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The tanh-coth method for some nonlinear pseudoparabolic equations with exact solutions

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Abstract

We studied mostly important four nonlinear pseudoparabolic physical models: the Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation, the Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation, the one-dimensional Oskolkov equation and the generalised hyperelastic-rod wave equation. By using the tanh-coth method and symbolic computation system Maple, we have obtained abundant new solutions of these equations. The exact solutions show that the tanh-coth method is a powerful mathematical tool for solving nonlinear pseudoparabolic equations.

Keywords: nonlinear pseudoparabolic equation; Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation; Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation; one-dimensional Oskolkov equation; generalised hyperelastic-rod wave equation; tanh-coth method

1 Introduction

Equations with a one-time derivative appearing in the highest order term are called pseudoparabolic and arise in many areas of mathematics and physics. They have been used, for instance, for fluid flow in fissured rock, consolidation of clay, shear in second-order fluids, thermodynamics and propagation of long waves of small amplitude. For more details, we refer the reader to [1–5] and references therein.

An important special case of pseudoparabolic-type equations is the generalised Benjamin-Bona-Mahony-Burgers (BBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + f(u)_x = 0, \tag{1}$$

where $u(x,t)$ represents the fluid velocity in the horizontal direction x, α is a positive constant, γ is any given real constant and $f(u)$ is a C^2-smooth nonlinear function. For $f(u)_x = uu_x$ with $\alpha = 0$, $\gamma = 1$ in equation (1) was proposed as an alternative regularised long-wave equation by Peregrine [6] and Benjamin *et al.* [7] for the well-known Korteweg-de Vries equation

$$u_t + u_{xxt} + u_x + uu_x = 0. \tag{2}$$

If we take $f(u)_x = \theta uu_x + \beta u_{xxx}$ in equation (1), then we obtain a general form of Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta uu_x + \beta u_{xxx} = 0. \tag{3}$$

Taking $\alpha = \beta = 0$ in (3), we get the general form of the BBM equation as follows:

$$u_t - u_{xxt} + \gamma u_x + \theta uu_x = 0, \tag{4}$$

where γ, θ are constants and $\theta \neq 0$. Equation (3) includes several types of the BBM equation as seen in the literature. For more details, we refer the reader to [6–15]. We will study the general form of Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation (3) using the tanh-coth method. We aim to extend the previous works especially in [6, 7] to make further progress for obtaining abundant new travelling wave solutions.

For $\beta = 0$ in equation (3), we obtain a general form of the Oskolkov-Benjamin-Bona-Mahony-Burgers equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta uu_x = 0. \tag{5}$$

This nonlinear, one-dimensional and pseudoparabolic equation describes nonlinear surface waves that spread along the axis Ox and αu_{xx} is the viscosity term [16, 17]. In the literature the inverse scattering method has been thoroughly used to derive the multiple soliton solutions of equation (5) [8, 18–22]. In this work we developed these solutions in a way that can be easily applied by using the tanh-coth method, which is less sophisticated than the inverse scattering method.

The equation

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + uu_x = 0 \tag{6}$$

is a one-dimensional analogue of the Oskolkov system

$$\left(1 - \lambda \nabla^2\right) u_t = \alpha \nabla^2 u - (u \cdot \nabla) u - \nabla^2 p + f, \quad \nabla \cdot u = 0. \tag{7}$$

This system describes the dynamics of an incompressible viscoelastic Kelvin-Voigt fluid. It was indicated in [23, 24] that the parameter λ can be negative and the negativeness of the parameter λ does not contradict the physical meaning of equation (7). We implemented the tanh-coth method to solve equation (6) and obtained new solutions which could not be attained in the past.

The generalised hyperelastic-rod wave equation

$$u_t - u_{xxt} + \alpha u_x + 2\beta uu_x + 3\theta u^2 u_x - \gamma u_x u_{xx} - uu_{xxx} = 0 \tag{8}$$

was first introduced in [25], in which the global existence of dissipative solutions were established, where α, β, θ and γ are constant parameters. This equation includes many important physical models in mathematical physics.

For $\beta = \frac{3}{2}$, $\theta = 0$, $\gamma = 2$, we obtain the Camassa-Holm (CH) equation

$$u_t - u_{xxt} + \alpha u_x + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{9}$$

where u is the fluid velocity in the direction x (or, equivalently, the height of the water's free surface above a flat bottom), α is a constant related to the critical shallow water wave speed. Camassa-Holm equation has been studied in [8, 18] and explicit travelling-wave solutions were sought [19]. Besides, solitary wave solutions for modified forms of this equation were developed by Wazwaz [20].

Taking $\beta = 2$, $\theta = 0$, $\gamma = 3$, equation (5) reduces to the Degasperis-Procesi (DP) equation

$$u_t - u_{xxt} + \alpha u_x + 4uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{10}$$

The recent study has revealed that the CH and DP equations can be used to describe the long-term dynamics of short surface waves [21, 22, 26].

For $\alpha = 1$, $\beta = \frac{1}{2}$, $\theta = 0$, $\gamma = 3$, the equation (5) leads to the Fornberg-Whitham (FW) equation

$$u_t - u_{xxt} + u_x + uu_x - 3u_x u_{xx} - uu_{xxx} = 0. \tag{11}$$

The FW equation was used to study the qualitative behaviour of wave-breaking. A peaked solitary wave solution $u(x,t) = Ae^{-\frac{1}{2}|x-\frac{4}{3}t|}$ of this type of equation was obtained by Fornberg and Whitham [27, 28]. Using the tanh-coth method, we consider equation (8), which is a combined form of CH, DP and FW equations, and obtain new exact solutions. These solutions can be seen as an improvement of the previously known data.

As stated before, pseudoparabolic-type equations arise in many areas of mathematics and physics to describe many physical phenomena. In recent years considerable attention has been paid to the study of pseudoparabolic-type equations, and to construct exact solutions for this type of equations, several methods, for instance, the tanh-coth method, have been developed. In [29, 30], we discussed some well-known Sobolev-type equations and pseudoparabolic equations and obtained new travelling wave solutions by using the tanh-coth method. Motivated by these studies, we employed the tanh-coth method to investigate new travelling wave solutions for the equations that were previously mentioned.

In what follows, we summarise the main features of the tanh-coth method as introduced in [31, 32], where more details and examples can be found.

2 Outline of the tanh-coth method

(i) First consider a general form of the nonlinear equation

$$P(u, u_t, u_x, u_{xx}, \dots) = 0. \tag{12}$$

(ii) To find the travelling wave solution of equation (12), the wave variable $\xi = x - Vt$ is introduced so that

$$u(x,t) = U(\mu\xi). \tag{13}$$

Based on this, one may use the following changes:

$$\begin{aligned}
\frac{\partial}{\partial t} &= -V\frac{d}{d\xi},\\
\frac{\partial}{\partial x} &= \mu\frac{d}{d\xi},\\
\frac{\partial^2}{\partial x^2} &= \mu^2\frac{d^2}{d\xi^2},\\
\frac{\partial^3}{\partial x^3} &= \mu^3\frac{d^3}{d\xi^3}
\end{aligned}\tag{14}$$

and so on for other derivatives. Using (14) changes PDE (12) to an ODE

$$Q\bigl(U,U',U'',\ldots\bigr) = 0.\tag{15}$$

(iii) If all terms of the resulting ODE contain derivatives in ξ, then by integrating this equation, and by considering the constant of integration to be zero, one obtains a simplified ODE.

