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New extensions concerned with results by Ponnusamy and Karunakaran

Mamoru Nunokawa[1], Kazuo Kuroki[2], Janusz Sokół[3] and Shigeyoshi Owa[2*]

*Correspondence:
shige21@ican.zaq.ne.jp
[2]Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
Full list of author information is available at the end of the article

Abstract
A subclass $\mathcal{A}(n,k)$ of analytic functions $f(z)$ in the open unit disk $\mathbb{U}$ is introduced. By means of the result due to Fukui and Sakaguchi (Bull. Fac. Edu. Wakayama Univ. Natur. Sci. 30:1-3, 1980), some interesting properties of $f(z)$ in $\mathcal{A}(n,k)$ concerned with Ponnusamy and Karunakaran (Complex Var. Theory Appl. 11:79-86, 1989) are discussed.

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1 Introduction

Let $\mathcal{A}(n,k)$ be a class of functions $f(z)$ of the form

$$f(z) = z^n + \sum_{m=n+k}^{\infty} a_m z^m \quad (n \geqq 1, k \geqq 1) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{A}(1,1)$, Sakaguchi [1] proved the following result.

Theorem A *Let $f(z) \in \mathcal{A}(1,1)$ and $g(z) \in \mathcal{A}(1,1)$ be starlike in $\mathbb{U}$. If $f(z)$ and $g(z)$ satisfy*

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0 \quad (z \in \mathbb{U}), \tag{1.2}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > 0 \quad (z \in \mathbb{U}). \tag{1.3}$$

After Theorem A, many mathematicians studying this field have applied this theorem to get some results (see [2]). In 1989, Ponnusamy and Karunakaran [3] improved Theorem A as follows.

Theorem B *Let α be a complex number with $\operatorname{Re}\alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n,k)$ and $g(z) \in \mathcal{A}(n,j)$ $(j \geqq 1)$ satisfy*

$$\operatorname{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right) > \delta \quad (z \in \mathbb{U}) \tag{1.4}$$

with $0 \leqq \delta < \frac{\operatorname{Re}\alpha}{n}$. *If* $f(z)$ *and* $g(z)$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} > \beta \quad (z \in \mathbb{U}), \tag{1.5}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > \frac{2\beta + \delta k}{2 + \delta k} \quad (z \in \mathbb{U}). \tag{1.6}$$

It is the purpose of the present paper to discuss Theorem B applying the lemma due to Fukui and Sakaguchi [4]. To discuss our problems, we need the following lemmas.

Lemma 1 *Let* $w(z) = \sum_{n=k}^{\infty} a_n z^n$ $(a_k \neq 0,\ k \geqq 1)$ *be analytic in* $\mathbb{U}$. *If the maximum value of* $|w(z)|$ *on the circle* $|z| = r < 1$ *is attained at* $z = z_0$, *then we have*

$$\frac{z_0 w'(z_0)}{w(z_0)} = \ell \geqq k, \tag{1.7}$$

which shows that $\frac{z_0 w'(z_0)}{w(z_0)}$ *is a positive real number.*

The proof of Lemma 1 can be found in [4], and we see that Lemma 1 is a generalization of Jack's lemma given by Jack [5]. Applying Lemma 1, we derive the following.

Lemma 2 *Let* $p(z) = 1 + \sum_{n=k}^{\infty} c_n z^n$ $(c_k \neq 0,\ k \geqq 1)$ *be analytic in* $\mathbb{U}$ *with* $p(z) \neq 0$ $(z \in \mathbb{U})$. *If there exists a point* $z_0 \in \mathbb{U}$ *such that*

$$\operatorname{Re} p(z) > 0 \quad \bigl(|z| < |z_0|\bigr)$$

and

$$\operatorname{Re} p(z_0) = 0,$$

then we have

$$-z_0 p'(z_0) \geqq \frac{\ell}{2}\bigl(1 + \bigl|p(z_0)\bigr|^2\bigr), \tag{1.8}$$

and so

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell, \tag{1.9}$$

where

$$k \leqq \frac{k}{2}\left(a + \frac{1}{a}\right) \leqq \ell \quad \left(\arg p(z_0) = \frac{\pi}{2}\right) \tag{1.10}$$

and

$$-k \geqq -\frac{k}{2}\left(a + \frac{1}{a}\right) \geqq \ell \quad \left(\arg p(z_0) = -\frac{\pi}{2}\right) \tag{1.11}$$

with $p(z_0) = \pm ia$ $(a > 0)$.

