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Uncountably many solutions of first-order neutral nonlinear differential equations

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Abstract
The article deals with the existence of uncountably many positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. Some examples are included to illustrate the results presented in this article.
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1 Introduction

In recent years, the study of existence and qualitative properties of solutions for various kinds of neutral delay differential equations has attracted much attention. For related results, we refer the reader to [1–13] and the references cited therein. The authors only considered the existence of solutions which are bounded by positive constants, *e.g.*, in [8, 9, 11–13]. For example, Erbe *et al.* [6] established a few oscillation and nonoscillation criteria for linear neutral delay differential equation

$$\bigl[x(t) - p(t)x(t-\tau)\bigr]' + q(t)x\bigl(t-\sigma(t)\bigr) = 0, \quad t \geq t_0.$$

Diblík and co-autors in [1–4] studied the existence of positive and oscillatory solutions of differential equations with delay and nonlinear systems in view of Ważievski's retract principle and later extended to retarded functional differential equations by Rybakowski. Zhou [12] deduced the existence of nonoscillatory solutions of the second-order nonlinear neutral differential equations and Lin *et al.* [9] discussed the existence of nonoscillatory solutions for a third-order nonlinear neutral delay differential equation, and by utilizing Krasnoselskii's fixed point theorem and Schauder's fixed point theorem, they developed some sufficient conditions for the existence of uncountably many nonoscillatory solutions bounded by positive constants. Some interesting results about the existence of nonoscillatory solutions of delay differential equations can also be found in [1, 5].

In this paper, we investigate the following nonlinear neutral differential delay differential equations:

$$\frac{d}{dt}\bigl[x(t) - a(t)x(t-\tau)\bigr] = p(t)f\bigl(x(t-\sigma)\bigr), \quad t \geq t_0, \tag{1}$$

where $\tau > 0$, $\sigma \geq 0$, $a \in C([t_0,\infty),(0,\infty))$, $p \in C(R,(0,\infty))$, $f \in C(R,R)$, f is a nondecreasing function for $x > 0$ and $f(x) > 0$, $x > 0$.

By a solution of Eq. (1), we mean a function $x \in C([t_1 - \tau,\infty),R)$ for some $t_1 \geq t_0$ such that $x(t) - a(t)x(t-\tau)$ is continuously differentiable on $[t_1,\infty)$ and such that Eq. (1) is satisfied for $t \geq t_1$.

As much as we know, in the literature there is no result for the existence of uncountably many solutions which are bounded below and above by positive functions. This problem is discussed and treated in this paper.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 ([6, 12] Krasnoselskii's fixed point theorem) *Let X be a Banach space, let Ω be a bounded closed convex subset of X and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is contractive and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x$$

has a solution in Ω.

2 The existence of positive solutions

In this section we consider the existence of uncountably many positive solutions for Eq. (1) which are bounded by two positive functions. We use the notation $m = \max\{\tau,\sigma\}$.

Theorem 2.1 *Suppose that there exist bounded from below and from above by the functions u and $v \in C^1([t_0,\infty),(0,\infty))$ constants $c > 0$, $K_2 > K_1 \geq 0$ and $t_1 \geq t_0 + m$ such that*

$$u(t) \leq v(t), \quad t \geq t_0, \tag{2}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1, \tag{3}$$

$$\begin{aligned} &\frac{1}{u(t-\tau)}\left(u(t) - K_1 + \int_t^\infty p(s)f\big(v(s-\sigma)\big)\,ds\right) \\ &\quad \leq a(t) \leq \frac{1}{v(t-\tau)}\left(v(t) - K_2 + \int_t^\infty p(s)f\big(u(s-\sigma)\big)\,ds\right) \leq c < 1, \quad t \geq t_1. \end{aligned} \tag{4}$$

Then Eq. (1) *has uncountably many positive solutions which are bounded by the functions u, v.*

Proof Let $C([t_0,\infty),R)$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t\geq t_0}|x(t)|$. Then $C([t_0,\infty),R)$ is a Banach space. We define a closed, bounded and convex subset Ω of $C([t_0,\infty),R)$ as follows:

$$\Omega = \big\{x = x(t) \in C\big([t_0,\infty),R\big) : u(t) \leq x(t) \leq v(t), t \geq t_0\big\}.$$

