Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Some differential subordinations using Ruscheweyh derivative and Sălăgean operator

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Abstract
In the present paper we study the operator defined by using the Ruscheweyh derivative $R^n f(z)$ and the Sălăgean operator $S^n f(z)$, denoted by $L^n_\alpha : \mathcal{A} \to \mathcal{A}$, $L^n_\alpha f(z) = (1-\alpha)R^n f(z) + \alpha S^n f(z)$, $z \in U$, where $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \cdots, z \in U\}$ is the class of normalized analytic functions with $\mathcal{A}_1 = \mathcal{A}$. We obtain several differential subordinations regarding the operator L^n_α.

MSC: 30C45; 30A20; 34A40

Keywords: differential subordination; convex function; best dominant; differential operator; Sălăgean operator; Ruscheweyh derivative

1 Introduction

Denote by U the unit disc of the complex plane, $U = \{z \in \mathbb{C} : |z| < 1\}$, and by $\mathcal{H}(U)$ the space of holomorphic functions in U. Let $\mathcal{A}_n = \{f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \cdots, z \in U\}$ with $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{H}[a,n] = \{f \in \mathcal{H}(U) : f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \cdots, z \in U\}$ for $a \in \mathbb{C}$ and $n \in \mathbb{N}$. Denote by $K = \{f \in \mathcal{A} : \operatorname{Re} \frac{zf''(z)}{f'(z)} + 1 > 0, z \in U\}$ the class of normalized convex functions in U.

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$, if there is a function w analytic in U, with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$, such that $f(z) = g(w(z))$ for all $z \in U$. If g is univalent, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be a univalent function in U. If p is analytic in U and satisfies the (second-order) differential subordination

$$\psi\left(p(z), zp'(z), z^2p''(z); z\right) \prec h(z), \quad z \in U, \tag{1.1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, or more simply a dominant, if $p \prec q$ for all p satisfying (1.1).

A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec q$ for all dominants q of (1.1) is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of U.

Definition 1.1 (Sălăgean [1]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator S^n is defined by $S^n : \mathcal{A} \to \mathcal{A}$,

$$S^0 f(z) = f(z),$$

$$S^1 f(z) = zf'(z),$$

$$\dots$$

$$S^{n+1}f(z) = z\big(S^n f(z)\big)', \quad z \in U.$$

Remark 1.1 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $S^n f(z) = z + \sum_{j=2}^{\infty} j^n a_j z^j$, $z \in U$.

Definition 1.2 (Ruscheweyh [2]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = zf'(z),$$

$$\dots$$

$$(n+1)R^{n+1}f(z) = z\big(R^n f(z)\big)' + nR^n f(z), \quad z \in U.$$

Remark 1.2 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} C_{n+j-1}^{n} a_j z^j$, $z \in U$.

Definition 1.3 ([3]) Let $\alpha \geq 0$, $n \in \mathbb{N}$. Denote by L_α^n the operator given by $L_\alpha^n : \mathcal{A} \to \mathcal{A}$,

$$L_\alpha^n f(z) = (1-\alpha)R^n f(z) + \alpha S^n f(z), \quad z \in U.$$

Remark 1.3 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $L_\alpha^n f(z) = z + \sum_{j=2}^{\infty}(\alpha j^n + (1-\alpha)C_{n+j-1}^{n})a_j z^j$, $z \in U$.

This operator was studied also in [3–5].

Lemma 1.1 (Hallenbeck and Ruscheweyh [6, Th. 3.1.6, p.71]) *Let h be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C}\backslash\{0\}$ be a complex number with* $\operatorname{Re}\gamma \geq 0$. *If $p \in \mathcal{H}[a,n]$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z) \prec h(z), \quad z \in U,$$

where $g(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1}\,dt$, $z \in U$.

Lemma 1.2 (Miller and Mocanu [6]) *Let g be a convex function in U and let $h(z) = g(z) + n\alpha zg'(z)$, for $z \in U$, where $\alpha > 0$ and n is a positive integer.*

If $p(z) = g(0) + p_n z^n + p_{n+1}z^{n+1} + \cdots$, $z \in U$, is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z), \quad z \in U,$$

then

$$p(z) \prec g(z), \quad z \in U,$$

and this result is sharp.

2 Main results

Theorem 2.1 *Let g be a convex function, $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\delta}g'(z)$, $z \in U$.*

If $\alpha, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} \left(L_\alpha^n f(z)\right)' \prec h(z), \quad z \in U, \tag{2.1}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta} \prec g(z), \quad z \in U,$$

and this result is sharp.

