \\
&\quad= e^{i(\theta-\psi+\pi)(\lambda+j)}\int_0^{+\infty} x^{2\lambda+k+j-1}\exp\big(-\big(e^{(\theta-\psi+\pi)i}+1\big)x\big)\,dx \\
&\quad= \frac{e^{i(j-k)(\theta-\psi+\pi)/2}\Gamma(2\lambda+k+j)}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k+j}}.
\end{aligned} \tag{3.7}$$

Set

$$P_n^{\lambda}(x;\theta,\psi) = \sum_{k=0}^{n} A_k(\lambda+ix)_k,$$

where

$$A_k = \frac{e^{ik\theta}(2\lambda)_k(-k)_k(1-e^{i(\psi-\theta)})^k}{k!(2\lambda)_k k!}.$$

Then

$$\begin{aligned}
J &:= \frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\big|\Gamma(\lambda+ix)\big|^2\,dx \\
&= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\sum_{j=0}^{n} A_j(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\big|\Gamma(\lambda+ix)\big|^2\,dx \\
&= \frac{1}{2\pi}\sum_{j=0}^{n} A_j\int_{-\infty}^{+\infty}(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\big|\Gamma(\lambda+ix)\big|^2\,dx \\
&= \frac{(2\lambda)_n e^{in\theta}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(1-e^{i(\psi-\theta)})^j}{(2\lambda)_j!}\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\lambda+ix)_j(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\big|\Gamma(\lambda+ix)\big|^2\,dx.
\end{aligned}$$

Using (3.7) and (2.2), we obtain

$$\begin{aligned}
J &= \frac{(2\lambda)_n e^{in\theta}}{n!}\sum_{j=0}^{n}\frac{(-n)_j(1-e^{i(\psi-\theta)})^j}{(2\lambda)_j!}\frac{e^{i(j-k)(\theta-\psi+\pi)/2}\Gamma(2\lambda+k+j)}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k+j}} \\
&= \frac{(2\lambda)_n e^{in\theta}}{n!}\Gamma(2\lambda+k)\frac{e^{-ik(\theta-\psi+\pi)/2}}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}
\end{aligned}$$

$$\times \sum_{j=0}^{n} \frac{(-n)_j(2\lambda+k)_j}{(2\lambda)_j!} \frac{(1-e^{i(\psi-\theta)})^j}{(e^{i(\theta-\psi+\pi)/2}+e^{-i(\theta-\psi+\pi)/2})^j(e^{-i(\theta-\psi+\pi)/2})^j}$$

$$= \frac{(2\lambda)_n e^{in\theta}}{n!}\Gamma(2\lambda+k)\frac{e^{-ik(\theta-\psi+\pi)/2}}{(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}\,{}_2F_1(-n,2\lambda+k;2\lambda;1).$$

By the formula (2.3), the above reduces to

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\left|\Gamma(\lambda+ix)\right|^2 dx$$
$$= \frac{e^{i(n\theta-k(\theta-\psi+\pi)/2)}\Gamma(2\lambda+k)}{n!(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}(-k)_n. \tag{3.8}$$

Since $(-k)_n = 0$ for $k < n$, then (3.8) is nonzero only for the case $k = n$. Then

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)(\lambda-ix)_k e^{(\theta-\psi+\pi)x}\left|\Gamma(\lambda+ix)\right|^2 dx$$
$$= \frac{e^{i(n\theta-k(\theta-\psi+\pi)/2)}\Gamma(2\lambda+k)}{n!(2\cos((\theta-\psi+\pi)/2))^{2\lambda+k}}(-n)_n.$$

From this and relation (3.7), it follows that

$$\int_{-\infty}^{+\infty} P_n^{\lambda}(x;\theta,\psi)\overline{P_m^{\lambda}(x;\theta,\psi)}w_{\theta}^{\lambda}(x)\,dx$$
$$= \delta_{nm}\frac{e^{-in\theta}(2\lambda)_n(-n)_n(1-e^{-i(\psi-\theta)})^n}{n!(2\lambda)_n n!}\frac{e^{i(n\theta-\frac{n}{2}(\theta-\psi+\pi))}\Gamma(2\lambda+n)}{n!(2\cos\frac{\theta-\psi+\pi}{2})^{2\lambda+n}}(-n)_n$$
$$= \delta_{nm}\frac{\Gamma(n+2\lambda)}{n!(2\cos\frac{\theta-\psi+\pi}{2})^{2\lambda}}.$$

