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Stability analysis and observer design for a class of nonlinear systems with multiple time-delays

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Abstract

In this paper, we propose a simple and useful approach to design an observer for multiple time-delays nonlinear systems in a triangular form. By constructing a new Lyapunov-Krasovskii functional and using the differential mean-value theorem, the sufficient conditions for the existence of such an observer are derived, which guarantee that the estimation error converges asymptotically towards zero. The observer gain is independent of the time-delay. A numerical example is provided to illustrate the result.

Keywords: time-delay; nonlinear systems; nonlinear observers; asymptotic stability

1 Introduction

Time-delay, as well as nonlinearities, is often encountered in various systems which render the control design more difficult [1]. During the past decades, a lot of significant advances have been proposed in stability analysis and feedback control for time-delay systems, *e.g.*, [1–7] and reference therein. Among these schemes, the system states are assumed to be precisely known for the control design, which is not true in some practical cases as some commercial control systems are not equipped with enough sensors. This inspires the issue of observer design for control systems, which is an active research topic in the control community.

Different types of observers have been proposed, *e.g.*, Luenberger observer [8], adaptive observer [9], high-gain observer [10]. The observer design problem for time-delay systems has been widely investigated in the recent years. For time-delay systems, most of the state observation methods developed in the literature concern the linear case; we refer the reader to some recent advances and their extensions [11–13]. However, the problem of state estimation of time-delay systems in the nonlinear case has been rarely studied. For an overview of recent works, see, *e.g.*, [14–16]. In [15], a new approach to the nonlinear observer design problem in the presence of delayed output measurements was presented. The proposed nonlinear observer possesses a state-dependent gain which is computed from the solution of a system of first-order singular partial differential equations. In [18], the authors established a new method for the observer design problem for a class of Lipschitz time-delay systems. The obtained synthesis conditions are expressed in terms of linear matrix inequalities (LMIs) easily tractable and are less restrictive than those obtained in [17]. In [19], the problem of observer design for a class of multi-output nonlinear system was considered. A new state observer design methodology for linear time-varying multi-output systems was presented. Furthermore, the same methodology was extended to a class of multi-output nonlinear systems and some sufficient conditions for the existence of the proposed observer were obtained, which guaranteed that the error of state estimation converged asymptotically to zero. For further results on observation of time-delay systems, we refer the reader to [20–23] and the references therein.

In this paper, we investigate observer design for nonlinear systems written in a triangular form. Our main task is to design the observer for a class of nonlinear systems with multiple time-delays. The observer is convergent, whatever the size of the delay. The design method of observer for the class of nonlinear systems with multiple time-delays is proposed, and the gain matrix is obtained. The observer gain is independent of the time-delay. The sufficient conditions are presented, which guarantee that the estimation error converges asymptotically towards zero.

This paper is arranged as follows. In Section 2, the system description and some lemmas are given. In Section 3, we present the observer synthesis method for a class of nonlinear systems with multiple time-delays. In Section 4, we propose an illustrative example in order to show the validity of our method. Finally, some conclusions are given in Section 5.

The notation used in this paper is fairly standard. Throughout this paper, R stands for the set of real numbers. The notation $A > 0$ (< 0) means that the matrix A is symmetric and positive definite (negative definite). A^T stands for the matrix transpose of matrix A. $\|\cdot\|$ denotes the Euclidean norm for a vector or a matrix. $\|\cdot\|_\infty$ denotes the infinity norm for a matrix.

2 System description

Consider the time-delays nonlinear system given in a lower-triangular form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f\big(x(t), u(t)\big) + \sum_{j=1}^{k} g_j\big(x(t-h_j), u(t)\big), \\ y &= Cx(t), \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in X \subset R^m$ is the bounded control input and $y(t) \in R$ is the system output. The delay h_j, $j = 1, 2, \ldots, k$, are constants, and $x(t) = \phi(t)$ for $-h \leq t \leq 0$, $h = \max_{1\leq j\leq k}\{h_j\}$. The functions $f(x(t), u(t))$ and $g_j(x(t-h), u(t))$, $j = 1, 2, \ldots, k$, are nonlinear and are assumed to be smooth, and

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \in R^{n\times n}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \in R^n,$$

$$f\big(x(t), u(t)\big) = \begin{bmatrix} f_1(x_1(t), u(t)) \\ f_2(x_1(t), x_2(t), u(t)) \\ \vdots \\ f_n(x(t), u(t)) \end{bmatrix} \in R^n, \tag{2}$$

$$g_j\big(x(t-h_j),u(t)\big)=\begin{bmatrix} g_{j1}(x_1(t-h_j),u(t)) \\ g_{j2}(x_1(t-h_j),x_2(t-h_j),u(t)) \\ \vdots \\ g_{jn}(x(t-h_j),u(t)) \end{bmatrix}\in R^n,\quad j=1,2,\ldots,k.$$

To complete the system description, the following assumptions are considered.

Assumption 1 For all $t \geq 0$, $\alpha \in R^n$, the entries of $\frac{\partial f(\alpha,u(t))}{\partial\alpha}$ are bounded.

Assumption 2 For all $t \geq 0$, $\beta_j \in R^n$, the entries of $\frac{\partial g_j(\beta_j,u(t))}{\partial\beta_j}$, $j=1,2,\ldots,k$, are bounded.