Proof Let us consider

$$\phi(z) = \frac{1-p(z)}{1+p(z)} = \frac{c_k}{2}z^k + \cdots \tag{1.12}$$

for $p(z)$. Then, it follows that $\phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0) = 0$, $|\phi(z)| < 1$ ($|z| < |z_0|$) and $|\phi(z_0)| = 1$. Therefore, applying Lemma 1, we have that

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = \frac{-2z_0p'(z_0)}{1-(p(z_0))^2} = \frac{-2z_0p'(z_0)}{1+|p(z_0)|^2} = \ell \geqq k. \tag{1.13}$$

This implies that $z_0p'(z_0)$ is a negative real number and

$$-z_0p'(z_0) \geqq \frac{k}{2}\left(1+\left|p(z_0)\right|^2\right). \tag{1.14}$$

Let us use the same method by Nunokawa [6]. If $\arg p(z_0) = \frac{\pi}{2}$, then we write $p(z_0) = ia$ ($a > 0$). This gives us that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right) = \operatorname{Im}\left(-\frac{iz_0p'(z_0)}{a}\right) \geqq \frac{k}{2}\left(a+\frac{1}{a}\right).$$

If $\arg p(z_0) = -\frac{\pi}{2}$, then we write $p(z_0) = -ia$ ($a > 0$). Thus we have that

$$\operatorname{Im}\left(\frac{z_0p'(z_0)}{p(z_0)}\right) = \operatorname{Im}\left(\frac{iz_0p'(z_0)}{a}\right) \leqq -\frac{k}{2}\left(a+\frac{1}{a}\right).$$

This completes the proof of Lemma 2. □

2 Main results

With the help of Lemma 2, we derive the following theorem.

Theorem 1 *Let* α *be a complex number with* $\operatorname{Re}\alpha > 0$ *and* $\beta < 1$. *Further, let* $f(z) \in \mathcal{A}(n,k)$ *and* $g(z) \in \mathcal{A}(n,j)$ ($j \geqq 1$) *satisfy*

$$\operatorname{Re}\left(\frac{\alpha g(z)}{zg'(z)}\right) > \delta \quad (z \in \mathbb{U}) \tag{2.1}$$

with $0 \leqq \delta < \frac{\operatorname{Re}\alpha}{n}$. *If* $f(z)$ *and* $g(z)$ *satisfy*

$$\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)}\right\} + \frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)} - \beta_1\right|^2 > \beta \quad (z \in \mathbb{U}), \tag{2.2}$$

then

$$\operatorname{Re}\left(\frac{f(z)}{g(z)}\right) > \beta_1 \quad (z \in \mathbb{U}), \tag{2.3}$$

where $\beta_1 = \frac{2\beta+\delta k}{2+\delta k}$.

Proof Defining the function $p(z)$ by

$$p(z) = \frac{\frac{f(z)}{g(z)} - \beta_1}{1 - \beta_1}, \tag{2.4}$$

we see that $p(0) = 1$ and

$$\begin{aligned}
&\operatorname{Re}\left\{(1-\alpha)\frac{f(z)}{g(z)} + \alpha\frac{f'(z)}{g'(z)} - \beta\right\} \\
&\quad = \operatorname{Re}\left\{(\beta_1 - \beta) + (1-\beta_1)\left(p(z) + \frac{\alpha g(z)}{zg'(z)} zp'(z)\right)\right\} \\
&\quad > -\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z)}{g(z)} - \beta_1\right|^2
\end{aligned} \tag{2.5}$$

for all $z \in \mathbb{U}$. Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\left|\arg p(z_0)\right| < \frac{\pi}{2} \quad \bigl(|z| < |z_0|\bigr)$$

and

$$\left|\arg p(z_0)\right| = \frac{\pi}{2}.$$

Then, by means of Lemma 2, we have that

$$-z_0 p'(z_0) \geqq \frac{k}{2}\left(1 + \left|p(z_0)\right|^2\right). \tag{2.6}$$