(iv) A new independent variable

$$Y = \tanh(\mu\xi)\tag{16}$$

is introduced that leads to the change of derivatives:

$$\begin{aligned}
\frac{d}{d\xi} &= \mu\bigl(1-Y^2\bigr)\frac{d}{dY},\\
\frac{d^2}{d\xi^2} &= -2\mu^2 Y\bigl(1-Y^2\bigr)\frac{d}{dY} + \mu^2\bigl(1-Y^2\bigr)^2\frac{d^2}{dY^2},\\
\frac{d^3}{d\xi^3} &= 2\mu^3\bigl(1-Y^2\bigr)\bigl(3Y^2-1\bigr)\frac{d}{dY} - 6\mu^3 Y\bigl(1-Y^2\bigr)^2\frac{d^2}{dY^2} + \mu^3\bigl(1-Y^2\bigr)^3\frac{d^3}{dY^3},
\end{aligned}\tag{17}$$

where other derivatives can be derived in a similar manner.

(v) The *ansatz* of the form

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}\tag{18}$$

is introduced where M is a positive integer, in most cases, that will be determined. If M is not an integer, then a transformation formula is used to overcome this difficulty. Substituting (17) and (18) into ODE (15) yields an equation in powers of Y.

(vi) To determine the parameter M, the linear terms of highest order in the resulting equation with the highest order nonlinear terms are balanced. With M determined, one collects all the coefficients of powers of Y in the resulting equation, where these coefficients have to vanish. This will give a system of algebraic equations involving a_k and b_k ($k = 0,\ldots,M$), V, and μ. Having determined these parameters, knowing that M is a positive integer in most cases and using (18), one obtains an analytic solution in a closed form.

3 The Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation

The Benjamin-Bona-Mahony-Peregrine-Burgers (BBMPB) equation is given by

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x + \beta u_{xxx} = 0, \tag{19}$$

where α is a positive constant, θ and β are nonzero real numbers. Using the wave variable $\xi = x - Vt$ in (19) then integrating this equation and considering the constant of integration to be zero, we obtain

$$(-V + \gamma)U - \alpha U' + \frac{\theta}{2}U^2 + (V + \beta)U'' = 0. \tag{20}$$

Balancing U^2 with U'' in (20) gives $M = 2$. The tanh-coth method admits the use of the finite expansion

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k}, \tag{21}$$

where $Y = \tanh(\mu\xi)$. Substituting (21) into (20) and collecting the coefficients of Y and setting it equal to zero, we find the system of equation:

$$\begin{aligned}
&Y^8\colon a_2^2\theta + 12Va_2\mu^2 + 12a_2\beta\mu^2 = 0,\\
&Y^7\colon 2a_1a_2\theta + 4a_2\alpha\mu + 4Va_1\mu^2 + 4a_1\beta\mu^2 = 0,\\
&Y^6\colon 2a_2\gamma + a_1^2\theta - 2Va_2 + 2a_0a_2\theta + 2a_1\alpha\mu - 16Va_2\mu^2 - 16a_2\beta\mu^2 = 0,\\
&Y^5\colon 2a_1\gamma - 2Va_1 + 2b_1a_2\theta + 2a_0a_1\theta - 4a_2\alpha\mu - 4Va_1\mu^2 - 4a_1\beta\mu^2 = 0,\\
&Y^4\colon 2a_0\gamma + a_0^2\theta - 2Va_0 + 2b_1a_1\theta + 2b_2a_2\theta - 2b_1\alpha\mu - 2a_1\alpha\mu + 4Vb_2\mu^2,\\
&\qquad + 4Va_2\mu^2 + 4b_2\beta\mu^2 + 4a_2\beta\mu^2 = 0,\\
&Y^3\colon 2b_1\gamma - 2Vb_1 + 2b_1a_0\theta + 2b_2a_1\theta - 4b_2\alpha\mu - 4Vb_1\mu^2 - 4b_1\beta\mu^2 = 0,\\
&Y^2\colon 2b_2\gamma + b_1^2\theta - 2Vb_2 + 2b_2a_0\theta + 2b_1\alpha\mu - 16Vb_2\mu^2 - 16b_2\beta\mu^2 = 0,\\
&Y^1\colon 2b_1b_2\theta + 4b_2\alpha\mu + 4Vb_1\mu^2 + 4b_1\beta\mu^2 = 0,\\
&Y^0\colon b_2^2\theta + 12Vb_2\mu^2 + 12b_2\beta\mu^2 = 0.
\end{aligned} \tag{22}$$

Using Maple gives eighteen sets of solutions:

$$\begin{aligned}
&a_0 = \frac{-\gamma - \beta}{\theta}, \quad a_1 = \frac{\gamma + \beta}{\theta}, \quad a_2 = b_1 = b_2 = 0, \quad V = -\beta, \quad \mu = \frac{-\gamma - \beta}{2\alpha},\\
&a_0 = a_1 = \frac{-\gamma - \beta}{\theta}, \quad a_2 = b_1 = b_2 = 0, \quad V = -\beta, \quad \mu = \frac{\gamma + \beta}{2\alpha},\\
&a_0 = \frac{-\gamma - \beta}{\theta}, \quad a_1 = b_1 = \frac{\gamma + \beta}{2\theta}, \quad a_2 = b_2 = 0, \quad V = -\beta, \quad \mu = \frac{-\gamma - \beta}{4\alpha},\\
&a_0 = \frac{-\gamma - \beta}{\theta}, \quad a_1 = a_2 = b_2 = 0, \quad b_1 = \frac{-\gamma - \beta}{\theta}, \quad V = -\beta, \quad \mu = \frac{\gamma + \beta}{2\alpha},\\
&a_0 = \frac{-\gamma - \beta}{\theta}, \quad a_1 = \frac{-\gamma - \beta}{2\theta}, \quad a_2 = b_2 = 0, \quad b_1 = -\frac{\gamma + \beta}{2\theta},
\end{aligned}$$

$$
\begin{aligned}
&V=-\beta, \qquad \mu=\frac{\gamma+\beta}{4\alpha},\\
&a_0=\frac{-\gamma-\beta}{\theta}, \qquad a_1=a_2=b_2=0, \qquad b_1=\frac{\gamma+\beta}{\theta},\\
&V=-\beta, \qquad \mu=-\frac{\gamma+\beta}{2\alpha},\\
&a_0=\frac{-10\gamma\mu-10\beta\mu+\alpha}{20\mu\theta}, \qquad a_1=-\frac{12\alpha\mu}{5\theta}, \qquad a_2=-\frac{6\alpha\mu}{5\theta}, \qquad b_1=b_2=0,\\
&V=\frac{\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{5\gamma+5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2-24\alpha^2}}{24\alpha},\\
&a_0=\frac{3(-10\gamma\mu-10\beta\mu+\alpha)}{20\mu\theta}, \qquad a_1=-\frac{12\alpha\mu}{5\theta}, \qquad a_2=-\frac{6\alpha\mu}{5\theta}, \qquad b_1=b_2=0,\\
&V=\frac{\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{-5\gamma-5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2+24\alpha^2}}{24\alpha},\\
&a_0=\frac{-10\gamma\mu-10\beta\mu-\alpha}{20\mu\theta}, \qquad a_1=-\frac{12\alpha\mu}{5\theta}, \qquad a_2=\frac{6\alpha\mu}{5\theta}, \qquad b_1=b_2=0,\\
&V=\frac{-\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{-5\gamma-5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2-24\alpha^2}}{24\alpha},\\
&a_0=\frac{3(-10\gamma\mu-10\beta\mu-\alpha)}{20\mu\theta}, \qquad a_1=-\frac{12\alpha\mu}{5\theta}, \qquad a_2=\frac{6\alpha\mu}{5\theta}, \qquad b_1=b_2=0,\\
&V=\frac{-\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{5\gamma+5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2+24\alpha^2}}{24\alpha},\\
&a_0=\frac{-10\gamma\mu-10\beta\mu+\alpha}{20\mu\theta}, \qquad a_1=a_2=0, \qquad b_1=-\frac{12\alpha\mu}{5\theta}, \qquad b_2=-\frac{6\alpha\mu}{5\theta},\\
&V=\frac{\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{5\gamma+5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2-24\alpha^2}}{24\alpha},\\
&a_0=\frac{3(-10\gamma\mu-10\beta\mu+\alpha)}{20\mu\theta}, \qquad a_1=a_2=0, \qquad b_1=-\frac{12\alpha\mu}{5\theta}, \qquad b_2=-\frac{6\alpha\mu}{5\theta},\\
&V=\frac{\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{-5\gamma-5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2+24\alpha^2}}{24\alpha},\\
&a_0=\frac{3(-10\gamma\mu-10\beta\mu-\alpha)}{20\mu\theta}, \qquad a_1=a_2=0, \qquad b_1=-\frac{12\alpha\mu}{5\theta}, \qquad b_2=\frac{6\alpha\mu}{5\theta},\\
&V=\frac{-\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{5\gamma+5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2+24\alpha^2}}{24\alpha},\\
&a_0=\frac{-10\gamma\mu-10\beta\mu-\alpha}{20\mu\theta}, \qquad a_1=a_2=0, \qquad b_1=-\frac{12\alpha\mu}{5\theta}, \qquad b_2=\frac{6\alpha\mu}{5\theta},\\
&V=\frac{-\frac{\alpha}{10}-\beta\mu}{\mu}, \qquad \mu=\frac{-5\gamma-5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2-24\alpha^2}}{24\alpha},\\
&a_0=\frac{3(-20\gamma\mu-20\beta\mu-\alpha)}{80\mu\theta}, \qquad a_1=b_1=-\frac{12\alpha\mu}{5\theta}, \qquad a_2=b_2=\frac{3\alpha\mu}{5\theta},\\
&V=\frac{-\frac{\alpha}{20}-\beta\mu}{\mu}, \qquad \mu=\frac{-5\gamma-5\beta\pm\sqrt{25\gamma^2+50\gamma\beta+25\beta^2-24\alpha^2}}{48\alpha},
\end{aligned}
\tag{23}
$$