We set $D(r)=\operatorname{diag}(1,r,\ldots,r^{n-1})$ and $\tilde{Q}(r)=r^{-2}D^{-1}(r)\bar{Q}D^{-1}(r)$.

The following lemmas are necessary for the proof of the main statement.

Lemma 1 *Let $P(r)$ and $\bar{P}$ be the solutions of the algebraic Riccati equations* (*AREs*):

$$\begin{aligned} &P(r)A^T+AP(r)-P(r)C^TCP(r)+\tilde{Q}(r)=0,\\ &\bar{P}A^T+A\bar{P}-\bar{P}C^TC\bar{P}+\bar{Q}=0, \end{aligned} \tag{3}$$

respectively, where A and C are given in an observable canonical form as in (2), *$\bar{Q}$ is any symmetric positive-definite matrix. Then $P(r)$ is positive-definite for $r>0$ and is given by*

$$P(r)=r^{-1}D^{-1}(r)\bar{P}D^{-1}(r).$$

Proof Let $\tilde{Q}(r)=B^T(r)B(r)$, where $B(r)\in R^{n\times n}$. Since $\tilde{Q}(r)$ is symmetric and positive-definite for all $r>0$, one gets that $B(r)$ is invertible. It is easy to verify that (A^T,C^T) is stabilizable and (B,A^T) is observable. According to ref. [24], we obtain that the matrix $P(r)$ is the unique solution of ARE (3) which is always symmetric and positive-definite for $r>0$.

Using the following properties:

$$AD(r)=rD(r)A,\qquad D(r)A^T=rA^TD(r),\qquad CD(r)=C,\qquad D(r)C^T=C^T,$$

we get

$$\begin{aligned} &D^{-1}(r)A=rAD^{-1}(r),\qquad A^TD^{-1}(r)=rD^{-1}(r)A^T,\\ &C=CD^{-1}(r),\qquad C^T=D^{-1}(r)C^T. \end{aligned} \tag{4}$$

Pre- and post-multiplying the second ARE in (3) by $r^{-1}D^{-1}(r)$, we have

$$\begin{aligned} &r^{-2}D^{-1}(r)\bar{P}A^TD^{-1}(r)+r^{-2}D^{-1}(r)A\bar{P}D^{-1}(r)\\ &\quad -r^{-2}D^{-1}(r)\bar{P}C^TC\bar{P}D^{-1}(r)+r^{-2}D^{-1}(r)\bar{Q}D^{-1}(r)=0. \end{aligned} \tag{5}$$

Using (4), (5) can be rewritten as

$$\begin{aligned} &\big[r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)\big]A^T+A\big[r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)\big]\\ &\quad -\big[r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)\big]C^TC\big[r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)\big]+r^{-2}D^{-1}(r)\bar{Q}D^{-1}(r)=0. \end{aligned}$$

By comparing the last ARE with the first ARE of (3), we conclude that

$$P(r) = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r). \tag{6}$$

□

Lemma 2 *If* $L = (l_{ij}) \in R^{n\times n}$ *is a lower-triangular matrix and* $D(r) = \operatorname{diag}(1, r, \ldots, r^{n-1})$, *then the following inequality holds for all* $0 < r \leq 1$:

$$\left\|D(r)LD^{-1}(r)\right\| \leq \delta_1 + r\delta_2, \tag{7}$$

where

$$\begin{aligned} \delta_1 &= \sqrt{n}\max\left\{|l_{11}|, |l_{22}|, \ldots, |l_{nn}|\right\}, \\ \delta_2 &= \sqrt{n}\max\left\{|l_{21}|, |l_{31}| + |l_{32}|, \ldots, |l_{n1}| + |l_{n2}| + \cdots + |l_{n,n-1}|\right\}. \end{aligned} \tag{8}$$

Proof Computing the product, we have

$$\begin{aligned} D(r)LD^{-1}(r) &= \begin{bmatrix} 1 & & & & \\ & r & & & \\ & & r^2 & & \\ & & & \ddots & \\ & & & & r^{n-1} \end{bmatrix} \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \\ &\quad\times \begin{bmatrix} 1 & & & & \\ & r^{-1} & & & \\ & & r^{-2} & & \\ & & & \ddots & \\ & & & & r^{-n+1} \end{bmatrix} \\ &= \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ rl_{21} & l_{22} & 0 & \cdots & 0 \\ r^2l_{31} & rl_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ r^{n-1}l_{n1} & r^{n-2}l_{n2} & r^{n-3}l_{n3} & \cdots & l_{nn} \end{bmatrix}. \end{aligned}$$

So, it follows that

$$\begin{aligned} \left\|D(r)LD^{-1}(r)\right\| &\leq \sqrt{n}\left\|D(r)LD^{-1}(r)\right\|_\infty \\ &= \sqrt{n}\max\left\{|l_{11}|, |l_{22}| + |rl_{21}|, |l_{33}| + |rl_{32}| + \left|r^2l_{31}\right|, \ldots, \right. \\ &\quad \left. |l_{nn}| + \left|r^{n-1}l_{n1}\right| + \left|r^{n-2}l_{n2}\right| + \cdots + |rl_{n,n-1}|\right\} \\ &= \sqrt{n}\max\left\{|l_{11}|, |l_{22}| + r|l_{21}|, |l_{33}| + r\left(|l_{32}| + |rl_{31}|\right), \ldots, \right. \\ &\quad \left. |l_{nn}| + r\left(\left|r^{n-2}l_{n1}\right| + \left|r^{n-3}l_{n2}\right| + \cdots + |l_{n,n-1}|\right)\right\}. \end{aligned}$$