□

Remark 1 For $x \in \mathbb{R}$, $\psi \in \mathbb{R}$, $\lambda > 0$ and $n \in \mathbb{N}$, the following explicité formula holds:

$$P_n^{\lambda}(x;\theta,\psi) = e^{i\theta n}\sum_{k=0}^{[\frac{n}{2}]}\frac{(\lambda-ix)_{n-2k}(\lambda+ix)_k}{(n-2k)!k!}\left(\frac{e^{i\psi}}{e^{2i\theta}}\right)^k. \tag{3.9}$$

Proof Consider the following:

$$\sum_{n=0}^{\infty} P_n^{\lambda}(x;\theta,\psi)z^n = \left(1-ze^{i\theta}\right)^{-\lambda+ix}\left(1-ze^{i\psi}\right)^{-\lambda-ix}$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} z^n\frac{(\lambda-ix)_{n-k}}{(n-k)!}e^{i\theta(n-k)}\frac{(\lambda+ix)_k}{k!}e^{i\psi k}$$
$$= \sum_{n=0}^{\infty} z^n\sum_{k=0}^{[\frac{n}{2}]}\frac{(\lambda-ix)_{n-2k}}{(n-2k)!}e^{i\theta(n-2k)}\frac{(\lambda+ix)_k}{k!}e^{i\psi k}.$$

Comparing both sides of the above, we get the equality (3.9). □

Proposition 1 *The family of generalized Meixner-Pollaczek polynomials* $P_n^\lambda(x;\theta,\psi)$ *can be extended to the case* $\lambda = 0$ *as follows*:

$$P_0^0(x;\theta,\psi) = 1,$$
$$nP_n^0(x;\theta,\psi) = \left(\frac{e^{i\theta} - e^{i\psi}}{i}\right)xP_{n-1}^1(x;\theta,\psi), \quad n \geq 1. \tag{3.10}$$

Proof Since

$$\begin{aligned}\lim_{\lambda\to 0} P_n^\lambda(x;\theta,\psi) &= \lim_{\lambda\to 0}(2\lambda)_n e^{in\theta}{}_2F_1\left(-n,\lambda+ix,2\lambda;1-\frac{e^{i\psi}}{e^{i\theta}}\right)\\ &= \frac{1}{n!}e^{in\theta}\left(1-e^{i(\psi-\theta)}\right)\Gamma(n)(-n)(ix){}_2F_1\left(-n+1,ix+1,2;1-e^{i(\psi-\theta)}\right)\\ &= \frac{1}{n!}e^{in\theta}\left(1-e^{i(\psi-\theta)}\right)\Gamma(n)(-n)(ix)P_{n-1}^1(x;\theta,\psi)\frac{n-1}{(2)_{n-1}}e^{-i(n-1)\theta}\\ &= \frac{1}{n}\left(\frac{e^{i\theta}-e^{i\psi}}{i}\right)xP_{n-1}^1(x;\theta,\psi),\end{aligned}$$

then (3.10) is a natural consequence. □

4 The case $\psi = \pi + \theta$

Let us consider now the case $\psi = \pi + \theta$. We observe that such a case leads to the very interesting family of symmetric polynomials. Some special cases of $P_n^\lambda(x;\theta,\pi+\theta;z)$ are known in the literature for $\theta = \frac{\pi}{2}$. These are the symmetric Meixner-Pollaczek polynomials, denoted by $P_n^\lambda(x/2;\theta)$, $\lambda > 0$. For instance, Bender *et al.* [15] and Koornwinder [16] have shown that there is a connection between the symmetric Meixner-Pollaczek polynomials $P_n^{\frac{1}{2}}(\frac{x}{2},\frac{\pi}{2})$ and the Heisenberg algebra. Another example is [19], where the symmetric Meixner-Pollaczek polynomials are considered.