For $K \in [K_1,K_2]$ we define two maps S_1 and $S_2 : \Omega \to C([t_0,\infty),R)$ as follows:

$$(S_1x)(t) = \begin{cases} K + a(t)x(t-\tau), & t \geq t_1, \\ (S_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases} \tag{5}$$

$$(S_2x)(t) = \begin{cases} -\int_t^\infty p(s)f(x(s-\sigma))\,ds, & t \ge t_1, \\ (S_2x)(t_1) + v(t) - v(t_1), & t_0 \le t \le t_1. \end{cases} \tag{6}$$

We will show that for any $x, y \in \Omega$, we have $S_1x + S_2y \in \Omega$. For every $x, y \in \Omega$ and $t \ge t_1$ with regard to (4), we obtain

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= K + a(t)x(t-\tau) - \int_t^\infty p(s)f\bigl(y(s-\sigma)\bigr)\,ds \\ &\le K + a(t)v(t-\tau) - \int_t^\infty p(s)f\bigl(u(s-\sigma)\bigr)\,ds \\ &\le K + v(t) - K_2 \le v(t). \end{aligned}$$

For $t \in [t_0, t_1]$ we have

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\le v(t_1) + v(t) - v(t_1) = v(t). \end{aligned}$$

Furthermore, for $t \ge t_1$ we get

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &\ge K + a(t)u(t-\tau) - \int_t^\infty p(s)f\bigl(v(s-\sigma)\bigr)\,ds \\ &\ge K + u(t) - K_1 \ge u(t). \end{aligned} \tag{7}$$

Let $t \in [t_0, t_1]$. With regard to (3), we get

$$v(t) - v(t_1) + u(t_1) \ge u(t), \quad t_0 \le t \le t_1.$$

Then, for $t \in [t_0, t_1]$ and any $x, y \in \Omega$, we obtain

$$\begin{aligned} (S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\ge u(t_1) + v(t) - v(t_1) \ge u(t). \end{aligned}$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \ge t_1$, we have

$$\bigl|(S_1x)(t) - (S_1y)(t)\bigr| = \bigl|a(t)\bigr|\bigl|x(t-\tau) - y(t-\tau)\bigr| \le c\|x - y\|.$$

This implies that

$$\|S_1x - S_1y\| \le c\|x - y\|.$$

Also, for $t \in [t_0, t_1]$ the inequality above is valid. We conclude that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First, we show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For

$t \geq t_1$ we have

$$\begin{aligned}\left|(S_2x_k)(t) - (S_2x)(t)\right| &\leq \left|\int_t^\infty p(s)\left[f\left(x_k(s-\sigma)\right) - f\left(x(s-\sigma)\right)\right] ds\right| \\ &\leq \int_{t_1}^\infty p(s)\left|f\left(x_k(s-\sigma)\right) - f\left(x(s-\sigma)\right)\right| ds.\end{aligned}$$

According to (7), we get

$$\int_{t_1}^\infty p(s)f\left(v(s-\sigma)\right) ds < \infty. \tag{8}$$

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we obtain that

$$\lim_{k\to\infty} \left\|(S_2x_k)(t) - (S_2x)(t)\right\| = 0.$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity, we only need to show, according to the Levitan result [7], that for any given $\varepsilon > 0$, the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε. Then, with regard to condition (8), for $x \in \Omega$ and any $\varepsilon > 0$, we take $t^* \geq t_1$ large enough so that

$$\int_{t^*}^\infty p(s)f\left(x(s-\sigma)\right) ds < \frac{\varepsilon}{2}.$$

Then, for $x \in \Omega$, $T_2 > T_1 \geq t^*$, we have

$$\begin{aligned}\left|(S_2x)(T_2) - (S_2x)(T_1)\right| &\leq \int_{T_2}^\infty p(s)f\left(x(s-\sigma)\right) ds \\ &\quad + \int_{T_1}^\infty p(s)f\left(x(s-\sigma)\right) ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