Proof By using the properties of the operator L_α^n, we have

$$L_\alpha^n f(z) = z + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^n\right) a_j z^j, \quad z \in U.$$

Consider $p(z) = (\frac{L_\alpha^n f(z)}{z})^\delta = (\frac{z + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha) C_{n+j-1}^n) a_j z^j}{z})^\delta = 1 + p_\delta z^\delta + p_{\delta+1} z^{\delta+1} + \cdots$, $z \in U$.
We deduce that $p \in \mathcal{H}[1, \delta]$.
Differentiating, we obtain $(\frac{L_\alpha^n f(z)}{z})^{\delta-1} (L_\alpha^n f(z))' = p(z) + \frac{1}{\delta} z p'(z)$, $z \in U$.
Then (2.1) becomes

$$p(z) + \frac{1}{\delta} z p'(z) \prec h(z) = g(z) + \frac{z}{\delta} g'(z), \quad z \in U.$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad \left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta} \prec g(z), \quad z \in U. \qquad \square$$

Theorem 2.2 *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}(1 + \frac{z h''(z)}{h'(z)}) > -\frac{1}{2}$, *$z \in U$, and $h(0) = 1$.*
If $\alpha, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1} \left(L_\alpha^n f(z)\right)' \prec h(z), \quad z \in U, \tag{2.2}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1}\, dt$. The function q is convex and it is the best dominant.

Proof Let

$$\begin{aligned} p(z) &= \left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta} = \left(\frac{z + \sum_{j=2}^{\infty} (\alpha j^n + (1-\alpha) C_{n+j-1}^n) a_j z^j}{z}\right)^{\delta} \\ &= \left(1 + \sum_{j=2}^{\infty} \left(\alpha j^n + (1-\alpha) C_{n+j-1}^n\right) a_j z^{j-1}\right)^{\delta} = 1 + \sum_{j=\delta+1}^{\infty} p_j z^{j-1} \end{aligned}$$

for $z \in U$, $p \in \mathcal{H}[1, \delta]$.

Differentiating, we obtain $(\frac{L_\alpha^n f(z)}{z})^{\delta-1}(L_\alpha^n f(z))' = p(z) + \frac{1}{\delta}zp'(z)$, $z \in U$, and (2.2) becomes

$$p(z) + \frac{1}{\delta}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad i.e., \quad \left(\frac{L_\alpha^n f(z)}{z}\right)^\delta \prec q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}\,dt, \quad z \in U,$$

and q is the best dominant. □

Corollary 2.3 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$.*

If $\alpha, \delta \ge 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^{\delta-1}\left(L_\alpha^n f(z)\right)' \prec h(z), \quad z \in U, \tag{2.3}$$

then

$$\left(\frac{L_\alpha^n f(z)}{z}\right)^\delta \prec q(z), \quad z \in U,$$

where q is given by $q(z) = (2\beta-1) + \frac{2(1-\beta)\delta}{z^\delta}\int_0^z \frac{t^{\delta-1}}{1+t}\,dt$, $z \in U$. The function q is convex and it is the best dominant.

Proof Following the same steps as in the proof of Theorem 2.2 and considering $p(z) = (\frac{L_\alpha^n f(z)}{z})^\delta$, the differential subordination (2.3) becomes

$$p(z) + \frac{z}{\delta}p'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

By using Lemma 1.1 for $\gamma = \delta$, we have $p(z) \prec q(z)$, *i.e.*,

$$\begin{aligned}\left(\frac{L_\alpha^n f(z)}{z}\right)^\delta &\prec q(z) = \frac{\delta}{z^\delta}\int_0^z h(t)t^{\delta-1}\,dt \\ &= \frac{\delta}{z^\delta}\int_0^z t^{\delta-1}\frac{1+(2\beta-1)t}{1+t}\,dt = \frac{\delta}{z^\delta}\int_0^z \left[(2\beta-1)t^{\delta-1} + 2(1-\beta)\frac{t^{\delta-1}}{1+t}\right]dt \\ &= (2\beta-1) + \frac{2(1-\beta)\delta}{z^\delta}\int_0^z \frac{t^{\delta-1}}{1+t}\,dt, \quad z \in U.\end{aligned}$$

□

Remark 2.1 For $n = 1$, $\alpha = 2$, $\delta = 1$, we obtain the same example as in [7, Example 2.2.1, p.26].