We define the symmetric generalized Meixner-Pollaczek (SGMP) polynomials $S_n^\lambda(x;\theta)$ by the following generating function:

$$\begin{aligned}G^\lambda(x;\theta,\pi+\theta;z) &= \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{i\theta})^{\lambda+ix}}\\ &= \frac{e^{-2x\arctan(ze^{i(\theta+\pi/2)})}}{(1-z^2e^{2i\theta})^\lambda}\\ &= \sum_{n=0}^{\infty} S_n^\lambda(x;\theta)z^n, \quad z\in\mathbb{D}.\end{aligned}$$

This sequence of polynomials has a hypergeometric representation

$$S_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}{}_2F_1(-n,\lambda+ix,2\lambda;2), \tag{4.1}$$

and an integral representation

$$S_n^\lambda(x;\theta) = \frac{1}{2\pi i}\int \frac{e^{-2x\arctan(ze^{i(\theta+\pi/2)})}}{(1-z^2e^{2i\theta})^\lambda}\frac{dz}{z^{n+1}}.$$

In this section we mainly consider the strip $\mathcal{S} = \{z \in C : |\Im z| < 1\}$. There are several reasons why the strip is of special interest. Let $w(x) = \frac{1}{2\cosh(\pi x/2)}$. The function $w(x)$ is a density function of a probability measure on $\mathcal{S}$. We describe an orthogonal basis for the basis in the Hilbert space $H^2(\mathcal{S},\mathcal{P})$, where $\mathcal{P}$ is the Poisson measure for 0. The inner product for any two functions $f, g \in H^2(\mathcal{S},\mathcal{P})$ is given by the formula

$$(f,g)_{H^2(\mathcal{S},\mathcal{P})} := \int_R \frac{f(x+i)\overline{g(x+i)} + f(x-i)\overline{g(x-i)}}{4\cosh\frac{\pi}{2}x}\,dx.$$

Now, we consider the system $\{\sigma_n(x)\}$ given by the recursion relation

$$\sigma_{-1} = 0, \sigma_0 = 1, \quad (n+1)\sigma_{n+1}(z) + ie^{i\theta}z\sigma_n(z) - e^{2i\theta}(n-1)\sigma_{n-1}(z) = 0. \tag{4.2}$$

Theorem 3 *Let the system* $\{\sigma_n(x)\}_{n=0}^{\infty}$ *be given by* (4.2), *then*:

(a) *the system satisfies*

$$G_\sigma(z,s) = \sum_{k=0}^{\infty}\sigma_k(z)s^k = e^{-z\arctan se^{i(\theta+\frac{\pi}{2})}},$$

(b) *the sequence of polynomials* $\{\sigma_n\}_0^\infty$ *is an orthogonal basis in the Hilbert space* $H^2(\mathcal{S},\mathcal{P})$,

(c) *the norm of polynomials* σ_n *is* $\sqrt{2}$ *if* $k \geq 1$ *and* 1 *if* $k = 0$.

Proof

(a) By (4.2) we have

$$(k+1)\sigma_{k+1}(z) + ie^{i\theta}z\sigma_k(z) - e^{2i\theta}(k-1)\sigma_{k-1}(z) = 0.$$

Multiplying the above relations by s^k, summing over k and simplifying, we obtain

$$\begin{aligned}
0 &= \sum_{k=0}^{\infty}\bigl[(k+1)\sigma_{k+1}(z) + ie^{i\theta}z\sigma_k(z) - e^{2i\theta}(k-1)\sigma_{k-1}(z)\bigr]s^k \\
&= \sum_{k=0}^{\infty}(k+1)\sigma_{k+1}(z)s^k + ie^{i\theta}z\sum_{k=0}^{\infty}\sigma_k(z)s^k - e^{2i\theta}\sum_{k=0}^{\infty}(k-1)\sigma_{k-1}(z)s^k \\
&= \frac{\partial G_\sigma(z,s)}{\partial s} + ie^{i\theta}zG_\sigma(z,s) - e^{2i\theta}s^2\frac{\partial G_\sigma(z,s)}{\partial s}.
\end{aligned}$$

This implies that

$$\bigl(1 - e^{2i\theta}s^2\bigr)\frac{\partial G_\sigma(z,s)}{\partial s} = -ie^{i\theta}zG_\sigma(z,s),$$

which in turn implies

$$\frac{\partial G_\sigma(z,s)}{\partial s} = \frac{-ie^{i\theta}z}{1 - e^{2i\theta}s^2}G_\sigma(z,s).$$

