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Extension of a quadratic transformation due to Whipple with an application

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Abstract
The aim of this research is to provide an extension of an interesting and useful quadratic transformation due to Whipple. The result is derived with the help of extension of classical Saalschütz's summation theorem recently added in the literature. The transformation is further used to obtain a new hypergeometric identity by employing the so-called beta integral method introduced and studied systematically by Krattenthaler and Rao.

MSC: 33C20; 33C05; 33B20

Keywords: Whipple's transformation; Saalschütz's summation theorem; extension of a quadratic transformation; beta integral

1 Introduction

The generalized hypergeometric function ${}_pF_q$, with p numerator and q denominator parameters is defined by [1]

$$\begin{aligned} {}_pF_q\left[\begin{matrix} a_1,\ldots,a_p; \\ b_1,\ldots,b_q; \end{matrix} \; z\right] &= {}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;z] \\ &= \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!}, \end{aligned} \tag{1.1}$$

where $(a)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined for any complex number a by

$$(a)_n = \begin{cases} 1, & n = 0, \\ a(a+1)\cdots(a+n-1), & n \in \mathbb{N}. \end{cases} \tag{1.2}$$

Using the fundamental relation $\Gamma(a+1) = a\Gamma(a)$, $(a)_n$ can be written in the form

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \quad (n \in \mathbb{N}\cup\{0\}), \tag{1.3}$$

where Γ is the well-known gamma function.

It is well known that whenever a generalized hypergeometric function reduces to quotient of the products of the gamma function, the results are very important from the application point of view. Thus in the theory of hypergeometric and generalized hypergeometric series, summation formulas and transformation formulas play an important role.

In a very popular, interesting and useful research article, Bailey [2], by employing classical summation theorems such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$; Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, established a large number of very interesting results (known as well as new) involving products of generalized hypergeometric series.

It is not out of place to mention here that recently a good deal of progress has been done in the direction of generalizing the above mentioned classical summation theorems. For details, we refer to [3–5].

In this research paper, we are interested in the following classical Saalschütz summation theorem [1]:

$$ {}_3F_2\left[\begin{matrix} a, b, -n; \\ c, 1+a+b-c-n; \end{matrix}\quad 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}. \tag{1.4}$$

By utilizing (1.4), Bailey [2] obtained the following three interesting quadratic transformations:

$$ (1-x)^{-2a}{}_2F_1\left[\begin{matrix} a, b; \\ a+b+\frac{1}{2}; \end{matrix}\quad -\tfrac{4x}{(1-x)^2}\right] = {}_2F_1\left[\begin{matrix} 2a, a-b+\frac{1}{2}; \\ a+b+\frac{1}{2}; \end{matrix}\quad x\right], \tag{1.5}$$

$$ (1-x)^{1-2a}{}_2F_1\left[\begin{matrix} a, b; \\ a+b+\frac{1}{2}; \end{matrix}\quad -\tfrac{4x}{(1-x)^2}\right] = {}_3F_2\left[\begin{matrix} 2a-1, a+\frac{1}{2}, a-b-\frac{1}{2}; \\ a-\frac{1}{2}, a+b+\frac{1}{2}; \end{matrix}\quad x\right] \tag{1.6}$$

and

$$ \begin{aligned} &(1-x)^{1-2a}{}_3F_2\left[\begin{matrix} a, a-\frac{1}{2}, e_1+e_2-2a; \\ e_1, e_2; \end{matrix}\quad -\tfrac{4x}{(1-x)^2}\right] \\ &\quad = {}_3F_2\left[\begin{matrix} 2a-1, 2a-e_1, 2a-e_2; \\ e_1, e_2; \end{matrix}\quad x\right]. \end{aligned} \tag{1.7}$$

The transformation formula (1.7) is originally due to Whipple [6] who obtained it by other means.

It is interesting to mention here that in (1.7), (i) if we replace a by $a+\frac{1}{2}$ and take $e_1 \to a+b+\frac{1}{2}$ and $e_2 \to a+\frac{1}{2}$ and (ii) if we take $e_1 \to a-\frac{1}{2}$ and $e_2 \to a+b+\frac{1}{2}$ and simplify, we respectively recover (1.5) and (1.6).