$$a_0 = \frac{4(-5\gamma\mu - 5\beta\mu - \alpha)}{16\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = \frac{3\alpha\mu}{5\theta},$$
$$V = \frac{-\frac{\alpha}{20} - \beta\mu}{\mu}, \qquad \mu = \frac{5\gamma + 5\beta \pm \sqrt{25\gamma^2 + 50\gamma\beta + 25\beta^2 + 24\alpha^2}}{48\alpha},$$
$$a_0 = \frac{3(-20\gamma\mu - 20\beta\mu + \alpha)}{80\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = -\frac{3\alpha\mu}{5\theta},$$
$$V = \frac{\frac{\alpha}{20} - \beta\mu}{\mu}, \qquad \mu = \frac{5\gamma + 5\beta \pm \sqrt{25\gamma^2 + 50\gamma\beta + 25\beta^2 - 24\alpha^2}}{48\alpha},$$
$$a_0 = \frac{-20\gamma\mu - 20\beta\mu + \alpha}{16\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = -\frac{3\alpha\mu}{5\theta},$$
$$V = \frac{\frac{\alpha}{20} - \beta\mu}{\mu}, \qquad \mu = \frac{-5\gamma - 5\beta \pm \sqrt{25\gamma^2 + 50\gamma\beta + 25\beta^2 + 24\alpha^2}}{48\alpha}.$$

These sets give the solutions respectively:

$$u_1(x,t) = \frac{-\gamma - \beta}{\theta} + \frac{\gamma + \beta}{\theta} \tanh \mu(x - Vt),$$
$$u_2(x,t) = \frac{-\gamma - \beta}{\theta} - \frac{\gamma + \beta}{\theta} \tanh \mu(x - Vt),$$
$$u_3(x,t) = \frac{-\gamma - \beta}{\theta} + \frac{\gamma + \beta}{2\theta} \tanh \mu(x - Vt) + \frac{\gamma + \beta}{2\theta} \coth \mu(x - Vt),$$
$$u_4(x,t) = \frac{-\gamma - \beta}{\theta} - \frac{\gamma + \beta}{\theta} \coth \mu(x - Vt),$$
$$u_5(x,t) = \frac{-\gamma - \beta}{\theta} - \frac{\gamma + \beta}{\theta} \tanh \mu(x - Vt) - \frac{\gamma + \beta}{2\theta} \coth \mu(x - Vt),$$
$$u_6(x,t) = \frac{-\gamma - \beta}{\theta} + \frac{\gamma + \beta}{\theta} \coth \mu(x - Vt),$$
$$u_7(x,t) = \frac{-10\gamma\mu - 10\beta\mu + \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \tanh \mu(x - Vt) - \frac{6\alpha\mu}{5\theta} \tanh^2 \mu(x - Vt),$$
$$u_8(x,t) = \frac{3(-10\gamma\mu - 10\beta\mu + \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \tanh \mu(x - Vt) - \frac{6\alpha\mu}{5\theta} \tanh^2 \mu(x - Vt),$$
$$u_9(x,t) = \frac{-10\gamma\mu - 10\beta\mu - \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \tanh \mu(x - Vt) + \frac{6\alpha\mu}{5\theta} \tanh^2 \mu(x - Vt)$$
$$- \frac{12\alpha\mu}{5} \coth \mu(x - Vt) - \frac{3\alpha\mu}{5} \coth^2 \mu(x - Vt), \tag{24}$$
$$u_{10}(x,t) = \frac{3(-10\gamma\mu - 10\beta\mu - \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \tanh \mu(x - Vt) + \frac{6\alpha\mu}{5\theta} \tanh^2 \mu(x - Vt),$$
$$u_{11}(x,t) = \frac{-10\gamma\mu - 10\beta\mu + \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \coth \mu(x - Vt) - \frac{6\alpha\mu}{5\theta} \coth^2 \mu(x - Vt),$$
$$u_{12}(x,t) = \frac{3(-10\gamma\mu - 10\beta\mu + \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \coth \mu(x - Vt) - \frac{6\alpha\mu}{5\theta} \coth^2 \mu(x - Vt),$$
$$u_{13}(x,t) = \frac{3(-10\gamma\mu - 10\beta\mu - \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \coth \mu(x - Vt) + \frac{6\alpha\mu}{5\theta} \coth^2 \mu(x - Vt),$$
$$u_{14}(x,t) = \frac{-10\gamma\mu - 10\beta\mu - \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta} \coth \mu(x - Vt) + \frac{6\alpha\mu}{5\theta} \coth^2 \mu(x - Vt),$$

$$u_{15}(x,t) = \frac{3(-20\gamma\mu - 20\beta\mu - \alpha)}{80\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x-Vt) + \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x-Vt) - \frac{12\alpha\mu}{5\theta}\coth\mu(x-Vt) + \frac{3\alpha\mu}{5\theta}\coth^2\mu(x-Vt),$$

$$u_{16}(x,t) = \frac{4(-5\gamma\mu - 5\beta\mu - \alpha)}{16\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x-Vt) + \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x-Vt) - \frac{12\alpha\mu}{5\theta}\coth\mu(x-Vt) + \frac{3\alpha\mu}{5\theta}\coth^2\mu(x-Vt),$$

$$u_{17}(x,t) = \frac{3(-20\gamma\mu - 20\beta\mu + \alpha)}{80\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x-Vt) - \frac{12\alpha\mu}{5\theta}\coth\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\coth^2\mu(x-Vt),$$

$$u_{18}(x,t) = \frac{-20\gamma\mu - 20\beta\mu + \alpha}{16\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x-Vt) - \frac{12\alpha\mu}{5\theta}\coth\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\coth^2\mu(x-Vt).$$

4 The Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation

We consider the Oskolkov-Benjamin-Bona-Mahony-Burgers (OBBMB) equation

$$u_t - u_{xxt} - \alpha u_{xx} + \gamma u_x + \theta u u_x = 0, \tag{25}$$

where α is positive and θ is a nonzero constant. Using the wave variable $\xi = x - Vt$ in (25) then integrating this equation and considering the constant of integration to be zero, we obtain

$$(-V+\gamma)U + \frac{\theta}{2}U^2 - \alpha U' + VU'' = 0. \tag{26}$$

Balancing the second term with the last term in (26) gives $M = 2$. Using the finite expansion