If follows from the above that

$$\begin{aligned}
&\operatorname{Re}\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\} \\
&\quad = (\beta_1 - \beta) + (1-\beta_1)\operatorname{Re}\left\{p(z_0) + \frac{\alpha g(z_0)}{z_0 g'(z_0)} z_0 p'(z_0)\right\} \\
&\quad = (\beta_1 - \beta) - (1-\beta_1)\operatorname{Re}\left\{\frac{\alpha g(z_0)}{z_0 g'(z_0)}\bigl(-z_0 p'(z_0)\bigr)\right\} \\
&\quad \leqq (\beta_1 - \beta) - (1-\beta_1)\frac{\delta k}{2}\left(1 + \left|p(z_0)\right|^2\right) \\
&\quad = -\frac{\delta k}{2(1-\beta_1)}\left|\frac{f(z_0)}{g(z_0)} - \beta_1\right|^2,
\end{aligned}$$

which contradicts (2.5). This completes the proof of the theorem. □

Remark 1 If $f(z)$ and $g(z)$ satisfy $f(z) = \beta_1 g(z)$ in Theorem 1, then Theorem 1 becomes Theorem B given by Ponnusamy and Karunakaran [3]. We also have the following theorem.

Theorem 2 *Let* α *be a complex number with* $\operatorname{Re}\alpha > 0$ *and* $\beta < 1$. *Further, let* $f(z) \in \mathcal{A}(n,k)$ *and* $g(z) \in \mathcal{A}(n,j)$ $(j \geqq 1)$ *satisfy the condition* (2.1) *with* $0 \leqq \delta < \frac{\operatorname{Re}\alpha}{n} \leqq 1 + \delta$. *If* $f(z)$ *and* $g(z)$

satisfy

$$\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}-\beta\right\}\right|<\frac{\pi}{2}+\mathrm{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2(\frac{2r}{1-r^2}+\frac{|\mathrm{Im}\,\alpha|}{n}+1)}\right) \tag{2.7}$$

for $|z|=r<1$, *then*

$$\left|\arg\left(\frac{f(z)}{g(z)}-\beta_1\right)\right|<\frac{\pi}{2}\quad(z\in\mathbb{U}) \tag{2.8}$$

or

$$\mathrm{Re}\left(\frac{f(z)}{g(z)}\right)>\beta_1\quad(z\in\mathbb{U}), \tag{2.9}$$

where $\beta_1=\frac{2\beta+\delta k}{2+\delta k}$ *and*

$$p(z)=\frac{\frac{f(z)}{g(z)}-\beta_1}{1-\beta_1}.$$

Proof Note that the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0)=1$. It follows that

$$\begin{aligned}\left|\arg\left\{(1-\alpha)\frac{f(z)}{g(z)}+\alpha\frac{f'(z)}{g'(z)}-\beta\right\}\right| &= \left|\arg\left\{(\beta_1-\beta)+(1-\beta_1)\left(p(z)+\frac{\alpha g(z)}{zg'(z)}zp'(z)\right)\right\}\right| \\ &< \frac{\pi}{2}+\mathrm{Tan}^{-1}\left(\frac{\delta k|p(z)|}{2(\frac{2r}{1-r^2}+\frac{|\mathrm{Im}\,\alpha|}{n}+1)}\right)\end{aligned}$$

for $|z|=r<1$. If there exists a point $z_0\in\mathbb{U}$ such that

$$\left|\arg p(z_0)\right|<\frac{\pi}{2}\quad\left(|z|<|z_0|\right)$$

and

$$\left|\arg p(z_0)\right|=\frac{\pi}{2},$$

then, by Lemma 2, we have that

$$\frac{z_0p'(z_0)}{p(z_0)}=i\ell,$$

where

$$\frac{k}{2}\left(a+\frac{1}{a}\right)\leqq\ell\quad\left(\arg p(z_0)=\frac{\pi}{2}\right)$$

and

$$-\frac{k}{2}\left(a+\frac{1}{a}\right)\geqq\ell\quad\left(\arg p(z_0)=-\frac{\pi}{2}\right)$$

with $p(z_0) = \pm ia$ $(a > 0)$. If $\arg p(z_0) = \frac{\pi}{2}$, then it follows that

$$\begin{aligned}
&\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\} \\
&\quad = \arg p(z_0)\left\{\frac{\beta_1-\beta}{p(z_0)} + (1-\beta_1)\left(1 + \frac{\alpha g(z_0)}{z_0 g'(z_0)}\frac{z_0 p'(z_0)}{p(z_0)}\right)\right\} \\
&\quad = \frac{\pi}{2} + \arg\left\{-\left(\frac{\beta_1-\beta}{a}\right)i + (1-\beta_1)\left(1 + i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right\} \\
&\quad = \frac{\pi}{2} + \arg I(z_0),
\end{aligned}$$

where

$$I(z_0) = -\left(\frac{\beta_1-\beta}{a}\right)i + (1-\beta_1)\left(1 + i\ell\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right). \tag{2.10}$$