Theorem 2.4 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + \frac{z}{\gamma}g'(z)$, $z \in U$, where $\gamma > 0$.*

If $\alpha \ge 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination

$$\frac{(\gamma+1)z}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1}f(z))^2} + \frac{z^2}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1}f(z))^2}\left[\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 2\frac{(L_\alpha^{n+1}f(z))'}{L_\alpha^{n+1}f(z)}\right] \prec h(z), \quad z \in U, \tag{2.4}$$

holds, then

$$z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \prec g(z), \quad z \in U,$$

and this result is sharp.

Proof For $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^\infty a_j z^j$, we have $L_\alpha^n f(z) = z + \sum_{j=2}^\infty (\alpha j^n + (1-\alpha) C_{n+j-1}^n) a_j z^j$, $z \in U$.

Consider $p(z) = z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2}$ and we obtain $p(z) + \frac{z}{\gamma} p'(z) = \frac{(\gamma+1)z}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} + \frac{z^2}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \times [\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 2\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)}]$.

Relation (2.4) becomes

$$p(z) + \frac{z}{\gamma} p'(z) \prec h(z) = g(z) + \gamma g'(z), \quad z \in U.$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \prec g(z), \quad z \in U.$$

□

Theorem 2.5 *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$.

If $\alpha \geq 0$, $\gamma \in \mathbb{C}\setminus\{0\}$ *is a complex number with* $\operatorname{Re}\gamma \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\frac{(\gamma+1)z}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} + \frac{z^2}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2}\left[\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 2\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)}\right] \prec h(z), \quad z \in U, \tag{2.5}$$

then

$$z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t) t^{\gamma-1}\,dt$. *The function q is convex and it is the best dominant.*

Proof Let $p(z) = z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2}$, $z \in U$, $p \in \mathcal{H}[1,1]$. Differentiating, we obtain $p(z) + \frac{z}{\gamma} p'(z) = \frac{(\gamma+1)z}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} + \frac{z^2}{\gamma}\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2}[\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} - 2\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)}]$, $z \in U$, and (2.5) becomes

$$p(z) + \frac{z}{\gamma} p'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.,} \quad z\frac{L_\alpha^n f(z)}{(L_\alpha^{n+1} f(z))^2} \prec q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t) t^{\gamma-1}\,dt, \quad z \in U,$$

and q is the best dominant. □

Theorem 2.6 *Let g be a convex function such that* $g(0) = 1$ *and let h be the function* $h(z) = g(z) + \frac{z}{\gamma}g'(z)$, $z \in U$, *where* $\gamma > 0$.

If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and the differential subordination*

$$\frac{(\gamma+2)z^2}{\gamma}\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma}\left[\frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left(\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}\right)^2\right] \prec h(z), \quad z \in U, \tag{2.6}$$

holds, then

$$z^2\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof Let $p(z) = z^2\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}$. We deduce that $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $p(z) + \frac{z}{\gamma}p'(z) = \frac{(\gamma+2)z^2}{\gamma}\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma}[\frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - (\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)})^2]$, $z \in U$.

Using the notation in (2.6), the differential subordination becomes

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z) = g(z) + \frac{z}{\gamma}g'(z).$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad z^2\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \prec g(z), \quad z \in U,$$

and this result is sharp. □

Theorem 2.7 *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$, $z \in U$, *and* $h(0) = 1$.

If $\alpha \geq 0$, $\gamma \in \mathbb{C}\backslash\{0\}$ *is a complex number with* $\operatorname{Re}\gamma \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ *and satisfies the differential subordination*

$$\frac{(\gamma+2)z^2}{\gamma}\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma}\left[\frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - \left(\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}\right)^2\right] \prec h(z), \quad z \in U, \tag{2.7}$$

then

$$z^2\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{\gamma}{z^\gamma}\int_0^z h(t)t^{\gamma-1}\,dt$. *The function q is convex and it is the best dominant.*

Proof Let $p(z) = z^2\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $p(z) + \frac{z}{\gamma}p'(z) = \frac{(\gamma+2)z^2}{\gamma}\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} + \frac{z^3}{\gamma}[\frac{(L_\alpha^n f(z))''}{L_\alpha^n f(z)} - (\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)})^2]$, $z \in U$, and (2.7) becomes

$$p(z) + \frac{1}{\gamma}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.,} \quad z^2 \frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)} \prec q(z) = \frac{\gamma}{z^\gamma} \int_0^z h(t) t^{\gamma-1}\, dt, \quad z \in U,$$

and q is the best dominant. □

Theorem 2.8 *Let g be a convex function such that $g(0) = 1$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$.*

If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination

$$1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} \prec h(z), \quad z \in U, \tag{2.8}$$

holds, then

$$\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof Let $p(z) = \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}$. We deduce that $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} = p(z) + zp'(z)$, $z \in U$.