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k}, \tag{27}$$

where $Y = \tanh(\mu\xi)$. Substituting (27) into (26) and collecting the coefficients of Y and setting it equal to zero, we find the system of equations:

$$\begin{aligned}
&Y^8: a_2^2\theta + 12Va_2\mu^2 = 0,\\
&Y^7: 2a_1a_2\theta + 4a_2\alpha\mu + 4Va_1\mu^2 = 0,\\
&Y^6: 2a_2\gamma + a_1^2\theta - 2Va_2 + 2a_0a_2\theta + 2a_1\alpha\mu - 16Va_2\mu^2 = 0,\\
&Y^5: 2a_1\gamma - 2Va_1 + 2b_1a_2\theta + 2a_0a_1\theta - 4a_2\alpha\mu - 4Va_1\mu^2 = 0,\\
&Y^4: 2a_0\gamma + a_0^2\theta - 2Va_0 + 2b_1a_1\theta + 2b_2a_2\theta - 2b_1\alpha\mu - 2a_1\alpha\mu\\
&\qquad + 4Vb_2\mu^2 + 4Va_2\mu^2 = 0, \qquad (28)\\
&Y^3: 2b_1\gamma - 2Vb_1 + 2b_1a_0\theta + 2b_2a_1\theta - 4b_2\alpha\mu - 4Vb_1\mu^2 = 0,
\end{aligned}$$

$$Y^2\colon 2b_2\gamma + b_1^2\theta - 2Vb_2 + 2b_2a_0\theta + 2b_1\alpha\mu - 16Vb_2\mu^2 = 0,$$
$$Y^1\colon 2b_1b_2\theta + 4b_2\alpha\mu + 4Vb_1\mu^2 = 0,$$
$$Y^0\colon b_2^2\theta + 12Vb_2\mu^2 = 0.$$

Maple gives twelve sets of solutions:

$$\begin{aligned}
&a_0 = \frac{-10\gamma\mu + \alpha}{20\mu\theta}, \qquad a_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = -\frac{6\alpha\mu}{5\theta}, \qquad b_1 = b_2 = 0,\\
&V = \frac{\alpha}{10\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{3(-10\gamma\mu + \alpha)}{20\mu\theta}, \qquad a_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = -\frac{6\alpha\mu}{5\theta}, \qquad b_1 = b_2 = 0,\\
&V = \frac{\alpha}{10\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{-10\gamma\mu - \alpha}{20\mu\theta}, \qquad a_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = \frac{6\alpha\mu}{5\theta}, \qquad b_1 = b_2 = 0,\\
&V = -\frac{\alpha}{10\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{3(-10\gamma\mu - \alpha)}{20\mu\theta}, \qquad a_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = \frac{6\alpha\mu}{5\theta}, \qquad b_1 = b_2 = 0,\\
&V = -\frac{\alpha}{10\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{-10\gamma\mu + \alpha}{20\mu\theta}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad b_2 = -\frac{6\alpha\mu}{5\theta},\\
&V = \frac{\alpha}{10\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{3(-10\gamma\mu + \alpha)}{20\mu\theta}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad b_2 = -\frac{6\alpha\mu}{5\theta},\\
&V = \frac{\alpha}{10\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{3(-10\gamma\mu - \alpha)}{20\mu\theta}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad b_2 = \frac{6\alpha\mu}{5\theta},\\
&V = -\frac{\alpha}{10\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{-10\gamma\mu - \alpha}{20\mu\theta}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad b_2 = \frac{6\alpha\mu}{5\theta},\\
&V = -\frac{\alpha}{10\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{24\alpha},\\
&a_0 = \frac{3(-20\gamma\mu - \alpha)}{80\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = \frac{3\alpha\mu}{5\theta},\\
&V = -\frac{\alpha}{20\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{48\alpha},
\end{aligned} \tag{29}$$

$$
\begin{aligned}
&a_0 = \frac{4(-5\gamma\mu - \alpha)}{16\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = \frac{3\alpha\mu}{5\theta}, \\
&V = -\frac{\alpha}{20\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{48\alpha}, \\
&a_0 = \frac{3(-20\gamma\mu + \alpha)}{80\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = -\frac{3\alpha\mu}{5\theta}, \\
&V = \frac{\alpha}{20\mu}, \qquad \mu = \frac{5\gamma \pm \sqrt{25\gamma^2 - 24\alpha^2}}{48\alpha}, \\
&a_0 = \frac{-20\gamma\mu + \alpha}{16\mu\theta}, \qquad a_1 = b_1 = -\frac{12\alpha\mu}{5\theta}, \qquad a_2 = b_2 = -\frac{3\alpha\mu}{5\theta}, \\
&V = \frac{\alpha}{20\mu}, \qquad \mu = \frac{-5\gamma \pm \sqrt{25\gamma^2 + 24\alpha^2}}{48\alpha}.
\end{aligned}
$$

These give the following solutions:

$$
\begin{aligned}
u_1(x,t) &= \frac{-10\gamma\mu - 10\beta\mu + \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) - \frac{6\alpha\mu}{5\theta}\tanh^2\mu(x - Vt), \\
u_2(x,t) &= \frac{3(-10\gamma\mu - 10\beta\mu + \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) - \frac{6\alpha\mu}{5\theta}\tanh^2\mu(x - Vt), \\
u_3(x,t) &= \frac{-10\gamma\mu - 10\beta\mu - \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) + \frac{6\alpha\mu}{5\theta}\tanh^2\mu(x - Vt) \\
&\quad - \frac{12\alpha\mu}{5}\coth\mu(x - Vt) - \frac{3\alpha\mu}{5}\coth^2\mu(x - Vt), \\
u_4(x,t) &= \frac{3(-10\gamma\mu - 10\beta\mu - \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) + \frac{6\alpha\mu}{5\theta}\tanh^2\mu(x - Vt), \\
u_5(x,t) &= \frac{-10\gamma\mu - 10\beta\mu + \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) - \frac{6\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \\
u_6(x,t) &= \frac{3(-10\gamma\mu - 10\beta\mu + \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) - \frac{6\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \\
u_7(x,t) &= \frac{3(-10\gamma\mu - 10\beta\mu - \alpha)}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) + \frac{6\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \\
u_8(x,t) &= \frac{-10\gamma\mu - 10\beta\mu - \alpha}{20\mu\theta} - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) + \frac{6\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \qquad (30) \\
u_9(x,t) &= \frac{3(-20\gamma\mu - 20\beta\mu - \alpha)}{80\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) + \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x - Vt) \\
&\quad - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) + \frac{3\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \\
u_{10}(x,t) &= \frac{4(-5\gamma\mu - 5\beta\mu - \alpha)}{16\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) + \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x - Vt) \\
&\quad - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) + \frac{3\alpha\mu}{5\theta}\coth^2\mu(x - Vt), \\
u_{11}(x,t) &= \frac{3(-20\gamma\mu - 20\beta\mu + \alpha)}{80\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x - Vt) - \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x - Vt) \\
&\quad - \frac{12\alpha\mu}{5\theta}\coth\mu(x - Vt) - \frac{3\alpha\mu}{5\theta}\coth^2\mu(x - Vt),
\end{aligned}
$$

$$u_{12}(x,t) = \frac{-20\gamma\mu - 20\beta\mu + \alpha}{16\mu\theta} - \frac{12\alpha\mu}{5\theta}\tanh\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\tanh^2\mu(x-Vt) - \frac{12\alpha\mu}{5\theta}\coth\mu(x-Vt) - \frac{3\alpha\mu}{5\theta}\coth^2\mu(x-Vt).$$

As can be seen easily, these solutions can be obtained by taking $\beta = 0$ in the solutions of previous equations.