Integrating both sides with respect to s with the condition $G_\sigma(0,0) = 1$, we obtain

$$G_\sigma(z,s) = \left(\frac{1-e^{i\theta}s}{1+e^{i\theta}s}\right)^{\frac{iz}{2}} = e^{-z\arctan se^{i(\theta+\frac{\pi}{2})}}.$$

(b) In order to prove the orthogonality of $\sigma_n(x)$ polynomials and compute their norms, it suffices to show that

$$\int_{\partial\mathcal{S}} G_\sigma(z,s)\overline{G_\sigma(z,t)}\,d\mathcal{P}_z = \frac{1+s\bar{t}}{1-s\bar{t}}. \tag{4.3}$$

To this end, let take $\alpha = -\arctan se^{i(\theta+\frac{\pi}{2})}$, $\beta = -\arctan \bar{t}e^{-i(\theta+\frac{\pi}{2})}$ and the formula $\int_{-\infty}^{\infty} \frac{e^{(\alpha+\beta)x}}{2\cosh\frac{\pi}{2}x}\,dx = \frac{1}{\cos(\alpha+\beta)}$. Then

$$\begin{aligned}\int_{\partial\mathcal{S}} G_\sigma(z,s)\overline{G_\sigma(z,t)}\,d\mathcal{P}_z &= \int_{-\infty}^{\infty} \frac{e^{(x+i)\alpha+(x-i)\beta}+e^{(x-i)\alpha+(x+i)\beta}}{4\cosh(\frac{\pi}{2}x)}\,dx \\ &= \frac{e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}}{2}\int_{-\infty}^{\infty}\frac{e^{(\alpha+\beta)x}}{2\cosh(\frac{\pi}{2}x)}\,dx = \frac{\cos(\alpha-\beta)}{\cos(\alpha+\beta)} \\ &= \frac{1+\tan\alpha\tan\beta}{1-\tan\alpha\tan\beta} = \frac{1+s\bar{t}}{1-s\bar{t}}.\end{aligned}$$

(c) In the light of (a) and equation (4.3), we have

$$\begin{aligned}&\int_{-\infty}^{+\infty} G_\sigma(z,s)\overline{G_\sigma(z,t)}\frac{dx}{2\cosh\frac{\pi}{2}x}\,dx \\ &\quad= \int_{-\infty}^{+\infty}\left(\sum_{k=0}^{\infty}\sigma_k(x)s^k\right)\left(\sum_{n=0}^{\infty}\sigma_n(x)\bar{t}^n\right)\frac{dx}{2\cosh\frac{\pi}{2}x}\,dx \\ &\quad= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}s^k\bar{t}^n\int_{-\infty}^{+\infty}\sigma_k(x)\sigma_n(x)\frac{dx}{2\cosh\frac{\pi}{2}x}\,dx \\ &\quad= -1+2\sum_{k=0}^{\infty}(s\bar{t})^k.\end{aligned}$$

Comparing the coefficients of the powers of s and $\bar{t}$, we obtain the desired result. □

Remark 2 Applying Cauchy's integral formula to the generating function of the system, one obtains the integral representation

$$\sigma_n(x) = \frac{1}{2\pi i}\int_K e^{-z\arctan(te^{i(\theta+\frac{\pi}{2})})}\frac{dt}{t^{n+1}}$$

around a closed contour K about the origin with radius less than 1.

Remark 3 Let $y(x) = \sigma_n(x)$. The function $y(x)$ satisfies the following difference equation:

$$\frac{y(x+i)-y(x-i)}{2i} = \frac{ny(x)}{x}.$$

Proposition 2 *The system* $\{\sigma_n\}$ *satisfies the following relation*:

$$\sigma_n(2x) = \lim_{\lambda\to 0^+} S_n^\lambda(x;\theta).$$

Proof By (4.1) and by the definition of ${}_2F_1(a,b,c;z)$, we have

$$\begin{aligned}\lim_{\lambda\to 0^+} S_n^\lambda(x;\theta) &= \lim_{\lambda\to 0^+} e^{in\theta}\frac{(2\lambda)_n}{n!}{}_2F_1(-n,\lambda+ix,2\lambda;2)\\ &= \lim_{\lambda\to 0^+} e^{in\theta}\frac{(2\lambda)_n}{n!}\sum_{k=0}^{n}\frac{(-n)_k(\lambda+ix)_k}{(2\lambda)_k}\frac{2^k}{k!}\\ &= \frac{e^{in\theta}}{n!}\lim_{\lambda\to 0^+}\sum_{k=0}^{n}(-n)_k(\lambda+ix)_k(2\lambda+k)_{n-k}\frac{2^k}{k!}\\ &= \frac{e^{in\theta}}{n!}\sum_{k=0}^{n}(-n)_k(ix)_k(k)_{n-k}\frac{2^k}{k!} = \sigma_n(2x).\end{aligned}$$