For $x \in \Omega$ and $t_1 \leq T_1 < T_2 \leq t^*$, we get

$$\begin{aligned}\left|(S_2x)(T_2) - (S_2x)(T_1)\right| &\leq \int_{T_1}^{T_2} p(s)f\left(x(s-\sigma)\right) ds \\ &\leq \max_{t_1\leq s\leq t^*}\left\{p(s)f\left(x(s-\sigma)\right)\right\}(T_2 - T_1).\end{aligned}$$

Thus there exists $\delta_1 = \frac{\varepsilon}{M}$, where $M = \max_{t_1\leq s\leq t^*}\{p(s)f(x(s-\sigma))\}$, such that

$$\left|(S_2x)(T_2) - (S_2x)(T_1)\right| < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_1.$$

Finally, for any $x \in \Omega$, $t_0 \le T_1 < T_2 \le t_1$, there exists a $\delta_2 > 0$ such that

$$\begin{aligned}\bigl|(S_2x)(T_2) - (S_2x)(T_1)\bigr| &= \bigl|v(T_1) - v(T_2)\bigr| = \left|\int_{T_1}^{T_2} v'(s)\,ds\right| \\ &\le \max_{t_0\le s\le t_1}\bigl\{\bigl|v'(s)\bigr|\bigr\}(T_2 - T_1) < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_2.\end{aligned}$$

Then $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0,\infty)$, and hence $S_2\Omega$ is a relatively compact subset of $C([t_0,\infty), R)$. By Lemma 1.1 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (1).

Next we show that Eq. (1) has uncountably many bounded positive solutions in Ω. Let the constant $\bar{K} \in [K_1, K_2]$ be such that $\bar{K} \ne K$. We infer similarly that there exist mappings $\bar{S}_1$, $\bar{S}_2$ satisfying (5), (6), where K, S_1, S_2 are replaced by $\bar{K}$, $\bar{S}_1$, $\bar{S}_2$, respectively. We assume that $x, y \in \Omega$, $S_1x + S_2x = x$, $\bar{S}_1y + \bar{S}_2y = y$, which are the bounded positive solutions of Eq. (1), that is,

$$\begin{aligned}x(t) &= K + a(t)x(t-\tau) - \int_t^\infty p(s)f\bigl(x(s-\sigma)\bigr)\,ds, \quad t \ge t_1, \\ y(t) &= \bar{K} + a(t)y(t-\tau) - \int_t^\infty p(s)f\bigl(y(s-\sigma)\bigr)\,ds, \quad t \ge t_1.\end{aligned}$$

From condition (8) it follows that there exists a $t_2 > t_1$ satisfying

$$\int_{t_2}^\infty p(s)\bigl[f\bigl(x(s-\sigma)\bigr) + f\bigl(y(s-\sigma)\bigr)\bigr]\,ds < |K - \bar{K}|. \tag{9}$$

In order to prove that the set of bounded positive solutions of Eq. (1) is uncountable, it is sufficient to verify that $x \ne y$. For $t \ge t_2$ we get

$$\begin{aligned}&\bigl|x(t) - y(t)\bigr| \\ &\quad = \left|K + a(t)x(t-\tau) - \int_t^\infty p(s)f\bigl(x(s-\sigma)\bigr)\,ds\right. \\ &\qquad \left. - \bar{K} - a(t)y(t-\tau) + \int_t^\infty p(s)f\bigl(y(s-\sigma)\bigr)\,ds\right| \\ &\quad \ge \left|K - \bar{K} + a(t)\bigl[x(t-\tau) - y(t-\tau)\bigr]\right. \\ &\qquad \left. - \int_t^\infty p(s)\bigl[f\bigl(x(s-\sigma)\bigr) - f\bigl(y(s-\sigma)\bigr)\bigr]\,ds\right| \\ &\quad \ge |K - \bar{K}| - a(t)\|x - y\| - \left|\int_t^\infty p(s)\bigl[f\bigl(x(s-\sigma)\bigr) - f\bigl(y(s-\sigma)\bigr)\bigr]\,ds\right| \\ &\quad \ge |K - \bar{K}| - c\|x - y\| - \int_t^\infty p(s)\bigl[f\bigl(x(s-\sigma)\bigr) + f\bigl(y(s-\sigma)\bigr)\bigr]\,ds.\end{aligned}$$