5 The one-dimensional Oskolkov equation

The one-dimensional Oskolkov equation is given by

$$u_t - \lambda u_{xxt} - \alpha u_{xx} + uu_x = 0. \tag{31}$$

We will investigate the equation for $\lambda \neq 0$ and $\alpha \in \mathbb{R}$. Using the wave variable $\xi = x - Vt$ in (31) then integrating this equation and considering the constant of integration to be zero, we obtain

$$-VU + \lambda U'' - \alpha U' + \frac{1}{2}U^2 = 0. \tag{32}$$

Balancing U^2 with U'' in (32) gives $M = 2$. The tanh-coth method admits the use of the finite expansion

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k}, \tag{33}$$

where $Y = \tanh(\mu\xi)$. Substituting (33) into (32) and collecting the coefficients of Y and setting it equal to zero, we find the system of equations:

$$\begin{aligned}
&Y^8: a_2^2 + 12\lambda a_2\mu^2 = 0,\\
&Y^7: 4a_1\lambda\mu^2 + 4a_2\alpha\mu + 2a_1a_2 = 0,\\
&Y^6: a_1^2 + 2\alpha a_1\mu - 16a_2\lambda\mu^2 - 2Va_2 + 2a_0a_2 = 0,\\
&Y^5: 2b_1a_2 - 4a_2\alpha\mu - 2Va_1 - 4a_1\lambda\mu^2 + 2a_0a_1 = 0,\\
&Y^4: 2b_1a_1 - 2Va_0 + 2b_2a_2 + a_0^2 - 2b_1\alpha\mu - 2a_1\alpha\mu + 4b_2\lambda\mu^2 + 4a_2\lambda\mu^2 = 0,\\
&Y^3: 2b_1a_0 - 4b_2\alpha\mu - 2Vb_1 - 4b_1\lambda\mu^2 + 2b_2a_1 = 0,\\
&Y^2: b_1^2 + 2\alpha b_1\mu - 16b_2\lambda\mu^2 - 2Vb_2 + 2b_2a_0 = 0,\\
&Y^1: 4b_1\lambda\mu^2 + 4b_2\alpha\mu + 2b_1b_2 = 0,\\
&Y^0: b_2^2 + 12\lambda b_2\mu^2 = 0.
\end{aligned} \tag{34}$$

Solving this system, we find the following sets of solutions:

$$a_0 = \frac{9\alpha^2}{25\lambda}, \qquad a_1 = -\frac{6\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{25\lambda}, \qquad b_1 = b_2 = 0,$$
$$V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{10\lambda},$$

$$\begin{aligned}
&a_0 = -\frac{3\alpha^2}{25\lambda}, \qquad a_1 = -\frac{6\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{25\lambda}, \qquad b_1 = b_2 = 0, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{10\lambda}, \\
&a_0 = \frac{9\alpha^2}{25\lambda}, \qquad a_1 = \frac{6\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{25\lambda}, \qquad b_1 = b_2 = 0, \\
&V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{10\lambda}, \\
&a_0 = -\frac{3\alpha^2}{25\lambda}, \qquad a_1 = \frac{6\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{25\lambda}, \qquad b_1 = b_2 = 0, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{10\lambda}, \\
&a_0 = \frac{9\alpha^2}{25\lambda}, \qquad a_1 = a_2 = 0, \qquad b_1 = \frac{6\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{25\lambda}, \\
&V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{10\lambda}, \\
&a_0 = -\frac{3\alpha^2}{25\lambda}, \qquad a_1 = a_2 = 0, \qquad b_1 = \frac{6\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{25\lambda}, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{10\lambda}, \qquad\qquad (35) \\
&a_0 = \frac{3\alpha^2}{10\lambda}, \qquad a_1 = -\frac{3\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{100\lambda}, \qquad b_1 = -\frac{3\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{100\lambda}, \\
&V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{20\lambda}, \\
&a_0 = -\frac{9\alpha^2}{50\lambda}, \qquad a_1 = -\frac{3\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{100\lambda}, \qquad b_1 = -\frac{3\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{100\lambda}, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{20\lambda}, \\
&a_0 = \frac{3\alpha^2}{10\lambda}, \qquad a_1 = \frac{3\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{100\lambda}, \qquad b_1 = \frac{3\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{100\lambda}, \\
&V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{20\lambda}, \\
&a_0 = -\frac{9\alpha^2}{50\lambda}, \qquad a_1 = \frac{3\alpha^2}{25\lambda}, \qquad a_2 = -\frac{3\alpha^2}{100\lambda}, \qquad b_1 = \frac{3\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{100\lambda}, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = -\frac{\alpha}{20\lambda}, \\
&a_0 = \frac{9\alpha^2}{25\lambda}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{6\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{25\lambda}, \\
&V = \frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{10\lambda}, \\
&a_0 = -\frac{3\alpha^2}{25\lambda}, \qquad a_1 = a_2 = 0, \qquad b_1 = -\frac{6\alpha^2}{25\lambda}, \qquad b_2 = -\frac{3\alpha^2}{25\lambda}, \\
&V = -\frac{6\alpha^2}{25\lambda}, \qquad \mu = \frac{\alpha}{10\lambda}.
\end{aligned}$$

These sets give the solutions respectively:

$$\begin{aligned}
u_1(x,t) &= \frac{9\alpha^2}{25\lambda} - \frac{6\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\tanh^2\mu(x-Vt),\\
u_2(x,t) &= -\frac{3\alpha^2}{25\lambda} - \frac{6\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\tanh^2\mu(x-Vt),\\
u_3(x,t) &= \frac{9\alpha^2}{25\lambda} + \frac{6\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\tanh^2\mu(x-Vt),\\
u_4(x,t) &= -\frac{3\alpha^2}{25\lambda} + \frac{6\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\tanh^2\mu(x-Vt),\\
u_5(x,t) &= \frac{9\alpha^2}{25\lambda} + \frac{6\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\coth^2\mu(x-Vt),\\
u_6(x,t) &= -\frac{3\alpha^2}{25\lambda} + \frac{6\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\coth^2\mu(x-Vt),\\
u_7(x,t) &= \frac{3\alpha^2}{10\lambda} - \frac{3\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\tanh^2\mu(x-Vt)\\
&\quad - \frac{3\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\coth^2\mu(x-Vt),\\
u_8(x,t) &= -\frac{9\alpha^2}{50\lambda} - \frac{3\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\tanh^2\mu(x-Vt)\\
&\quad - \frac{3\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\coth^2\mu(x-Vt),\\
u_9(x,t) &= \frac{3\alpha^2}{10\lambda} + \frac{3\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\tanh^2\mu(x-Vt)\\
&\quad + \frac{3\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\coth^2\mu(x-Vt),\\
u_{10}(x,t) &= -\frac{9\alpha^2}{50\lambda} + \frac{3\alpha^2}{25\lambda}\tanh\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\tanh^2\mu(x-Vt)\\
&\quad + \frac{3\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{100\lambda}\coth^2\mu(x-Vt),\\
u_{11}(x,t) &= \frac{9\alpha^2}{25\lambda} - \frac{6\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\coth^2\mu(x-Vt),\\
u_{12}(x,t) &= -\frac{3\alpha^2}{25\lambda} - \frac{6\alpha^2}{25\lambda}\coth\mu(x-Vt) - \frac{3\alpha^2}{25\lambda}\coth^2\mu(x-Vt).
\end{aligned} \tag{36}$$

6 The generalised hyperelastic-rod wave equation

The generalised hyperelastic-rod wave equation reads as follows:

$$u_t - u_{xxt} + \alpha u_x + 2\beta u u_x + 3\theta u^2 u_x - \gamma u_x u_{xx} - u u_{xxx} = 0, \tag{37}$$

where α, β, θ and γ are constant parameters, and we assume that θ is nonzero. The wave variable $\xi = x - Vt$ carries (37) into the ODE

$$-VU' + VU''' + \alpha U' + 2\beta UU' + 3\theta U^2 U' - \gamma U'U'' - UU''' = 0. \tag{38}$$