Very recently, Rakha and Rathie [7] established the extension of Saalschütz summation theorem (1.4) in the form

$$ {}_4F_3\left[\begin{matrix} a, b, d+1, -n; \\ c+1, 1+a+b-c-n, d; \end{matrix}\quad 1\right] = \frac{(c-a)_n(c-b)_n(g+1)_n}{(c+1)_n(c-a-b)_n(g)_n}, \tag{1.8}$$

where

$$ g = \frac{f(b-c)}{b-f} \qquad \text{and} \qquad f = \frac{d(a-c)}{a-d}.$$

Also, in the same paper [7], as an applications, they have obtained the following results:

$$(1-x)^{-2a}{}_3F_2\left[\begin{matrix} a,b,d+1; \\ a+b+\frac{3}{2},d; \end{matrix}\quad -\frac{4x}{(1-x)^2}\right]$$
$$= {}_4F_3\left[\begin{matrix} 2a,a-b-\frac{1}{2},1+a-A,1+a+A; \\ a+b+\frac{3}{2},a-A,a+A; \end{matrix}\quad x\right], \tag{1.9}$$

where

$$A^2 = \frac{1}{b-d}\left(a^2b - abd - \frac{1}{2}bd - \frac{1}{4}d\right) \tag{1.10}$$

and

$$(1-x)^{1-2a}{}_3F_2\left[\begin{matrix} a,b,d+1; \\ a+b+\frac{3}{2},d; \end{matrix}\quad -\frac{4x}{(1-x)^2}\right]$$
$$= {}_5F_4\left[\begin{matrix} 2a-1,a+\frac{1}{2},a-b-\frac{3}{2},a+\frac{1}{2}-A,a+\frac{1}{2}+A; \\ a-\frac{1}{2},a+b+\frac{3}{2},a-\frac{1}{2}-A,a-\frac{1}{2}+A; \end{matrix}\quad x\right], \tag{1.11}$$

where

$$A^2 = \frac{1}{b-d}\left(a^2b - ab - d - abd - \frac{1}{2}bd + \frac{1}{4}b\right). \tag{1.12}$$

The results (1.9) and (1.11) may be regarded as the extensions of (1.5) and (1.6) as it can be seen by taking $d = a + b + \frac{1}{2}$.

The aim of this research is twofold. First, by utilizing the extension of Saalschütz's summation theorem (1.8), we obtain a natural extension of Whipple's transformation (1.7). Then, by employing the beta integral method, we obtain a new hypergeometric identity. The results derived in this paper are simple, easily established and may be potentially useful.

2 Demonstration of the beta integral method

The beta function $\mathrm{B}(\alpha,\beta)$ is defined by the first integral and is known to be evaluated as the second one as follows:

$$\mathrm{B}(\alpha,\beta) = \begin{cases} \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt & (\Re(\alpha) > 0; \Re(\beta) > 0); \\ \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\beta \in \mathbb{C}\setminus\mathbb{Z}_0^-). \end{cases} \tag{2.1}$$

Krattenthaler and Rao [8] made a systematic use of the so-called *beta integral method*, a method of deriving *new* hypergeometric identities from *old* ones by mainly using the beta integral in (2.1) based on the *Mathematica Package* HYP, to illustrate several interesting identities for the hypergeometric series and Kampé de Fériet series in most cases of unit arguments.

In this section, we also apply the beta integral method to the known results (1.7), (1.9) and (1.11) to get new hypergeometric identities. However, we shall derive one identity in detail and others can be obtained similarly.

Thus, for example, let us consider (1.7) in the form

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, a-\frac{1}{2}, e_1+e_2-2a; \\ e_1, e_2; \end{matrix} \; -\frac{4x}{(1-x)^2}\right] \\
&\quad = (1-x)^{2a-1}{}_3F_2\left[\begin{matrix} 2a-1, 2a-e_1, 2a-e_2; \\ e_1, e_2; \end{matrix} \; x\right].
\end{aligned}
\tag{2.2}
$$

Now, multiplying both sides of equation (2.2) by $x^{c-1}(1-x)^{e-c-1}$, integrating the resulting equation with respect to x from 0 to 1, expressing the involved ${}_3F_2$ as series, changing the order of integration and summation (which is easily seen to be justified due to the uniform convergence of the series involved in the process) and using the beta integral (2.1), then after some simplification, summing up the resulting series, we get the following identity (presumably new).

Corollary 1 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$
\begin{aligned}
&{}_5F_4\left[\begin{matrix} a, a-\frac{1}{2}, e_1+e_2-2a, c, 1-e; \\ e_1, e_2, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e; \end{matrix} \; 1\right] \\
&\quad = \frac{\Gamma(e)\Gamma(2a+e-c-1)}{\Gamma(e-c)\Gamma(2a+e-1)}{}_4F_3\left[\begin{matrix} 2a-1, 2a-e_1, 2a-e_2, c; \\ e_1, e_2, 2a+e-1; \end{matrix} \; 1\right].
\end{aligned}
\tag{2.3}
$$

Following the same procedure, from (2.1) and known results (1.5) and (1.6), we get the following identities (presumably new).