Then we have

$$(1 + c)\|x - y\| \ge |K - \bar{K}| - \int_t^\infty p(s)\bigl[f\bigl(x(s-\sigma)\bigr) + f\bigl(y(s-\sigma)\bigr)\bigr]\,ds, \quad t \ge t_2.$$

According to (9) we get that $x \neq y$. Since the interval $[K_1, K_2]$ contains uncountably many constants, then Eq. (1) has uncountably many positive solutions which are bounded by the functions $u(t)$, $v(t)$. This completes the proof. □

Corollary 2.1 *Suppose that there exist bounded from below and from above by the functions u and $v \in C^1([t_0, \infty), (0, \infty))$ constants $c > 0$, $K_2 > K_1 \geq 0$ and $t_1 \geq t_0 + m$ such that* (2), (4) *hold and*

$$v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1. \tag{10}$$

Then Eq. (1) *has uncountably many positive solutions which are bounded by the functions u, v.*

Proof We only need to prove that condition (10) implies (3). Let $t \in [t_0, t_1]$ and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1).$$

Then, with regard to (10), it follows that

$$H'(t) = v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1.$$

Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, this implies that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1.$$

Thus all the conditions of Theorem 2.1 are satisfied. □

Example 2.1 Consider the nonlinear neutral differential equation

$$\bigl[x(t) - a(t)x(t-2)\bigr]' = p(t)x^3(t-1), \quad t \geq t_0, \tag{11}$$

where $p(t) = e^{-t}$. We will show that the conditions of Corollary 2.1 are satisfied. The functions $u(t) = 0.5$, $v(t) = 2$ satisfy (2) and also condition (10) for $t \in [t_0, t_1] = [0, 4]$. For the constants $K_1 = 0.5$, $K_2 = 1$, condition (4) has the form

$$16e^{-t} \leq a(t) \leq \frac{1}{2} + \frac{1}{16}e^{-t}, \quad t \geq t_1 = 4. \tag{12}$$

If the function $a(t)$ satisfies (12), then Eq. (11) has uncountably many positive solutions which are bounded by the functions u, v.

Example 2.2 Consider the nonlinear neutral differential equation

$$\bigl[x(t) - a(t)x(t-\tau)\bigr]' = p(t)x^2(t-\sigma), \quad t \geq t_0, \tag{13}$$

where $\tau, \sigma \in (0, \infty)$, $p(t) = e^{-3t}$. We will show that the conditions of Corollary 2.1 are satisfied. The functions $u(t) = e^{-2t}$, $v(t) = e^{\tau} + e^{-t}$, $t \geq 1$, satisfy (2) and since

$$v'(t) - u'(t) = e^{-t}\bigl(2e^{-t} - 1\bigr) < 0 \quad \text{for } t \in [1, 2],$$

condition (10) is also satisfied. For the constants $K_1 = 0$, $K_2 = e^{\tau} - 1$, condition (4) has the form

$$e^{-2\tau} + \frac{1}{3}e^{-t} + \frac{1}{2}e^{-2t+\sigma-\tau} + \frac{1}{5}e^{-3t+2(\sigma-\tau)} \le a(t) \le e^{-\tau} + \frac{e^{-7t+4\sigma-\tau}}{7(1+e^{-t})}, \quad t \ge 2.$$

For $\tau = \sigma = 1$ we get

$$e^{-2} + \frac{1}{3}e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{5}e^{-3t} \le a(t) \le e^{-1} + \frac{e^{-7t+3}}{7(1+e^{-t})}, \quad t \ge t_1 = 2. \tag{14}$$

If the function $a(t)$ satisfies (14), then Eq. (13) has uncountably many solutions which are bounded by the functions u, v.