Integrating this equation and considering the constant of integration to be zero, we obtain

$$(-V+\alpha)U + VU'' + \beta U^2 + \theta U^3 - \frac{\gamma-1}{2}\left(U'\right)^2 - UU'' = 0. \tag{39}$$

Balancing U^3 with UU'' in (39) gives $M = 2$. As stated before, the tanh-coth method uses a finite series

$$U(\mu\xi) = S(Y) = \sum_{k=0}^{2} a_k Y^k + \sum_{k=1}^{2} b_k Y^{-k} \tag{40}$$

to express the solution $u(x,t)$. Substituting (40) into (39) and collecting the coefficients of Y gives the system of algebraic equations:

$$\begin{aligned}
&Y^{12}\colon a_2^3\theta - 4a_2^2\mu^2 - 2a_2^2\gamma\mu^2 = 0,\\
&Y^{11}\colon 3a_1a_2^2\theta - 6a_1a_2\mu^2 - 2a_1a_2\gamma\mu^2 = 0,\\
&Y^{10}\colon 8a_2^2\mu^2 - 3a_1^2\mu^2 + 2a_2^2\beta - a_1^2\gamma\mu^2 + 8a_2^2\gamma\mu^2 + 12Va_2\mu^2 + 6a_0a_2^2\theta\\
&\qquad + 6a_1^2a_2\theta - 12a_0a_2\mu^2 = 0,\\
&Y^{9}\colon a_1^3\theta + 2a_1a_2\beta + 2Va_1\mu^2 + 3b_1a_2^2\theta - 8b_1a_2\mu^2 - 2a_0a_1\mu^2 + 6a_1a_2\mu^2\\
&\qquad + 6a_0a_1a_2\theta + 2b_1a_2\gamma\mu^2 + 4a_1a_2\gamma\mu^2 = 0,\\
&Y^{8}\colon 2a_1^2\mu^2 + 2a_2\alpha + 2a_1^2\beta - 2Va_2 + 4a_0a_2\beta + 2a_1^2\gamma\mu^2 - 4a_2^2\gamma\mu^2 - 16Va_2\mu^2\\
&\qquad - 6b_1a_1\mu^2 + 6b_2a_2^2\theta - 24b_2a_2\mu^2 + 6a_0a_1^2\theta + 6a_0^2a_2\theta\\
&\qquad + 16a_0a_2\mu^2 + 12b_1a_1a_2\theta + 2b_1a_1\gamma\mu^2 + 8b_2a_2\gamma\mu^2 = 0,\\
&Y^{7}\colon a_1\alpha - Va_1 + 2b_1a_2\beta + 2a_0a_1\beta - 2Va_1\mu^2 + 3b_1a_1^2\theta + 14b_1a_2\mu^2 - 6b_2a_1\mu^2\\
&\qquad + 3a_0^2a_1\theta + 2a_0a_1\mu^2 + 6b_1a_0a_2\theta + 6b_2a_1a_2\theta\\
&\qquad - 4b_1a_2\gamma\mu^2 + 2b_2a_1\gamma\mu^2 - 2a_1a_2\gamma\mu^2 = 0,\\
&Y^{6}\colon b_1^2\mu^2 + a_1^2\mu^2 + 2a_0\alpha + 2a_0^3\theta + 2a_0^2\beta - 2Va_0 + 4b_1a_1\beta + 4b_2a_2\beta - b_1^2\gamma\mu^2\\
&\qquad - a_1^2\gamma\mu^2 + 4Vb_2\mu^2 + 4Va_2\mu^2 + 6b_2a_1^2\theta + 6b_1^2a_2\theta + 12b_1a_1\mu^2 - 4b_2a_0\mu^2\\
&\qquad + 48b_2a_2\mu^2 - 4a_0a_2\mu^2 + 12b_1a_0a_1\theta + 12b_2a_0a_2\theta\\
&\qquad - 4b_1a_1\gamma\mu^2 - 16b_2a_2\gamma\mu^2 = 0, \qquad\qquad (41)\\
&Y^{5}\colon b_1\alpha - Vb_1 + 2b_1a_0\beta + 2b_2a_1\beta - 2Vb_1\mu^2 + 3b_1a_0^2\theta + 3b_1^2a_1\theta\\
&\qquad + 2b_1a_0\mu^2 - 6b_1a_2\mu^2 + 14b_2a_1\mu^2 + 6b_1b_2a_2\theta + 6b_2a_0a_1\theta\\
&\qquad - 2b_1b_2\gamma\mu^2 + 2b_1a_2\gamma\mu^2 - 4b_2a_1\gamma\mu^2 = 0,\\
&Y^{4}\colon 2b_1^2\mu^2 + 2b_2\alpha + 2b_1^2\beta - 2Vb_2 + 4b_2a_0\beta + 2b_1^2\gamma\mu^2 - 4b_2^2\gamma\mu^2 - 16Vb_2\mu^2\\
&\qquad + 6b_1^2a_0\theta + 6b_2a_0^2\theta - 6b_1a_1\mu^2 + 16b_2a_0\mu^2 + 6b_2^2a_2\theta\\
&\qquad - 24b_2a_2\mu^2 + 12b_1b_2a_1\theta + 2b_1a_1\gamma\mu^2 + 8b_2a_2\gamma\mu^2 = 0,\\
&Y^{3}\colon b_1^3\theta + 2b_1b_2\beta + 2Vb_1\mu^2 + 6b_1b_2\mu^2 - 2b_1a_0\mu^2 + 3b_2^2a_1\theta - 8b_2a_1\mu^2
\end{aligned}$$

$$+ 6b_1b_2a_0\theta + 4b_1b_2\gamma\mu^2 + 2b_2a_1\gamma\mu^2 = 0,$$

$$Y^2\colon 8b_2^2\mu^2 - 3b_1^2\mu^2 + 2b_2^2\beta - b_1^2\gamma\mu^2 + 8b_2^2\gamma\mu^2 + 12Vb_2\mu^2$$

$$+ 6b_1^2b_2\theta + 6b_2^2a_0\theta - 12b_2a_0\mu^2 = 0,$$

$$Y^1\colon 3b_1b_2^2\theta - 6b_1b_2\mu^2 - 2b_1b_2\gamma\mu^2 = 0,$$

$$Y^0\colon b_2^3\theta - 4b_2^2\mu^2 - 2b_2^2\gamma\mu^2 = 0.$$

The last system gives the three sets of solutions as follows.
The first:

$$\begin{aligned}
a_0 = {}& \bigl(-8\gamma\mu^2 + 12\beta - 8\mu^2 - 40\beta\mu^2 + 50\gamma^3\mu^2 + 18\theta\alpha - 80\mu^2\beta^2 - 16\beta\gamma^2\mu^2 \\
& - 88\mu^2\gamma\beta^2 + 24\beta^2 + 32\gamma\beta^2 + 28\gamma\beta + 26\gamma^2\mu^2 + 14\gamma^2\beta^2 + 33\theta\alpha\gamma + 36\beta\theta\alpha \\
& + 18\theta\alpha\gamma^2 - 24\beta^2\gamma^2\mu^2 + 20\beta\gamma^3\mu^2 - 68\mu^2\gamma\beta + 23\beta\gamma^2 + 24\mu^2\theta\alpha + 12\mu^2\theta\alpha\gamma \\
& + 30\beta\theta\alpha\gamma - 24\mu^2\theta\alpha\gamma^2 + 6\beta\theta\alpha\gamma^2 - 12\gamma^3\mu^2\theta\alpha + 8\beta\gamma^4\mu^2 + 3\theta\alpha\gamma^3 + 30\gamma^4\mu^2 \\
& + 2\beta^2\gamma^3 + \beta\gamma^4 + 6\gamma^5\mu^2 + 8\beta\gamma^3\bigr) \\
& /\bigl\{2\theta\bigl(-12\gamma^2 - 8\gamma - 2\beta^2 - 3\theta\alpha - 7\beta - 8\gamma^3 - 2\gamma^4 + 24\beta\gamma^2\mu^2 + 4\beta\gamma^3\mu^2 \\
& + 44\mu^2\gamma\beta + 24\beta\mu^2 - 2 + 34\gamma^2\mu^2 + 14\gamma^3\mu^2 + 2\gamma^4\mu^2 + 34\gamma\mu^2 + 12\mu^2 - 4\gamma\beta^2 \\
& - 2\gamma^2\beta^2 + 3\theta\alpha\gamma^2 - 16\gamma\beta - 13\beta\gamma^2 - 4\beta\gamma^3\bigr)\bigr\},
\end{aligned}$$