Using the notation in (2.8), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec g(z), \quad z \in U,$$

and this result is sharp. □

Theorem 2.9 *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$, *$z \in U$, and $h(0) = 1$.*

If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$1 - \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2} \prec h(z), \quad z \in U, \tag{2.9}$$

then

$$\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{z} \int_0^z h(t)\, dt$. The function q is convex and it is the best dominant.

Proof Let $p(z) = \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $1-\frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2}=p(z)+zp'(z)$, $z\in U$, and (2.9) becomes

$$p(z)+zp'(z)\prec h(z),\quad z\in U.$$

Using Lemma 1.1, we have

$$p(z)\prec q(z),\quad z\in U,\quad \textit{i.e.},\quad \frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}\prec q(z)=\frac{1}{z}\int_0^z h(t)\,dt,\quad z\in U,$$

and q is the best dominant. □

Corollary 2.10 *Let* $h(z)=\frac{1+(2\beta-1)z}{1+z}$ *be a convex function in* U, *where* $0\le\beta<1$.

If $\alpha\ge 0$, $n\in\mathbb{N}$, $f\in\mathcal{A}$ *and satisfies the differential subordination*

$$1-\frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))''}{[(L_\alpha^n f(z))']^2}\prec h(z),\quad z\in U, \tag{2.10}$$

then

$$\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}\prec q(z),\quad z\in U,$$

where q *is given by* $q(z)=(2\beta-1)+2(1-\beta)\frac{\ln(1+z)}{z}$, $z\in U$. *The function* q *is convex and it is the best dominant.*

Proof Following the same steps as in the proof of Theorem 2.9 and considering $p(z)=\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}$, the differential subordination (2.10) becomes

$$p(z)+zp'(z)\prec h(z)=\frac{1+(2\beta-1)z}{1+z},\quad z\in U.$$

By using Lemma 1.1 for $\gamma=1$, we have $p(z)\prec q(z)$, *i.e.*,

$$\begin{aligned}\frac{L_\alpha^n f(z)}{z(L_\alpha^n f(z))'}\prec q(z)&=\frac{1}{z}\int_0^z h(t)\,dt=\frac{1}{z}\int_0^z\frac{1+(2\beta-1)t}{1+t}\,dt\\&=\frac{1}{z}\int_0^z\left[(2\beta-1)+\frac{2(1-\beta)}{1+t}\right]dt=(2\beta-1)+2(1-\beta)\frac{\ln(1+z)}{z},\quad z\in U.\end{aligned}$$ □

Example 2.1 Let $h(z)=\frac{1-z}{1+z}$ be a convex function in U with $h(0)=1$ and $\mathrm{Re}(\frac{zh''(z)}{h'(z)}+1)>-\frac{1}{2}$.

Let $f(z)=z+z^2$, $z\in U$. For $n=1$, $\alpha=2$, we obtain $L_2^1 f(z)=-R^1 f(z)+2S^1 f(z)=-zf'(z)+2zf'(z)=zf'(z)=z+2z^2$.

Then $(L_2^1 f(z))'=1+4z$,

$$\frac{L_2^1 f(z)}{z(L_2^1 f(z))'}=\frac{z+2z^2}{z(1+4z)}=\frac{1+2z}{1+4z},$$

$$1-\frac{L_2^1 f(z)\cdot(L_2^1 f(z))''}{[(L_2^1 f(z))']^2}=1-\frac{(z+2z^2)\cdot 4}{(1+4z)^2}=\frac{8z^2+4z+1}{(1+4z)^2}.$$

We have $q(z)=\frac{1}{z}\int_0^z\frac{1-t}{1+t}\,dt=-1+\frac{2\ln(1+z)}{z}$.