When $0 < r \le 1$, we have

$$\begin{aligned}\|D(r)LD^{-1}(r)\| &\le \sqrt{n}\max\{|l_{11}|, |l_{22}| + r|l_{21}|, |l_{33}| + r(|l_{32}| + |l_{31}|), \ldots, \\ &\qquad |l_{nn}| + r(|l_{n1}| + |l_{n2}| + \cdots + |l_{n,n-1}|)\} \\ &\le \delta_1 + r\delta_2.\end{aligned}$$

□

Remark 1 If Assumptions 1 and 2 hold and $0 < r \le 1$, then there are $c_i > 0$, $i = 1, 2$, $c_{j3} > 0$, $c_{j4} > 0$, $j = 1, 2, \ldots, k$, such that

$$\left\|D(r)\frac{\partial f(\alpha, u(t))}{\partial \alpha}D^{-1}(r)\right\| \le c_1 + rc_2,$$
$$\left\|D(r)\frac{\partial g_j(\beta_j, u(t))}{\partial \beta_j}D^{-1}(r)\right\| \le c_{j3} + rc_{j4}, \quad j = 1, 2, \ldots, k.$$

Lemma 3 [25] *For any real vectors a, b and any matrix $Q > 0$ with appropriate dimensions, it follows that*

$$2a^T b \le a^T Qa + b^T Q^{-1} b.$$

Consider the following functional differential equation of retarded type:

$$\dot{x}(t) = f(t, x_t), \tag{9}$$

where $x(t) \in R^n$, $f : R \times \mathbb{C} \to R^n$.

Lemma 4 (Lyapunov-Krasovskii stability theorem [1]) *Suppose that $f : R \times \mathbb{C} \to R^n$ given in* (9) *maps every $R \times$ (bounded set in $\mathbb{C}$) into a bounded set in R^n, and that $u, v, w : \bar{R}_+ \to \bar{R}_+$ are continuous nondecreasing functions, where additionally $u(s)$ and $v(s)$ are positive for $s > 0$ and $u(0) = v(0) = 0$. If there exists a continuous differentiable functional $V : R \times \mathbb{C} \to R$ such that*

$$u(\|\phi(0)\|) \le V(t, \phi) \le v(\|\phi\|_c)$$

and

$$\dot{V}(t, \phi) \le -w(\|\phi(0)\|),$$

then the trivial solution of (9) *is uniformly stable. If $w(s) > 0$ for $s > 0$, then it is uniformly asymptotically stable. In addition, if $\lim_{s\to\infty} u(s) = \infty$, then it is globally uniformly asymptotically stable.*

3 Observer design

Now, for the time-delay system described by (1), we propose the following state observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + f(\hat{x}(t), u(t)) + \sum_{j=1}^{k} g_j(\hat{x}(t - h_j), u(t)) - L(C\hat{x}(t) - Cx(t)). \tag{10}$$

Our aim is to find the gain L such that the estimation error $e(t) = \hat{x}(t) - x(t)$ asymptotically converges towards zero. The estimation error dynamics is governed by

$$\dot{e}(t) = Ae(t) - LCe(t) + \Delta f_1 + \sum_{j=1}^{k} \Delta g_j, \tag{11}$$

where

$$\Delta f = f\big(\hat{x}(t), u(t)\big) - f\big(x(t), u(t)\big),$$
$$\Delta g_j = g_j\big(\hat{x}(t-h), u(t)\big) - g_j\big(x(t-h), u(t)\big), \quad j = 1, 2, \ldots, k.$$

In the sequel, we introduce our main contribution which consists of a new feasibility condition for the observer synthesis problem of a class of nonlinear time-delays systems. The convergence analysis is performed by the use of a Lyapunov-Krasovskii functional.

For any symmetric positive-definite matrix $\bar{Q} > 0$, let $\bar{P} > 0$ be the solutions of the algebraic Riccati equations (AREs):

$$\bar{P}A^T + A\bar{P} - \bar{P}C^T C\bar{P} + \bar{Q} = 0. \tag{12}$$

Theorem 1 *Assume that Assumptions* 1 *and* 2 *hold and* $L = P(r)C^T$, *where* $P(r) = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)$. *Then for any*

$$0 < r < \min\left\{1, \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2\lambda_{\max}(\bar{P}^{-1})[k+1+(c_1+c_2)^2]}, \min_{1\le j\le k} \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2k\lambda_{\max}(\bar{P}^{-1})(c_{j3}+c_{j4})^2}\right\}, \tag{13}$$

the observer error $e(t) = \hat{x}(t) - x(t)$ *that results from* (1) *and* (10) *converges asymptotically towards zero.*

Proof From Lemma 1, we known that the ARE

$$P(r)A^T + AP(r) - P(r)C^T CP(r) + kQ(r) = 0,$$

has the solution

$$P(r) = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r),$$

where $Q(r) = (1/k)r^{-2}D^{-1}(r)\bar{Q}D^{-1}(r)$.