$$a_1 = b_1 = b_2 = 0, \qquad a_2 = \frac{2\mu^2(2+\gamma)}{\theta},$$

$$\begin{aligned}
V = {}& -\bigl(-8\gamma\mu^2 + 4\beta - 8\mu^2 + 8\beta\mu^2 + 50\gamma^3\mu^2 + 18\theta\alpha + 16\mu^2\beta^2 + 76\beta\gamma^2\mu^2 \\
& + 40\mu^2\gamma\beta^2 - 4\beta^2 - 8\beta^3 - 18\gamma\beta^2 + 26\gamma^2\mu^2 - 20\gamma^2\beta^2 - 12\gamma\beta^3 + 33\theta\alpha\gamma \\
& + 24\beta\theta\alpha + 18\theta\alpha\gamma^2 + 32\beta^2\gamma^2\mu^2 + 52\beta\gamma^3\mu^2 + 44\mu^2\gamma\beta - 13\beta\gamma^2 + 24\mu^2\theta\alpha \\
& + 12\mu^2\theta\alpha\gamma + 36\beta\theta\alpha\gamma - 24\mu^2\theta\alpha\gamma^2 + 12\beta\theta\alpha\gamma^2 - 12\gamma^3\mu^2\theta\alpha + 12\beta\mu^2\gamma^4 \\
& + 3\theta\alpha\gamma^3 + 8\gamma^3\mu^2\beta^2 + 30\gamma^4\mu^2 - 6\beta^2\mu^3 - 3\beta\gamma^4 + 6\gamma^5\mu^2 - 12\beta\gamma^3 - 4\gamma^2\beta^3\bigr) \\
& /\bigl\{6\theta\bigl(-2\gamma^3 + 2\gamma^3\mu^2 - 6\gamma^2 + 12\gamma^2\mu^2 - 4\beta\gamma^2 + 4\beta\gamma^2\mu^2 - 6\gamma + 3\theta\alpha\gamma - 9\gamma\beta \\
& - 2\gamma\beta^2 + 22\gamma\mu^2 + 20\mu^2\mu\beta - 7\beta - 2\beta^2 - 3\theta\alpha + 12\mu^2 + 24\beta\mu^2 - 2\bigr)\bigr\},
\end{aligned}$$

$$\begin{aligned}
\mu_1 = {}& \frac{\pm 1}{4(\gamma^2+3\gamma+2)}\bigl(-\bigl(2\gamma^2 + 6\gamma + 4\bigr)\bigl(-2 - \gamma^2 - 4\beta - 3\gamma - 2\gamma\beta \\
& + \bigl(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
& - 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\bigr)^{\frac{1}{2}}\bigr)\bigr)^{\frac{1}{2}},
\end{aligned}$$

$$\begin{aligned}
\mu_2 = {}& \frac{\pm 1}{4(\gamma^2+3\gamma+2)}\bigl(\bigl(2\gamma^2 + 6\gamma + 4\bigr)\bigl(2 + \gamma^2 + 4\beta + 3\gamma + 2\gamma\beta \\
& + \bigl(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
& - 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\bigr)^{\frac{1}{2}}\bigr)\bigr)^{\frac{1}{2}}.
\end{aligned}$$

The second:

$$\begin{aligned}
a_0 = {} & \big(-8\gamma\mu^2 + 12\beta - 8\mu^2 - 40\beta\mu^2 + 50\gamma^3\mu^2 + 18\theta\alpha - 80\mu^2\beta^2 - 16\beta\gamma^2\mu^2 \\
& - 88\mu^2\gamma\beta^2 + 24\beta^2 + 32\gamma\beta^2 + 28\gamma\beta + 26\gamma^2\mu^2 + 14\gamma^2\beta^2 + 33\theta\alpha\gamma + 36\beta\theta\alpha \\
& + 18\theta\alpha\gamma^2 - 24\beta^2\gamma^2\mu^2 + 20\beta\gamma^3\mu^2 - 68\mu^2\gamma\beta + 23\beta\gamma^2 + 24\mu^2\theta\alpha + 12\mu^2\theta\alpha\gamma \\
& + 30\beta\theta\alpha\gamma - 24\mu^2\theta\alpha\gamma^2 + 6\beta\theta\alpha\gamma^2 - 12\gamma^3\mu^2\theta\alpha + 8\beta\gamma^4\mu^2 + 3\theta\alpha\gamma^3 + 30\gamma^4\mu^2 \\
& + 2\beta^2\gamma^3 + \beta\gamma^4 + 6\gamma^5\mu^2 + 8\beta\gamma^3\big) \\
& /\big\{2\theta\big(-12\gamma^2 - 8\gamma - 2\beta^2 - 3\theta\alpha - 7\beta - 8\gamma^3 - 2\gamma^4 + 24\beta\gamma^2\mu^2 + 4\beta\gamma^3\mu^2 \\
& + 44\mu^2\gamma\beta + 24\beta\mu^2 - 2 + 34\gamma^2\mu^2 + 14\gamma^3\mu^2 + 2\gamma^4\mu^2 + 34\gamma\mu^2 + 12\mu^2 \\
& - 4\gamma\beta^2 - 2\gamma^2\beta^2 + 3\theta\alpha\gamma^2 - 16\gamma\beta - 13\beta\gamma^2 - 4\beta\gamma^3\big)\big\}, \\
a_1 = {} & a_2 = b_1 = 0, \qquad b_2 = \frac{2\mu^2(2+\gamma)}{\theta}, \\
V = {} & -\big(-8\gamma\mu^2 + 4\beta - 8\mu^2 + 8\beta\mu^2 + 50\gamma^3\mu^2 + 18\theta\alpha + 16\mu^2\beta^2 + 76\beta\gamma^2\mu^2 + 40\mu^2\gamma\beta^2 \\
& - 4\beta^2 - 8\beta^3 - 18\gamma\beta^2 + 26\gamma^2\mu^2 - 20\gamma^2\beta^2 - 12\gamma\beta^3 + 33\theta\alpha\gamma + 24\beta\theta\alpha + 18\theta\alpha\gamma^2 \\
& + 32\beta^2\gamma^2\mu^2 + 52\beta\gamma^3\mu^2 + 44\mu^2\gamma\beta - 13\beta\gamma^2 + 24\mu^2\theta\alpha + 12\mu^2\theta\alpha\gamma + 36\beta\theta\alpha\gamma \\
& - 24\mu^2\theta\alpha\gamma^2 + 12\beta\theta\alpha\gamma^2 - 12\gamma^3\mu^2\theta\alpha + 12\beta\mu^2\gamma^4 + 3\theta\alpha\gamma^3 + 8\gamma^3\mu^2\beta^2 \\
& + 30\gamma^4\mu^2 - 6\beta^2\mu^3 - 3\beta\gamma^4 + 6\gamma^5\mu^2 - 12\beta\gamma^3 - 4\gamma^2\beta^3\big) \\
& /\big\{6\theta\big(-2\gamma^3 + 2\gamma^3\mu^2 - 6\gamma^2 + 12\gamma^2\mu^2 - 4\beta\gamma^2 + 4\beta\gamma^2\mu^2 - 6\gamma + 3\theta\alpha\gamma - 9\gamma\beta \\
& - 2\gamma\beta^2 + 22\gamma\mu^2 + 20\mu^2\mu\beta - 7\beta - 2\beta^2 - 3\theta\alpha + 12\mu^2 + 24\beta\mu^2 - 2\big)\big\}, \\
\mu_1 = {} & \frac{\pm 1}{4(\gamma^2+3\gamma+2)}\big(-\big(2\gamma^2 + 6\gamma + 4\big)\big(-2 - \gamma^2 - 4\beta - 3\gamma - 2\gamma\beta \\
& + \big(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
& - 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\big)^{\frac{1}{2}}\big)\big)^{\frac{1}{2}}, \\
\mu_2 = {} & \frac{\pm 1}{4(\gamma^2+3\gamma+2)}\big(\big(2\gamma^2 + 6\gamma + 4\big)\big(2 + \gamma^2 + 4\beta + 3\gamma + 2\gamma\beta \\
& + \big(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
& - 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\big)^{\frac{1}{2}}\big)\big)^{\frac{1}{2}}.
\end{aligned}$$