Note that

$$\begin{aligned}
\operatorname{Im} I(z_0) &= \frac{\beta-\beta_1}{a} + (1-\beta_1)\ell \operatorname{Re}\frac{\alpha g(z_0)}{z_0 g'(z_0)} \\
&\geqq (1-\beta_1)\delta\ell + \frac{\beta-\beta_1}{a} \\
&\geqq \frac{\delta k}{2}(1-\beta_1)\left(a + \frac{1}{a}\right) + \frac{\beta-\beta_1}{a} \\
&= \frac{\delta k}{2}(1-\beta_1)a > 0
\end{aligned} \tag{2.11}$$

and

$$\operatorname{Re} I(z_0) = (1-\beta_1)\left(1 - \ell \operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right) \leqq (1-\beta_1)\left(1 + \ell\left|\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right|\right). \tag{2.12}$$

Letting

$$q(z) = \frac{\alpha g(z)}{z g'(z)} + 1 - \frac{\alpha}{n}, \tag{2.13}$$

we know that $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$ and $\operatorname{Re} q(z) > 0$ $(z \in \mathbb{U})$. Therefore, applying the subordinations, we can write that

$$q(z) = \frac{1-w(z)}{1+w(z)}$$

with the Schwarz function $w(z)$ analytic in $\mathbb{U}$, $w(0) = 0$ and $|w(z)| \leqq |z|$. This leads us to

$$|w(z)| = \left|\frac{1-q(z)}{1+q(z)}\right| \leqq r \quad \bigl(|z| \leqq r < 1\bigr),$$

which is equivalent to

$$\left|q(z) - \frac{1+r^2}{1-r^2}\right| \leqq \frac{2r}{1-r^2}.$$

This gives us that

$$\left|\operatorname{Im} q(z)\right| = \left|\operatorname{Im}\left(\frac{\alpha g(z)}{zg'(z)} + 1 - \frac{\alpha}{n}\right)\right| \leqq \frac{2r}{1-r^2} \tag{2.14}$$

for $|z| = r < 1$. Thus we have that

$$\left|\operatorname{Im}\left(\frac{\alpha g(z_0)}{z_0 g'(z_0)}\right)\right| \leqq \frac{2r}{1-r^2} + \frac{|\operatorname{Im}\alpha|}{n} \quad \bigl(|z| = r < 1\bigr). \tag{2.15}$$

Using (2.12) and (2.15), we obtain that

$$\arg I(z_0) = \operatorname{Tan}^{-1}\left(\frac{\operatorname{Im} I(z_0)}{\operatorname{Re} I(z_0)}\right) \geqq \operatorname{Tan}^{-1}\left(\frac{\delta k a}{2(\frac{2r}{1-r^2} + \frac{|\operatorname{Im}\alpha|}{n} + 1)}\right),$$

which contradicts our condition (2.7).

If $\arg p(z_0) = -\frac{\pi}{2}$, using the same way, we also have that

$$\arg\left\{(1-\alpha)\frac{f(z_0)}{g(z_0)} + \alpha\frac{f'(z_0)}{g'(z_0)} - \beta\right\} \leqq -\left\{\frac{\pi}{2} + \operatorname{Tan}^{-1}\left(\frac{\delta k a}{2(\frac{2r}{1-r^2} + \frac{|\operatorname{Im}\alpha|}{n} + 1)}\right)\right\},$$

which contradicts (2.7). □

Competing interests
The authors did not provide this information.

Authors' contributions
The authors did not provide this information.

Author details
[1]University of Gunma, 798-8 Hoshikuki, Chuou-Ward, Chiba 260-0808, Japan. [2]Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan. [3]Department of Mathematics, Rzeszow University of Technology, Al. Powstańców, Warszawy 12, Rzeszów, 35-959, Poland.

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