So, we have

$$A^T P^{-1}(r) + P^{-1}(r)A - C^T C + kP^{-1}(r)Q(r)P^{-1}(r) = 0.$$

For positive definite matrices $P^{-1}(r)$, let us consider the Lyapunov-Krasovskii functional candidate:

$$V\big(t, e(t)\big) = e^T(t)P^{-1}(r)e(t) + \frac{1}{2}\sum_{j=1}^{k}\int_{t-h_j}^{t} e^T(s)P^{-1}(r)Q(r)P^{-1}(r)e(s)\,ds. \tag{14}$$

Then we have

$$\begin{aligned}\dot{V}\big(t,e(t)\big) &= \dot{e}^T(t)P^{-1}(r)e(t) + e^T(t)P^{-1}(r)\dot{e}(t) \\ &\quad + \frac{1}{2}\sum_{j=1}^{k}\big[e^T(t)P^{-1}(r)Q(r)P^{-1}(r)e(t) - e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j)\big] \\ &= e^T(t)\left(A^TP^{-1}(r) + P^{-1}(r)A - 2C^TC + \frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\right)e(t) \\ &\quad + 2e^T(t)P^{-1}(r)\left(\Delta f + \sum_{j=1}^{k}\Delta g_j\right) - \frac{1}{2}\sum_{j=1}^{k}e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j) \\ &= e^T(t)\left(-C^TC - \frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\right)e(t) + 2e^T(t)P^{-1}(r)\left(\Delta f + \sum_{j=1}^{k}\Delta g_j\right) \\ &\quad - \frac{1}{2}\sum_{j=1}^{k}e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j).\end{aligned}$$

Using the differential mean-value theorem, we can write that

$$\begin{aligned}\Delta f &= f\big(\hat{x}(t),u(t)\big) - f\big(x(t),u(t)\big) \\ &= \int_0^1 \frac{\partial f(\alpha,u(t))}{\partial\alpha}\bigg|_{\alpha=\alpha(\lambda)}\big(\hat{x}(t)-x(t)\big)\,d\lambda, \\ \Delta g_j &= g_j\big(\hat{x}(t-h_j),u(t)\big) - g_j\big(x(t-h_j),u(t)\big) \\ &= \int_0^1 \frac{\partial g_j(\beta_j,u(t))}{\partial\beta_j}\bigg|_{\beta_j=\beta_j(\lambda)}\big(\hat{x}(t-h_j)-x(t-h_j)\big)\,d\lambda, \quad j=1,2,\ldots,k,\end{aligned} \tag{15}$$

where

$$\begin{aligned}\alpha(\lambda) &= x(t) + \lambda\big(\hat{x}(t)-x(t)\big), \\ \beta_j(\lambda) &= x(t-h_j) + \lambda\big(\hat{x}(t-h_j)-x(t-h_j)\big).\end{aligned}$$

Let us denote

$$\Psi_\alpha(\lambda) = \frac{\partial f(\alpha,u(t))}{\partial\alpha}\bigg|_{\alpha=\alpha(\lambda)}, \qquad \Psi_{\beta_j}(\lambda) = \frac{\partial g_j(\beta_j,u(t))}{\partial\beta_j}\bigg|_{\beta_j=\beta_j(\lambda)}.$$

This immediately gives

$$\begin{aligned}\dot{V}\big(t,e(t)\big) &\le \int_0^1 e^T(t)\left(-\frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\right)e(t)\,d\lambda + 2\int_0^1 e^T(t)P^{-1}(r)\Psi_\alpha(\lambda)e(t)\,d\lambda \\ &\quad + 2\sum_{j=1}^{k}\int_0^1 e^T(t)P^{-1}(r)\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda \\ &\quad - \frac{1}{2}\sum_{j=1}^{k}e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j).\end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned}
&2\int_0^1 e^T(t)P^{-1}(r)\Psi_\alpha(\lambda)e(t)\,d\lambda \\
&\quad\le \int_0^1 e^T(t)P^{-1}(r)e(t)\,d\lambda + \int_0^1 e^T(t)\Psi_\alpha^T(\lambda)P^{-1}(r)\Psi_\alpha(\lambda)e(t)\,d\lambda, \\
&2\int_0^1 e^T(t)P^{-1}(r)\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda \\
&\quad\le \int_0^1 e^T(t)P^{-1}(r)e(t)\,d\lambda + \int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)P^{-1}(r)\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda.
\end{aligned}$$

This implies that

$$\begin{aligned}
\dot V\bigl(t,e(t)\bigr) \le{}& \int_0^1 e^T(t)\biggl(-\frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\biggr)e(t)\,d\lambda + (k+1)\int_0^1 e^T(t)P^{-1}(r)e(t)\,d\lambda \\
&+ \int_0^1 e^T(t)\Psi_\alpha^T(\lambda)P^{-1}(r)\Psi_\alpha(\lambda)e(t)\,d\lambda \\
&+ \sum_{j=1}^k \int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)P^{-1}(r)\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda \\
&- \frac{1}{2}\sum_{j=1}^k e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j) \\
\le{}& -\frac{1}{2}\int_0^1 e^T(t)D(r)\bar P^{-1}\bar Q\bar P^{-1}D(r)e(t)\,d\lambda + (k+1)\int_0^1 e^T(t)rD(r)\bar P^{-1}D(r)e(t)\,d\lambda \\
&+ \int_0^1 e^T(t)\Psi_\alpha^T(\lambda)rD(r)\bar P^{-1}D(r)\Psi_\alpha(\lambda)e(t)\,d\lambda \\
&+ \sum_{j=1}^k \biggl[\int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)rD(r)\bar P^{-1}D(r)\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda\biggr] \\
&- \frac{1}{2k}\sum_{j=1}^k \biggl[\int_0^1 e^T(t-h_j)D(r)\bar P^{-1}\bar Q\bar P^{-1}D(r)e(t-h_j)\,d\lambda\biggr].
\end{aligned}$$