Using Theorem 2.9, we obtain

$$\frac{8z^2+4z+1}{(1+4z)^2} \prec \frac{1-z}{1+z}, \quad z \in U,$$

induce

$$\frac{1+2z}{1+4z} \prec -1 + \frac{2\ln(1+z)}{z}, \quad z \in U.$$

Theorem 2.11 *Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + zg'(z)$, $z \in U$.*

If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination

$$\left[\left(L_\alpha^n f(z)\right)'\right]^2 + L_\alpha^n f(z) \cdot \left(L_\alpha^n f(z)\right)'' \prec h(z), \quad z \in U, \tag{2.11}$$

holds, then

$$\frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} \prec g(z), \quad z \in U.$$

This result is sharp.

Proof Let $p(z) = \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z}$. We deduce that $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $[(L_\alpha^n f(z))']^2 + L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'' = p(z) + zp'(z)$, $z \in U$.

Using the notation in (2.11), the differential subordination becomes

$$p(z) + zp'(z) \prec h(z) = g(z) + zg'(z).$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \textit{i.e.}, \quad \frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} \prec g(z), \quad z \in U,$$

and this result is sharp. □

Theorem 2.12 *Let h be a holomorphic function which satisfies the inequality* $\operatorname{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$, *$z \in U$, and $h(0) = 0$.*

If $\alpha \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left[\left(L_\alpha^n f(z)\right)'\right]^2 + L_\alpha^n f(z) \cdot \left(L_\alpha^n f(z)\right)'' \prec h(z), \quad z \in U, \tag{2.12}$$

then

$$\frac{L_\alpha^n f(z) \cdot (L_\alpha^n f(z))'}{z} \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1}{z}\int_0^z h(t)\,dt$. The function q is convex and it is the best dominant.

Proof Let $p(z) = \frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'}{z}$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $[(L_\alpha^n f(z))']^2 + L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'' = p(z) + zp'(z)$, $z \in U$, and (2.12) becomes

$$p(z) + zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'}{z} \prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt, \quad z \in U,$$

and q is the best dominant. □

Corollary 2.13 *Let $h(z) = \frac{1+(2\beta-1)z}{1+z}$ be a convex function in U, where $0 \le \beta < 1$.*

If $\alpha \ge 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left[\left(L_\alpha^n f(z)\right)'\right]^2 + L_\alpha^n f(z)\cdot\left(L_\alpha^n f(z)\right)'' \prec h(z), \quad z \in U, \tag{2.13}$$

then

$$\frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'}{z} \prec q(z), \quad z \in U,$$

where q is given by $q(z) = (2\beta-1) + 2(1-\beta)\frac{\ln(1+z)}{z}$, $z \in U$. The function q is convex and it is the best dominant.

Proof Following the same steps as in the proof of Theorem 2.12 and considering $p(z) = \frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'}{z}$, the differential subordination (2.13) becomes

$$p(z) + zp'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.$$

By using Lemma 1.1 for $\gamma = 1$, we have $p(z) \prec q(z)$, *i.e.*,

$$\begin{aligned}\frac{L_\alpha^n f(z)\cdot(L_\alpha^n f(z))'}{z} &\prec q(z) = \frac{1}{z}\int_0^z h(t)\,dt = \frac{1}{z}\int_0^z \frac{1+(2\beta-1)t}{1+t}\,dt \\ &= \frac{1}{z}\int_0^z \left[(2\beta-1) + \frac{2(1-\beta)}{1+t}\right]dt \\ &= (2\beta-1) + 2(1-\beta)\frac{\ln(1+z)}{z}, \quad z \in U.\end{aligned}$$

□

Example 2.2 Let $h(z) = \frac{1-z}{1+z}$ be a convex function in U with $h(0) = 1$ and $\operatorname{Re}(\frac{zh''(z)}{h'(z)} + 1) > -\frac{1}{2}$.

Let $f(z) = z + z^2$, $z \in U$. For $n = 1$, $\alpha = 2$, we obtain $L_2^1 f(z) = -R^1 f(z) + 2S^1 f(z) = -zf'(z) + 2zf'(z) = zf'(z) = z + 2z^2$, $z \in U$.

Then $(L_2^1 f(z))' = 1 + 4z$,

$$\frac{L_2^1 f(z)\cdot(L_2^1 f(z))'}{z} = \frac{(z+2z^2)(1+4z)}{z} = 8z^2 + 6z + 1,$$

$$\left[\left(L_2^1 f(z)\right)'\right]^2 + L_2^1 f(z)\cdot\left(L_2^1 f(z)\right)'' = (1+4z)^2 + (z+2z^2)\cdot 4 = 24z^2 + 12z + 1.$$

We have $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}\,dt = -1 + \frac{2\ln(1+z)}{z}$.