The third:

$$\begin{aligned}
a_0 = {} & \bigg(-8\gamma\mu^2 + 3\beta - 8\mu^2 - 40\beta\mu^2 + 50\gamma^3\mu^2 + \frac{9}{2}\theta\alpha - 80\mu^2\beta^2 - 16\beta\gamma^2\mu^2 \\
& - 88\mu^2\gamma\beta^2 + 6\beta^2 + 8\gamma\beta^2 + 7\gamma\beta + 26\gamma^2\mu^2 + \frac{7}{2}\gamma^2\beta^2 + \frac{33}{4}\theta\alpha\gamma + 9\beta\theta\alpha \\
& - 24\beta^2\gamma^2\mu^2 + 20\beta\gamma^3\mu^2 - 68\mu^2\gamma\beta + \frac{23}{4}\beta\gamma^2 + 24\mu^2\theta\alpha + 12\mu^2\theta\alpha\gamma + \frac{15}{2}\beta\theta\alpha\gamma \\
& - 24\mu^2\theta\alpha\gamma^2 + \frac{3}{2}\beta\theta\alpha\gamma^2 - 12\gamma^3\mu^2\theta\alpha + 8\beta\gamma^4\mu^2 + \frac{3}{4}\theta\alpha\gamma^3 + 30\gamma^4\mu^2 + \frac{1}{2}\beta^2\gamma^3
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{4}\beta\gamma^4 + 6\gamma^5\mu^2 + 2\beta\gamma^3\Big) \\
&/\big\{\theta\big(-2 + 16\beta\gamma^3\mu^2 + 176\mu^2\gamma\beta - 12\gamma^2 - 8\gamma - 3\theta\alpha - 2\beta^2 - 8\gamma^3 - 2\gamma^4 - 7\beta \\
&+ 48\mu^2 + 96\beta\gamma^2\mu^2 + 3\theta\alpha\gamma^2 - 16\gamma\beta - 13\beta\gamma^2 - 4\beta\gamma^3 - 4\gamma\beta^2 - 2\gamma^2\beta^2 \\
&+ 136\gamma^2\mu^2 + 56\gamma^3\mu^2 + 8\gamma^4\mu^2 + 136\gamma\mu^2 + 96\beta\mu^2\big)\big\},
\end{aligned}
$$

$$
a_1 = b_1 = 0, \qquad a_2 = b_2 = \frac{2\mu^2(2+\gamma)}{\theta},
$$

$$
\begin{aligned}
V = {}&-\big(-32\gamma\mu^2 + 4\beta - 32\mu^2 + 32\beta\mu^2 + 200\gamma^3\mu^2 + 18\theta\alpha + 64\mu^2\beta^2 + 304\beta\gamma^2\mu^2 \\
&+ 160\mu^2\gamma\beta^2 - 4\beta^2 - 8\beta^3 - 18\gamma\beta^2 + 104\gamma^2\mu^2 - 20\gamma^2\beta^2 - 12\gamma\beta^3 + 33\theta\alpha\gamma \\
&+ 24\beta\theta\alpha + 18\theta\alpha\gamma^2 + 128\beta^2\gamma^2\mu^2 + 208\beta\gamma^3\mu^2 + 176\mu^2\gamma\beta - 13\beta\gamma^2 + 96\mu^2\theta\alpha \\
&+ 48\mu^2\theta\alpha\gamma + 36\beta\theta\alpha\gamma - 96\mu^2\theta\alpha\gamma^2 + 12\beta\theta\alpha\gamma^2 - 48\gamma^3\mu^2\theta\alpha + 48\beta\mu^2\gamma^4 \\
&+ 3\theta\alpha\gamma^3 + 32\gamma^3\mu^2\beta^2 + 120\gamma^4\mu^2 - 6\beta^2\mu^3 - 3\beta\gamma^4 + 24\gamma^5\mu^2 - 12\beta\gamma^3 - 4\gamma^2\beta^3\big) \\
&/\big\{\big(6\theta 8\gamma^3\mu^2 - 2\gamma^3 - 6\gamma^2 + 48\gamma^2\mu^2 - 4\beta\gamma^2 + 16\beta\gamma^2\mu^2 - 6\gamma + 3\theta\alpha\gamma - 9\gamma\beta \\
&- 2\gamma\beta^2 + 88\gamma\mu^2 + 80\mu^2\mu\beta - 7\beta - 2\beta^2 - 3\theta\alpha + 48\mu^2 + 96\beta\mu^2 - 2\big)\big\},
\end{aligned}
$$

$$
\begin{aligned}
\mu_1 = {}&\frac{\pm 1}{8(\gamma^2+3\gamma+2)}\big(-\big(2\gamma^2 + 6\gamma + 4\big)\big(-2 - \gamma^2 - 4\beta - 3\gamma - 2\gamma\beta \\
&+ \big(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
&- 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\big)^{\frac{1}{2}}\big)\big)^{\frac{1}{2}},
\end{aligned}
$$

$$
\begin{aligned}
\mu_2 = {}&\frac{\pm 1}{8(\gamma^2+3\gamma+2)}\big(\big(2\gamma^2 + 6\gamma + 4\big)\big(2 + \gamma^2 + 4\beta + 3\gamma + 2\gamma\beta \\
&+ \big(4 - 16\beta + 12\mu + 16\beta^2 + 13\gamma^2 + 6\gamma^3 + \gamma^4 + 16\gamma\beta^2 - 32\gamma\beta + 4\gamma^2\beta^2 \\
&- 20\beta\gamma^2 - 4\beta\gamma^3 - 48\theta\alpha - 72\theta\alpha\gamma - 24\theta\alpha\gamma^2\big)^{\frac{1}{2}}\big)\big)^{\frac{1}{2}}.
\end{aligned}
$$

These in turn give the following three solutions:

$$
\begin{aligned}
u_1(x,t) &= a_0 + \frac{2\mu^2(2+\gamma)}{\theta}\tanh^2\mu(x-Vt), \\
u_2(x,t) &= a_0 + \frac{2\mu^2(2+\gamma)}{\theta}\coth^2\mu(x-Vt), \\
u_3(x,t) &= a_0 + \frac{2\mu^2(2+\gamma)}{\theta}\tanh^2\mu(x-Vt) + \frac{2\mu^2(2+\gamma)}{\theta}\coth^2\mu(x-Vt).
\end{aligned}
\tag{42}
$$

7 Conclusion

In this paper, we focused on travelling wave solutions of the general form of Benjamin-Bona-Mahony-Peregrine-Burgers equation, the general form of the Oskolkov-Benjamin-Bona-Mahony-Burgers equation, the one-dimensional Oskolkov equation and the generalised hyperelastic-rod wave equation. We derived various exact travelling wave solutions of these physical structures by using the tanh-coth method. Throughout the work, Maple was used to deal with the tedious algebraic operations.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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