□

Remark 4 From Proposition 1 we get

$$\sigma_n(x) = \frac{xe^{i\theta}}{in}S_{n-1}^1\left(\frac{x}{2};\theta\right).$$

5 The case $\psi = \pi - \theta$

We define quasi-symmetric Meixner-Pollaczek (QMP) polynomials $Q_n^\lambda(x;\theta)$ by the generating function

$$G^\lambda(x;\theta,\pi-\theta;z) = \frac{1}{(1-ze^{i\theta})^{\lambda-ix}(1+ze^{-i\theta})^{\lambda+ix}} = \sum_{n=0}^{\infty} Q_n^\lambda(x;\theta)z^n, \quad z\in\mathbb{D}.$$

Remark 5

(a) The QMP polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ satisfy the three-term recurrence relation

$$\begin{aligned}&Q_{-1}^\lambda = 0,\\ &Q_0^\lambda = 1,\\ &nQ_n^\lambda = 2i\big[(\lambda+n-1)\sin\theta - x\cos\theta\big]Q_{n-1}^\lambda + (2\lambda+n-2)Q_{n-2}^\lambda, \quad n\ge 1.\end{aligned}$$

(b) The polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ are given by the formula

$$Q_n^\lambda(x;\theta) = e^{in\theta}\sum_{j=0}^{n}(-1)^j\frac{(\lambda+ix)_j(\lambda-ix)_{n-j}}{j!(n-j)!}e^{-2ij\theta}, \quad n\in\mathbb{N}\cup\{0\}.$$

(c) The polynomials $Q_n^\lambda = Q_n^\lambda(x;\theta)$ have the hypergeometric representation

$$Q_n^\lambda(x;\theta) = e^{in\theta}\frac{(2\lambda)_n}{n!}{}_2F_1\big(-n,\lambda+ix,2\lambda;1+e^{-2i\theta}\big). \tag{5.1}$$

(d) The polynomials $y(x) = Q_n^\lambda(x;\theta)$ satisfy the following difference equation:

$$e^{i\theta}(\lambda-ix)y(x+i) - 2\big[x\sin\theta + (n+\lambda)\cos\theta\big]y(x) + e^{-i\theta}(\lambda+ix)y(x-i) = 0.$$

(e) The polynomials $Q_n^\lambda(x;\theta)$ are orthogonal on $(-\infty,+\infty)$ with the weight

$$w_\theta^\lambda(x) = \frac{1}{2\pi} e^{2\theta x} \left|\Gamma(\lambda + ix)\right|^2$$

for $\lambda > 0$ and $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{2\theta x} \left|\Gamma(\lambda + ix)\right|^2 Q_n^\lambda(x;\theta) \overline{Q_m^\lambda(x;\theta)}\, dx = \delta_{mn} \frac{\Gamma(n+2\lambda)}{(\cos\theta)^{2\lambda} n!}. \tag{5.2}$$

The Fisher information $I_\theta(\mu)$ of a random variable X with distribution $\mu(x;\theta)$, where θ is a continuous parameter, is defined by [20]

$$I_\theta(\mu) = \mathbb{E}\left\{ \left[\frac{\partial}{\partial x} \ln(\mu) \right]^2 \right\}.$$

It is named after RA Fisher who invented the concept of maximum likelihood estimator and discovered several of its properties. Over the years, the concept of Fisher information has found many application in physics [21], biology [22], engineering, *etc.* In [23] Dominici considered a sequence $P_n(x)$ of orthogonal polynomials with respect to the weight function $\rho(x)$ satisfying

$$\sum_{x=0}^{\infty} P_n(x) P_m(x) \rho(x) = h_n \delta_{nm}, \quad n, m = 0, 1, \ldots.$$