Using Theorem 2.12, we obtain

$$24z^2 + 12z + 1 \prec \frac{1-z}{1+z}, \quad z \in U,$$

induce

$$8z^2 + 6z + 1 \prec -1 + \frac{2\ln(1+z)}{z}, \quad z \in U.$$

Theorem 2.14 *Let g be a convex function such that $g(0) = 0$ and let h be the function $h(z) = g(z) + \frac{z}{1-\delta}g'(z)$, $z \in U$.*

If $\alpha \geq 0$, $\delta \in (0,1)$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination

$$\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta}\left(\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}\right) \prec h(z), \quad z \in U, \tag{2.14}$$

holds, then

$$\frac{L_\alpha^{n+1} f(z)}{z}\cdot\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \prec g(z), \quad z \in U.$$

This result is sharp.

Proof Let $p(z) = \frac{L_\alpha^{n+1} f(z)}{z}\cdot(\frac{z}{L_\alpha^n f(z)})^\delta$. We deduce that $p \in \mathcal{H}[1,1]$.

Differentiating, we obtain $(\frac{z}{L_\alpha^n f(z)})^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta}(\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$.

Using the notation in (2.14), the differential subordination becomes

$$p(z) + \frac{1}{1-\delta}zp'(z) \prec h(z) = g(z) + \frac{z}{1-\delta}g'(z).$$

By using Lemma 1.2, we have

$$p(z) \prec g(z), \quad z \in U, \quad \text{i.e.,} \quad \frac{L_\alpha^{n+1} f(z)}{z}\cdot\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \prec g(z), \quad z \in U,$$

and this result is sharp. □

Theorem 2.15 *Let h be a holomorphic function which satisfies the inequality* $\mathrm{Re}(1 + \frac{zh''(z)}{h'(z)}) > -\frac{1}{2}$, *$z \in U$, and $h(0) = 1$.*

If $\alpha \geq 0$, $\delta \in (0,1)$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination

$$\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \frac{L_\alpha^{n+1} f(z)}{1-\delta}\left(\frac{(L_\alpha^{n+1} f(z))'}{L_\alpha^{n+1} f(z)} - \delta\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}\right) \prec h(z), \quad z \in U, \tag{2.15}$$

then

$$\frac{L_\alpha^{n+1} f(z)}{z}\cdot\left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \prec q(z), \quad z \in U,$$

where $q(z) = \frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}\,dt$. The function q is convex and it is the best dominant.

Proof Let $p(z) = \frac{L_\alpha^{n+1}f(z)}{z} \cdot (\frac{z}{L_\alpha^n f(z)})^\delta$, $z \in U$, $p \in \mathcal{H}[0,1]$.

Differentiating, we obtain $(\frac{z}{L_\alpha^n f(z)})^\delta \frac{L_\alpha^{n+1}f(z)}{1-\delta}(\frac{(L_\alpha^{n+1}f(z))'}{L_\alpha^{n+1}f(z)} - \delta\frac{(L_\alpha^n f(z))'}{L_\alpha^n f(z)}) = p(z) + \frac{1}{1-\delta}zp'(z)$, $z \in U$, and (2.15) becomes

$$p(z) + \frac{1}{1-\delta}zp'(z) \prec h(z), \quad z \in U.$$

Using Lemma 1.1, we have

$$p(z) \prec q(z), \quad z \in U, \quad i.e.,$$

$$\frac{L_\alpha^{n+1}f(z)}{z} \cdot \left(\frac{z}{L_\alpha^n f(z)}\right)^\delta \prec q(z) = \frac{1-\delta}{z^{1-\delta}}\int_0^z h(t)t^{-\delta}\,dt, \quad z \in U,$$

and q is the best dominant. □

Competing interests
The author declares that she has no competing interests.

Authors' contributions
The author drafted the manuscript, read and approved the final manuscript.

Acknowledgements
Dedicated to Professor Hari M Srivastava.
The author thanks the referee for his/her valuable suggestions to improve the present article.

Received: 9 April 2013 Accepted: 10 May 2013 Published: 28 May 2013

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doi:10.1186/1687-1847-2013-150
Cite this article as: Alb Lupaş: **Some differential subordinations using Ruscheweyh derivative and Sălăgean operator.** *Advances in Difference Equations* 2013 **2013**:150.