Let $\eta(t) = D(r)e(t)$, we have

$$\begin{aligned}
\dot V\bigl(t,e(t)\bigr) \le{}& -\frac{1}{2}\int_0^1 \eta^T(t)\bar P^{-1}\bar Q\bar P^{-1}\eta(t)\,d\lambda + (k+1)r\int_0^1 \eta^T(t)\bar P^{-1}\eta(t)\,d\lambda \\
&+ \int_0^1 \eta^T(t)D^{-1}(r)\Psi_\alpha^T(\lambda)rD(r)\bar P^{-1}D(r)\Psi_\alpha(\lambda)D^{-1}(r)\eta(t)\,d\lambda \\
&+ \sum_{j=1}^k \int_0^1 \eta^T(t-h_j)D^{-1}(r)\Psi_{\beta_j}^T(\lambda)rD(r)\bar P^{-1}D(r)\Psi_{\beta_j}(\lambda)D^{-1}(r)\eta(t-h_j)\,d\lambda \\
&- \frac{1}{2k}\sum_{j=1}^k \int_0^1 \eta^T(t-h_j)\bar P^{-1}\bar Q\bar P^{-1}\eta^T(t-h_j)\,d\lambda \\
\le{}& \int_0^1 \biggl[-\frac{1}{2}\lambda_{\min}\bigl(\bar P^{-1}\bar Q\bar P^{-1}\bigr)
\end{aligned}$$

$$
\begin{aligned}
&+ r\lambda_{\max}(\bar{P}^{-1})\bigl(k+1+\|D(r)\Psi_\alpha(\lambda)D^{-1}(r)\|^2\bigr)\Bigr]\|\eta(t)\|^2\,d\lambda \\
&+ \sum_{j=1}^{k}\int_0^1\Bigl[-\frac{1}{2k}\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}) \\
&+ r\lambda_{\max}(\bar{P}^{-1})\|D(r)\Psi_{\beta_j}(\lambda)D^{-1}(r)\|^2\Bigr]\|\eta(t-h_j)\|^2\,d\lambda \\
\le{}& \int_0^1\Bigl[-\frac{1}{2}\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}) + r\lambda_{\max}(\bar{P}^{-1})\bigl(k+1+(c_1+rc_2)^2\bigr)\Bigr]\|\eta(t)\|^2\,d\lambda \\
&+ \sum_{j=1}^{k}\int_0^1\Bigl[-\frac{1}{2k}\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}) + r\lambda_{\max}(\bar{P}^{-1})(c_{j3}+rc_{j4})^2\Bigr]\|\eta(t-h_j)\|^2\,d\lambda \\
\le{}& \int_0^1\Bigl[-\frac{1}{2}\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}) + r\lambda_{\max}(\bar{P}^{-1})\bigl(k+1+(c_1+c_2)^2\bigr)\Bigr]\|\eta(t)\|^2\,d\lambda \\
&+ \sum_{j=1}^{k}\int_0^1\Bigl[-\frac{1}{2k}\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}) + r\lambda_{\max}(\bar{P}^{-1})(c_{j3}+c_{j4})^2\Bigr]\|\eta(t-h_j)\|^2\,d\lambda.
\end{aligned}
$$

From (13), we have $\dot{V} < 0$. According to Lemma 4, we deduce that the observer error converges asymptotically towards zero. This ends the proof of Theorem 1. □

Consider the following nonlinear systems:

$$
\begin{aligned}
&\dot{x}(t) = Ax(t) + \sum_{j=1}^{k} M_j g_j\bigl(x(t-h_j), u(t)\bigr), \\
&y = Cx(t),
\end{aligned}
\tag{16}
$$

where A, C and $g_j(x(t-h_j), u(t))$ are given by (2), and M_j is a lower-triangular matrix.

Remark 2 If Assumption 2 holds and $0 < r \le 1$, then there are $\mu_{j1} > 0$, $\mu_{j2} > 0$, $j = 1, 2, \ldots, k$, such that

$$
\left\| D(r)M_j \frac{\partial g_j(\beta_j, u(t))}{\partial \beta_j} D^{-1}(r) \right\| \le \mu_{j1} + r\mu_{j2}, \quad j = 1, 2, \ldots, k.
$$

Consider the following observer:

$$
\dot{\hat{x}}(t) = A\hat{x}(t) + \sum_{j=1}^{k} M_j g_j\bigl(\hat{x}(t-h_j), u(t)\bigr) - L\bigl(C\hat{x}(t) - Cx(t)\bigr).
\tag{17}
$$