Corollary 2 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, b, c, 1-e; \\ a+b+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e; \end{matrix} \; 1\right] \\
&\quad = \frac{\Gamma(e)\Gamma(2a+e-c)}{\Gamma(e-c)\Gamma(2a+e)}{}_3F_2\left[\begin{matrix} 2a, a-b+\frac{1}{2}, c; \\ a+b+\frac{1}{2}, 2a+e; \end{matrix} \; 1\right].
\end{aligned}
\tag{2.4}
$$

Corollary 3 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, b, c, 1-e; \\ a+b+\frac{1}{2}, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e; \end{matrix} \; 1\right] \\
&\quad = \frac{\Gamma(e)\Gamma(2a+e-c-1)}{\Gamma(e-c)\Gamma(2a+e-1)}{}_4F_3\left[\begin{matrix} 2a-1, a+\frac{1}{2}, a-b-\frac{1}{2}, c; \\ a-\frac{1}{2}, a+b+\frac{1}{2}, 2a+e-1; \end{matrix} \; 1\right].
\end{aligned}
\tag{2.5}
$$

We conclude this section by remarking that the results (2.4) and (2.5) can also be obtained from (2.3) by (i) replacing a by $a+\frac{1}{2}$ and taking $e_1 \to a+b+\frac{1}{2}$ and $e_2 \to a+\frac{1}{2}$ and (ii) taking $e_1 \to a-\frac{1}{2}$ and $e_2 \to a+b+\frac{1}{2}$ respectively.

3 Extension of Whipple's transformation (1.7)

The extension of Whipple's quadratic transformation (1.7) to be established in this paper is given in the following theorem.

Theorem 1 *The following extension of Whipple's transformation* (1.7) *holds true*.

$$(1-x)^{1-2a}{}_4F_3\left[\begin{matrix} a, a-\frac{1}{2}, e_1+e_2-2a, d+1; \\ e_1+1, e_2, d; \end{matrix} \quad -\frac{4x}{(1-x)^2}\right]$$
$$= {}_5F_4\left[\begin{matrix} 2a-1, 2a-e_1-1, 2a-e_2, a+\frac{1}{2}-A, a+\frac{1}{2}+A; \\ e_1+1, e_2, a-\frac{1}{2}-A, a-\frac{1}{2}+A; \end{matrix} \quad x\right], \tag{3.1}$$

where A is given by

$$A^2 = \left(a-\frac{1}{2}\right)^2 - \frac{d(e_2-2a)(2a-e_1-1)}{e_1+e_2-2a-d}. \tag{3.2}$$

Proof In order to establish (3.1), we proceed as follows. Denote the left-hand side of (3.1) by $\mathbf{S}$, then upon expressing the ${}_4F_3$ as a series given by the definition (1.1), after some simplification, we have

$$\mathbf{S} = \sum_{n=0}^{\infty} \frac{(a)_n(a-\frac{1}{2})_n(e_1+e_2-2a)_n(d+1)_n(-1)^n 2^{2n}}{(e_1+1)_n(e_2)_n(d)_n n!} x^n(1-x)^{1-2a-2n}.$$

Using the well-known binomial theorem

$${}_1F_0\left[\begin{matrix} a; \\ -; \end{matrix} \quad z\right] = (1-z)^{-a} = \sum_{m=0}^{\infty} \frac{(a)_m}{m!} z^m,$$

we have

$$\mathbf{S} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(a)_n(a-\frac{1}{2})_n(e_1+e_2-2a)_n(d+1)_n(2a+2n-1)_m(-1)^n 2^{2n}}{(e_1+1)_n(e_2)_n(d)_n n! m!} x^{n+m}.$$

Now, replacing m by $m-n$, making use of the result [1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k)$$

and then using the identity

$$(m-n)! = (-1)^n \frac{m!}{(-m)_n},$$

we have, after some simplification,

$$\mathbf{S} = \sum_{m=0}^{\infty} \frac{(2a-1)_m}{m!} x^m \sum_{n=0}^{m} \frac{(-m)_n(e_1+e_2-2a)_n(2a+m-1)_n(d+1)_n}{(e_1+1)_n(e_2)_n(d)_n n!}.$$

Summing up the inner series, we have

$$\mathbf{S} = \sum_{m=0}^{\infty} \frac{(2a-1)_m}{m!} x^m {}_4F_3 \left[\begin{matrix} -m, e_1+e_2-2a, 2a+m-1, d+1; \\ e_1+1, e_2, d; \end{matrix} \quad 1 \right].$$

Finally, using the extension of Saalschütz's summation theorem (1.8) and after *much* simplification, we get

$$\mathbf{S} = \sum_{m=0}^{\infty} \frac{(2a-1)_m(2a-e_1-1)_m(2a-e_2)_m(a+\frac{1}{2}-A)_m(a+\frac{1}{2}+A)_m}{(e_1+1)_m(e_2)_m(a-\frac{1}{2}-A)_m(a-\frac{1}{2}+A)_m m!} x^m,$$

where A is the same as given in (3.2). Finally, summing up the series with the definition (1.1), we easily arrive at the right-hand side of (3.1). This completes the proof of Theorem 1. □