Example 2.3 Consider the nonlinear neutral differential equation

$$\left[x(t) - a(t)x(t-\tau)\right]' = p(t)x^3(t-\sigma), \quad t \ge t_0, \tag{15}$$

where $\tau, \sigma \in (0, \infty)$, $p(t) = e^{-t}$. We will show that the conditions of Corollary 2.1 are satisfied. The functions $u(t) = e^{-t}$, $v(t) = e^{\tau} + 2e^{-t}$, $t \ge 1$ satisfy (2) and also (10)

$$v'(t) - u'(t) = -e^{-t} < 0 \quad \text{for } t \in [1, 3.2].$$

For the constants $K_1 = 1$, $K_2 = e^{\tau} - 1$, where $\tau > \ln 2$, $t \ge 3.2$, condition (4) has the form

$$e^{-\tau}\left(1 - e^{t} + e^{3\tau} + 3e^{2\tau+\sigma-t} + 4e^{\tau+2\sigma-2t} + 2e^{3\sigma-3t}\right) \le a(t) \le e^{-\tau} + \frac{e^{3\sigma-\tau-4t}}{4(1+2e^{-t})}.$$

For $\tau = \sigma = 1$ and $t \ge 3.2$, we have

$$e^{-1}\left(1 - e^{t} + e^{3} + 3e^{3-t} + 4e^{3-2t} + 2e^{3(1-t)}\right) < 0.$$

Then for $a(t)$, which satisfies the inequalities

$$0 < a(t) \le e^{-1} + \frac{e^{2(1-2t)}}{4(1+2e^{-t})}, \quad t \ge t_1 \ge 3.2, \tag{16}$$

Eq. (11) has uncountably many solutions which are bounded by the functions u, v.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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References

1. Boichuk, A, Diblík, J, Khusainov, D, Růžičková, M: Fredholm's boundary-value problems for differential systems with a single delay. Nonlinear Anal. **72**, 2251-2258 (2010)
2. Diblík, J, Svoboda, Z, Šmarda, Z: Retract principle for neutral functional differential equations. Nonlinear Anal., Theory Methods Appl. **71**, 1393-1400 (2009)
3. Diblík, J, Růžičková, M: Existence of positive solutions of a singular initial problem for a nonlinear system of differential equations. Rocky Mt. J. Math. **3**, 923-944 (2004)
4. Diblík, J, Růžičková, M, Šutá, Z: Asymptotical convergence of the solutions of a linear differential equation with delays. Adv. Differ. Equ. **2010**, Article ID 749852 (2010)
5. Dorociaková, B, Najmanová, A, Olach, R: Existence of nonoscillatory solutions of first-order neutral differential equations. Abstr. Appl. Anal. **2011**, Article ID 346745 (2011). doi:10.1155/2011/346745
6. Erbe, LH, Kong, QK, Zhang, BG: Oscillation Theory for Functional Differential Equations. Dekker, New York (1995)
7. Levitan, BM: Some problems of the theory of almost periodic functions I. Usp. Mat. Nauk **2**(5), 133-192 (1947)
8. Lin, X: Oscillation of second-order nonlinear neutral differential equations. J. Math. Anal. Appl. **309**, 442-452 (2005)
9. Lin, Z, Chen, L, Kang, SM, Cho, SY: Existence of nonoscillatory solutions for a third order nonlinear neutral delay differential equation. Abstr. Appl. Anal. **2011**, Article ID 693890 (2011). doi:10.1155/2011/693890
10. Špániková, E, Šamajová, H: Asymptotic properties of solutions to n-dimensional neutral differential systems. Nonlinear Anal., Theory Methods Appl. **71**, 2877-2885 (2009)
11. Wang, X, Liao, L: Asymptotic behaviour of solutions of neutral differential equations with positive and negative coefficients. J. Math. Anal. Appl. **279**, 326-338 (2003)
12. Zhou, Y: Existence for nonoscillatory solutions of second-order nonlinear differential equations. J. Math. Anal. Appl. **331**, 91-96 (2007)
13. Zhou, Y, Zhang, BG: Existence of nonoscillatory solutions of neutral differential equations with positive and negative coefficients. Appl. Math. Lett. **15**, 867-874 (2002)

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