Our aim is to find the gain L such that the estimation error $e(t) = \hat{x}(t) - x(t)$ asymptotically converges towards zero. The estimation error dynamics is governed by

$$
\dot{e}(t) = Ae(t) - LCe(t) + \sum_{j=1}^{k} M_j \Delta g_j,
\tag{18}
$$

where

$$\Delta g_j = g_j\big(\hat{x}(t-h), u(t)\big) - g_j\big(x(t-h), u(t)\big), \quad j = 1, 2, \ldots, k.$$

Theorem 2 *Assume that Assumption 2 holds and $L = P(r)C^T$, where $P(r) = r^{-1}D^{-1}(r)\bar{P} \times D^{-1}(r)$. Then for any*

$$0 < r < \min\left\{1, \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2k\lambda_{\max}(\bar{P}^{-1})}, \min_{1\le j\le k} \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2k\lambda_{\max}(\bar{P}^{-1})(\mu_{j1}+\mu_{j2})^2}\right\}, \tag{19}$$

the observer error $e(t) = \hat{x}(t) - x(t)$ that results from (16) *and* (17) *converges asymptotically towards zero.*

Proof From Lemma 1, we known that the ARE

$$P(r)A^T + AP(r) - P(r)C^TCP(r) + kQ(r) = 0$$

has the solution

$$P(r) = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r),$$

where $Q(r) = (1/k)r^{-2}D^{-1}(r)\bar{Q}D^{-1}(r)$.

So, we have

$$A^TP^{-1}(r) + P^{-1}(r)A - C^TC + kP^{-1}(r)Q(r)P^{-1}(r) = 0.$$

For positive definite matrices $P^{-1}(r)$, let us consider the Lyapunov-Krasovskii functional candidate

$$V\big(t, e(t)\big) = e^T(t)P^{-1}(r)e(t) + \frac{1}{2}\sum_{j=1}^{k}\int_{t-h_j}^{t} e^T(s)P^{-1}(r)Q(r)P^{-1}(r)e(s)\,ds. \tag{20}$$

Then we have

$$\begin{aligned}
\dot{V}\big(t, e(t)\big) &= \dot{e}^T(t)P^{-1}(r)e(t) + e^T(t)P^{-1}(r)\dot{e}(t) + \frac{1}{2}\sum_{j=1}^{k}\big[e^T(t)P^{-1}(r)Q(r)P^{-1}(r)e(t) \\
&\quad - e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j)\big] \\
&= e^T(t)\left(A^TP^{-1}(r) + P^{-1}(r)A - 2C^TC + \frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\right)e(t) \\
&\quad + 2e^T(t)P^{-1}(r)\left(\sum_{j=1}^{k}M_j\Delta g_j\right) - \frac{1}{2}\sum_{j=1}^{k}e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j) \\
&= e^T(t)\left(-C^TC - \frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\right)e(t) + 2e^T(t)P^{-1}(r)\left(\sum_{j=1}^{k}M_j\Delta g_j\right) \\
&\quad - \frac{1}{2}\sum_{j=1}^{k}e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j).
\end{aligned}$$

Using the differential mean-value theorem, we can write that

$$\begin{aligned}\Delta g_j &= g_j\big(\hat{x}(t-h_j),u(t)\big) - g_j\big(x(t-h_j),u(t)\big)\\ &= \int_0^1 \frac{\partial g_j(\beta_j,u(t))}{\partial \beta_j}\bigg|_{\beta_j=\beta_j(\lambda)} \big(\hat{x}(t-h_j)-x(t-h_j)\big)\,d\lambda, \quad j=1,2,\ldots,k,\end{aligned} \tag{21}$$

where

$$\beta_j(\lambda) = x(t-h_j) + \lambda\big(\hat{x}(t-h_j)-x(t-h_j)\big).$$

Let us denote

$$\Psi_{\beta_j}(\lambda) = \frac{\partial g_j(\beta_j,u(t))}{\partial \beta_j}\bigg|_{\beta_j=\beta_j(\lambda)}.$$

This immediately gives

$$\begin{aligned}\dot{V}\big(t,e(t)\big) \le{}& \int_0^1 e^T(t)\bigg(-\frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\bigg)e(t)\,d\lambda\\ &+ 2\sum_{j=1}^{k}\int_0^1 e^T(t)P^{-1}(r)M_j\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda\\ &- \frac{1}{2}\sum_{j=1}^{k} e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j).\end{aligned}$$

Using Lemma 3, we have

$$\begin{aligned}&2\int_0^1 e^T(t)P^{-1}(r)M_j\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda\\ &\quad\le \int_0^1 e^T(t)P^{-1}(r)e(t)\,d\lambda\\ &\qquad+ \int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)M_j^T P^{-1}(r)M_j\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda.\end{aligned}$$