Remark If we equate the coefficients of x^n on both sides of equation (3.1), we get the following interesting identity (presumably new):

$$\begin{aligned} &{}_4F_3 \left[\begin{matrix} -n, e_1+e_2-2a, 2a+n-1, d+1; \\ e_1+1, e_2, d; \end{matrix} \quad 1 \right] \\ &\quad = \frac{(2a-e_1-1)_n(2a-e_2)_n(a+\frac{1}{2}-A)_n(a+\frac{1}{2}+A)_n}{(e_1+1)_n(e_2)_n(a-\frac{1}{2}-A)_n(a-\frac{1}{2}+A)_n}, \end{aligned} \tag{3.3}$$

where A is the same as given in (3.2).

Further, in (3.3), if we take $d = e_1$, we get the following identity (presumably new):

$${}_3F_2 \left[\begin{matrix} -n, e_1+e_2-2a, 2a+n-1; \\ e_1, e_2; \end{matrix} \quad 1 \right] = \frac{(2a-e_1)_n(2a-e_2)_n}{(e_1)_n(e_2)_n}, \tag{3.4}$$

which can also be obtained directly by equating coefficients of x^n in Whipple's transformation (1.7).

We remark in passing that if in Theorem 1 we take $d = e_1$, so that $A = a - e_1 - \frac{1}{2}$, after little simplification, we recover Whipple's transformation (1.7).

4 Application

As already explained in detail in Section 2, using the beta integral method to our main result (3.1), it is not difficult to obtain the following hypergeometric identity which is given here without a proof.

Corollary 4 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$\begin{aligned} &{}_6F_5 \left[\begin{matrix} a, a-\frac{1}{2}, e_1+e_2-2a, d+1, c, 1-e; \\ e_1+1, e_2, d, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e; \end{matrix} \quad 1 \right] \\ &\quad = \frac{\Gamma(e)\Gamma(2a+e-c-1)}{\Gamma(e-c)\Gamma(2a+e-1)} \end{aligned}$$

$$\times {}_6F_5\left[\begin{matrix}2a-1, 2a-e_1-1, 2a-e_2, a+\frac{1}{2}-A, a+\frac{1}{2}+A, c; \\ e_1+1, e_2, a-\frac{1}{2}-A, a-\frac{1}{2}+A, 2a+e-1;\end{matrix}\quad 1\right], \tag{4.1}$$

where, of course, A is the same as defined in (3.2).

Here we mention two interesting special cases of (4.1).

(1) In (4.1), if we replace a by $a+\frac{1}{2}$ and take $e_1 \to a+b+\frac{1}{2}$ and $e_2 \to a+\frac{1}{2}$, we get the following transformation.

Corollary 5 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}a, b, d+1, c, 1-e; \\ a+b+\frac{3}{2}, d, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e;\end{matrix}\quad 1\right] \\ &\quad = \frac{\Gamma(e)\Gamma(2a+e-c)}{\Gamma(e-c)\Gamma(2a+e)}{}_5F_4\left[\begin{matrix}2a, a-b-\frac{1}{2}, 1+a-A, 1+a+A, c; \\ a+b+\frac{3}{2}, a-A, a+A, 2a+e;\end{matrix}\quad 1\right].\end{aligned} \tag{4.2}$$

(2) In (4.1), if we take $e_1 \to a+b+\frac{1}{2}$ and $e_2 \to a-\frac{1}{2}$, we get the following result.

Corollary 6 *For a to be a negative integer, the following hypergeometric transformation holds true.*

$$\begin{aligned}&{}_5F_4\left[\begin{matrix}a, b, d+1, c, 1-e; \\ a+b+\frac{3}{2}, d, \frac{1}{2}+\frac{1}{2}c-\frac{1}{2}e, 1+\frac{1}{2}c-\frac{1}{2}e;\end{matrix}\quad 1\right] \\ &\quad = \frac{\Gamma(e)\Gamma(2a+e-c-1)}{\Gamma(e-c)\Gamma(2a+e-1)} \\ &\qquad \times {}_6F_5\left[\begin{matrix}2a-1, a+\frac{1}{2}, a-b-\frac{3}{2}, a+\frac{1}{2}-A, a+\frac{1}{2}+A, c; \\ a-\frac{1}{2}, a+b+\frac{3}{2}, a-A-\frac{1}{2}, a+A-\frac{1}{2}, 2a+e-1;\end{matrix}\quad 1\right].\end{aligned} \tag{4.3}$$

We conclude this section by remarking that the results (4.2) and (4.3) can also be obtained from (1.9) and (1.11) by applying the beta integral method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to this paper. They read and approved the final paper.

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