Introducing the functions

$$\rho_n(x) = \frac{[P_n(x)]^2 \rho(x)}{h_n}, \quad n \in \mathbb{N}_0, \tag{5.3}$$

the Fisher information corresponding to the functions (5.3) may be described as follows:

$$I_\theta(P_n) = \sum_{x=0}^{\infty} \left[\frac{\partial}{\partial\theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)}, \quad n \in \mathbb{N}_0.$$

For the family $P_n(x)$ of polynomials defined by

$$P_n(x) = {}_2F_1\left[-n, -x, c; z(\theta)\right], \quad n \in \mathbb{N}_0,$$

in [23], it was computed that

$$\frac{\partial P_n}{\partial\theta} = \frac{n}{z} \frac{\partial z}{\partial\theta} \left[P_n(x) - P_{n-1}(x)\right], \quad n \in \mathbb{N}_0. \tag{5.4}$$

In this work we use the ideas of [23] to compute the Fisher information of QMP polynomials.

Theorem 4 *The Fisher information of QMP polynomials is given by*

$$I_\theta\left(Q_n^\lambda\right) = \int_{-\infty}^{\infty} \left[\frac{\partial}{\partial\theta} \rho_n(x) \right]^2 \frac{1}{\rho_n(x)}\, dx = \frac{-2[n^2 + (2n+1)\lambda]}{\cos^2\theta}, \quad n \in \mathbb{N}_0,$$

with $\rho_n(x)$ defined as in (5.3).

Proof For GMP we have $\rho(x) = w_\theta^\lambda(x) = \frac{1}{2\pi}e^{2\theta x}|\Gamma(\lambda + ix)|^2$.

From (5.4) and (5.1), we have

$$\frac{\partial Q_n^\lambda}{\partial\theta} = -n\tan(\theta)Q_n^\lambda + i\frac{2\lambda + n - 1}{\cos\theta}Q_{n-1}^\lambda,$$

while (5.3) and (5.2) give

$$\rho_n(x) = \frac{e^{2\theta x}|\Gamma(\lambda + ix)|^2(\cos\theta)^{2\lambda}n![Q_n(x)]^2}{2\pi\Gamma(n + 2\lambda)}. \tag{5.5}$$

Note that

$$\int_{-\infty}^{\infty}\rho_n(x)\,dx = 1, \quad n \in \mathbb{N}_0. \tag{5.6}$$

Differentiating (5.5) with respect to θ, we obtain

$$\frac{\partial\rho_n(x)}{\partial\theta} = \frac{i\rho_n(x)}{\cos\theta Q_n^\lambda}\big[(n + 1)Q_{n+1}^\lambda - (2\lambda + n - 1)Q_{n-1}^\lambda\big].$$

Therefore

$$\begin{aligned}
&\left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2\frac{1}{\rho_n(x)} \\
&\quad = \frac{-\rho_n(x)}{\cos^2\theta(Q_n^\lambda)^2} \\
&\qquad \times \big[(n + 1)^2\big(Q_{n+1}^\lambda\big)^2 - 2(n + 1)(2\lambda + n - 1)Q_{n+1}^\lambda Q_{n-1}^\lambda + (2\lambda + n - 1)\big(Q_{n-1}^\lambda\big)^2\big] \\
&\quad = \frac{1}{\cos^2\theta}\bigg[(n + 1)(n + 2\lambda)\rho_{n+1}(x) + n(n + 2\lambda - 1)\rho_{n-1} \\
&\qquad - 2(n + 1)(2\lambda + n - 1)\frac{(\cos\theta)^{2\lambda}n!}{\Gamma(n + 2\lambda)\rho(x)Q_{n+1}^\lambda Q_{n-1}^\lambda}\bigg].
\end{aligned} \tag{5.7}$$

Integrating (5.7) and using the orthogonality relation (5.2), and (5.6), we get

$$I_\theta\big(Q_n^\lambda\big) = \int_{-\infty}^{\infty}\left[\frac{\partial}{\partial\theta}\rho_n(x)\right]^2\frac{1}{\rho_n(x)}\,dx = \frac{-1}{\cos^2\theta}\big[(n + 1)(n + 2\lambda) + n(n + 2\lambda - 1)\big]$$

and the result follows. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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