This implies that

$$\begin{aligned}\dot{V}\big(t,e(t)\big) \le{}& \int_0^1 e^T(t)\bigg(-\frac{k}{2}P^{-1}(r)Q(r)P^{-1}(r)\bigg)e(t)\,d\lambda + k\int_0^1 e^T(t)P^{-1}(r)e(t)\,d\lambda\\ &+ \sum_{j=1}^{k}\int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)M_j^T P^{-1}(r)M_j\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda\\ &- \frac{1}{2}\sum_{j=1}^{k} e^T(t-h_j)P^{-1}(r)Q(r)P^{-1}(r)e(t-h_j)\\ \le{}& -\frac{1}{2}\int_0^1 e^T(t)D(r)\bar{P}^{-1}\bar{Q}\bar{P}^{-1}D(r)e(t)\,d\lambda + k\int_0^1 e^T(t)rD(r)\bar{P}^{-1}D(r)e(t)\,d\lambda\end{aligned}$$

$$+\sum_{j=1}^{k}\biggl[\int_0^1 e^T(t-h_j)\Psi_{\beta_j}^T(\lambda)M_j^T rD(r)\bar{P}^{-1}D(r)M_j\Psi_{\beta_j}(\lambda)e(t-h_j)\,d\lambda$$
$$-\int_0^1 \frac{1}{2k}e^T(t-h_j)D(r)\bar{P}^{-1}\bar{Q}\bar{P}^{-1}D(r)e(t-h_j)\,d\lambda\biggr].$$

Let $\eta(t)=D(r)e(t)$, we have

$$\begin{aligned}
\dot{V}\bigl(t,e(t)\bigr) &\le -\frac{1}{2}\int_0^1 \eta^T(t)\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\eta(t)\,d\lambda + kr\int_0^1 \eta^T(t)\bar{P}^{-1}\eta(t)\,d\lambda \\
&\quad + \sum_{j=1}^{k}\int_0^1 \eta^T(t-h_j)D^{-1}(r)\Psi_{\beta_j}^T(\lambda)M_j^T r \\
&\quad \times D(r)\bar{P}^{-1}D(r)M_j\Psi_{\beta_j}(\lambda)D^{-1}(r)\eta(t-h_j)\,d\lambda \\
&\quad - \frac{1}{2k}\sum_{j=1}^{k}\int_0^1 \eta^T(t-h_j)\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\eta^T(t-h_j)\,d\lambda \\
&\le \int_0^1\biggl[-\frac{1}{2}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr)+kr\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)\biggr]\bigl\|\eta(t)\bigr\|^2\,d\lambda \\
&\quad + \sum_{j=1}^{k}\int_0^1\biggl[-\frac{1}{2k}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr) \\
&\quad + r\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)\bigl\|D(r)M_j\Psi_{\beta_j}(\lambda)D^{-1}(r)\bigr\|^2\biggr]\bigl\|\eta(t-h_j)\bigr\|^2\,d\lambda \\
&\le \int_0^1\biggl[-\frac{1}{2}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr)+kr\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)\biggr]\bigl\|\eta(t)\bigr\|^2\,d\lambda \\
&\quad + \sum_{j=1}^{k}\int_0^1\biggl[-\frac{1}{2k}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr)+r\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)(\mu_{j1}+r\mu_{j2})^2\biggr]\bigl\|\eta(t-h_j)\bigr\|^2\,d\lambda \\
&\le \int_0^1\biggl[-\frac{1}{2}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr)+kr\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)\biggr]\bigl\|\eta(t)\bigr\|^2\,d\lambda \\
&\quad + \sum_{j=1}^{k}\int_0^1\biggl[-\frac{1}{2k}\lambda_{\min}\bigl(\bar{P}^{-1}\bar{Q}\bar{P}^{-1}\bigr)+r\lambda_{\max}\bigl(\bar{P}^{-1}\bigr)(\mu_{j1}+\mu_{j2})^2\biggr]\bigl\|\eta(t-h_j)\bigr\|^2\,d\lambda.
\end{aligned}$$

From (19), we have $\dot{V}<0$. This ends the proof of Theorem 2. □

Remark 3 In (16), the nonlinear term is the function of $u(t)$ and $x(t-h_j)$, $j=1,2,\ldots,k$. But it does not contain $x(t)$. If $f(x(t),u(t))=0$, then (1) can be written as

$$\begin{aligned}
\dot{x}(t) &= Ax(t)+\sum_{j=1}^{k}g_j\bigl(x(t-h_j),u(t)\bigr), \\
y &= Cx(t).
\end{aligned} \tag{22}$$

When $M_j=I$, $j=1,2,\ldots,k$, (16) becomes (22). So, (22) is the special case of (16).

Consider the following time-delay system:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_1x(t) + \sum_{j=1}^{k} B_jx(t-h_j) + \omega\bigl(u(t)\bigr), \\ y &= Cx(t), \end{aligned} \tag{23}$$

where A and C are defined as in (2). $A_1 = (a_{ij})$ and $B_j = (b^j_{ik})$, $j = 1, 2, \dots, k$, are real and lower-triangular matrices and $\omega(u(t))$ is an input-injection vector of dimension n.

From Lemma 2, we have

$$\begin{aligned} &\bigl\|D(r)A_1D^{-1}(r)\bigr\| \le \upsilon_1 + r\upsilon_2, \\ &\bigl\|D(r)B_jD^{-1}(r)\bigr\| \le \upsilon_{j3} + r\upsilon_{j4}, \quad j = 1, 2, \dots, k, \end{aligned} \tag{24}$$

where

$$\begin{aligned} \upsilon_1 &= \sqrt{n}\max\bigl\{|a_{11}|, |a_{22}|, \dots, |a_{nn}|\bigr\}, \\ \upsilon_2 &= \sqrt{n}\max\bigl\{|a_{21}|, |a_{31}| + |a_{32}|, \dots, |a_{n1}| + |a_{n2}| + \cdots + |a_{n,n-1}|\bigr\}, \\ \upsilon_{j3} &= \sqrt{n}\max\bigl\{\bigl|b^j_{11}\bigr|, \bigl|b^j_{22}\bigr|, \dots, \bigl|b^j_{nn}\bigr|\bigr\}, \\ \upsilon_{j4} &= \sqrt{n}\max\bigl\{\bigl|b^j_{21}\bigr|, \bigl|b^j_{31}\bigr| + \bigl|b^j_{32}\bigr|, \dots, \bigl|b^j_{n1}\bigr| + \bigl|b^j_{n2}\bigr| + \cdots + \bigl|b^j_{n,n-1}\bigr|\bigr\}. \end{aligned} \tag{25}$$

Consider the following observer:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + A_1\hat{x}(t) + \sum_{j=1}^{k} B_j\hat{x}(t-h_j) - L\bigl(C\hat{x}(t) - Cx(t)\bigr) + \omega\bigl(u(t)\bigr). \tag{26}$$

The estimation error is $e(t) = \hat{x}(t) - x(t)$. The estimation error dynamics is governed by

$$\dot{e}(t) = (A + A_1 - LC)e(t) + \sum_{j=1}^{k} B_je(t-h_j). \tag{27}$$

Corollary 1 *Consider the nonlinear system* (23). *Assume that* $L = P(r)C^T$, *where* $P(r) = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)$. *Then for any*

$$0 < r < \min\biggl\{1, \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2\lambda_{\max}(\bar{P}^{-1})[k + 1 + (\upsilon_1 + \upsilon_2)^2]}, \min_{1\le j\le k} \frac{\lambda_{\min}(\bar{P}^{-1}\bar{Q}\bar{P}^{-1})}{2k\lambda_{\max}(\bar{P}^{-1})[(\upsilon_{j3} + \upsilon_{j4})^2]}\biggr\},$$

the estimation error $e(t) = \hat{x}(t) - x(t)$ *that results from* (23) *and* (26) *converges asymptotically towards zero.*

Proof The matrices A and B_j can be seen as the matrix Jacobian. Therefore, the proof becomes straightforward as it was developed before. □

Remark 4 Those results obtained can be extended to multiple time-delays nonlinear systems in upper-triangular form.

Remark 5 In [26], the sufficient conditions which guarantee that the estimation error converges asymptotically towards zero are given in terms of a linear matrix inequality. Comparing with [26], our results are less conservative and more convenient to use.

4 Numerical example

Let us consider the time delay system

$$\begin{aligned}
&\dot{x}_1(t) = x_2(t) + \sin\bigl(x_1(t-1) + u(t)\bigr),\\
&\dot{x}_2(t) = -0.15\sin\bigl(x_2(t-0.5)\bigr),\\
&y(t) = x_1(t),
\end{aligned}$$

where

$$\begin{aligned}
&A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \qquad h_1 = 1, \qquad h_2 = 0.5,\\
&x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \qquad M_1 = M_2 = I,\\
&g_1\bigl(x(t-h_1), u(t)\bigr) = \begin{pmatrix} \sin(x_1(t-1) + u(t)) \\ 0 \end{pmatrix},\\
&g_2\bigl(x(t-h_2), u(t)\bigr) = \begin{pmatrix} 0 \\ -0.15\sin(x_2(t-0.5)) \end{pmatrix}.
\end{aligned}$$

Take

$$\bar{Q} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving the following equation:

$$\bar{P}A^T + A\bar{P} - \bar{P}C^T C\bar{P} + \bar{Q} = 0,$$

we get

$$\bar{P} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad \bar{P}^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

It is easy to obtain that $\mu_{11} = \mu_{12} = 1$, $\mu_{21} = \mu_{22} = 0.5$. Let $r = 0.025$, $D(r) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & r \end{smallmatrix}\right]$. It is easy to verify that (19) holds.

We get

$$L = r^{-1}D^{-1}(r)\bar{P}D^{-1}(r)C^T|_{r=0.025} = \begin{bmatrix} 80 \\ 1{,}600 \end{bmatrix}.$$

The observer is given by

$$\dot{\hat{x}}_1(t) = \hat{x}_2(t) + \sin\bigl(\hat{x}_1(t-1) + u(t)\bigr) - 80\bigl(\hat{x}_1(t) - x_1(t)\bigr),$$
$$\dot{\hat{x}}_2(t) = -0.15\sin\bigl(\hat{x}_2(t-0.5)\bigr) - 1{,}600\bigl(\hat{x}_1(t) - x_1(t)\bigr).$$

According to Theorem 2, the estimation error $e(t) = \hat{x}(t) - x(t)$ converges asymptotically towards zero.

5 Conclusion

The main purpose of this paper is to offer a systematic algorithm for designing an observer for a class of nonlinear systems with multiple time-delays. By using an improved Lyapunov-Krasovskii functional and the differential mean-value theorem, we present the sufficient conditions for the existence of the observer, which guarantee that the estimation error converges asymptotically towards zero. The new design plays an important role in obtaining a nonrestrictive synthesis condition and rendering our approach application to a broader class of systems, namely the class of nonlinear time-delay systems in a lower-triangular form. The proposed design is valid whatever the size of the delay. Finally, the efficiency of the proposed method is shown by a numerical example.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YD carried out the main part of this manuscript. FY participated in the discussion and gave the example. All authors read and approved the final manuscript.

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