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Numerical oscillations for first-order nonlinear delay differential equations in a hematopoiesis model

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Abstract
In this paper, we consider the oscillations of numerical solutions for the nonlinear delay differential equations in a hematopoiesis model. Using two θ-methods, namely the linear θ-method and the one-leg θ-method, several conditions, under which the numerical solutions oscillate, are obtained. Moreover, it is proved that every non-oscillatory numerical solution tends to an equilibrium point of the original system. Some numerical experiments are provided to support the theoretical results.
MSC: 65L05; 65L20

Keywords: nonlinear delay differential equations; numerical solutions; oscillations; non-oscillations

1 Introduction

In recent years, there has been much research activity concerning the oscillatory behavior of solutions of difference equations [1, 2], differential equations with piecewise continuous arguments (EPCA) [3, 4], dynamic equations [5, 6] and partial differential equations [7, 8]. Among these investigations, oscillations of solutions of delay differential equations (DDEs) have also been the subject of many recent studies [9–12]. The strong interest in this study is motivated by the fact that it has many useful applications in some mathematical models such as biology, ecology, spread of some infectious diseases in humans and so on. For more information on this study, the reader can see [13, 14] and the references therein.

Relative to the investigation of the oscillations of the analytic solutions, much research has been focused on the corresponding behavior of the numerical solutions for DDEs. In [15, 16], oscillations of numerical solutions in θ-methods and Runge-Kutta methods for a linear EPCA $x'(t) + ax(t) + a_1x([t-1]) = 0$ were considered, respectively. Wang *et al.* [17] studied numerical oscillations of alternately advanced and retarded EPCA, the conditions of oscillations for the θ-methods are obtained. To the best of our knowledge, until now very few results dealing with the oscillations of the numerical solutions for nonlinear DDEs have been reported except for [18]. Different from [18], in our paper, we consider another nonlinear DDEs in a hematopoiesis model and get some new results.

In this paper, we consider the following equation:

$$x'(t) + px(t) - \frac{qx(t)}{r + x^{\alpha}(t-\tau)} = 0, \quad t \geq 0, \tag{1}$$

with the conditions

$$p, q, r, \tau \in (0, \infty), \qquad \alpha \in \mathbf{N} = \{1, 2, \ldots\}, \qquad q/p > r. \tag{2}$$

Equation (1) is one of the models proposed by Nazarenko [19] to study the control of a single population of cells. Here $x(t)$ is the size of the population of cells, and cells are lost at a rate p, and the function

$$F(u) = \frac{qu}{r + u^{\alpha}(t - \tau)}$$

is the flux function, which depends on the size of cells $x(t)$ and $x(t - \tau)$ at times t and $t - \tau$, respectively, and τ is the time of maturation. The model (1) has been recently investigated by several researchers. Kubiaczyk and Saker [20] considered Equation (1) and gave a sufficient condition for oscillations of all solutions about the positive unique equilibrium point K and proved that every non-oscillatory positive solution of Equation (1) tends to K as $t \to \infty$. Following up the investigation in [20], Saker and Agarwal [21] studied the oscillations and global attractivity of Equation (1) with time periodic coefficients. Song *et al.* [22] considered the existence of local and global Hopf bifurcations of Equation (1). Up to now, few results on the properties of numerical solutions for Equation (1) were established. In the present paper, we investigate some sufficient conditions under which the numerical solutions are oscillatory. We also consider the asymptotic behavior of non-oscillatory numerical solutions.

The remainder of this paper is organized as follows. In the next section, some necessary concepts and results for oscillations of the analytic solutions are given. In Section 3, we obtain a recurrence relation by applying the θ-methods to the simplified form which comes from making an invariant oscillation transformation on Equation (1). Moreover, the oscillations of the numerical solutions are discussed and conditions under which the numerical solutions oscillate are obtained. In Section 4, we investigate the asymptotic behavior of non-oscillatory solutions, and the results of some numerical experiments are given in Section 5. Finally, conclusions are drawn in Section 6.

2 Preliminaries

Definition 2.1 A solution $x(t)$ of Equation (1) is said to oscillate about K if $x(t) - K$ has arbitrarily large zeros. Otherwise, $x(t)$ is called non-oscillatory. When $K = 0$, we say that $x(t)$ oscillates about zero or simply oscillates.

Definition 2.2 A sequence $\{x_n\}$ is said to oscillate about $\{y_n\}$ if $\{x_n - y_n\}$ is neither eventually positive nor eventually negative. Otherwise, $\{x_n\}$ is called non-oscillatory. If $\{y_n\} = \{y\}$ is a constant sequence, we simply say that $\{x_n\}$ oscillates about $\{y\}$. When $\{y_n\} = \{0\}$, we say that $\{x_n\}$ oscillates about zero or simply oscillates.

Definition 2.3 We say Equation (1) oscillates if all of its solutions are oscillatory.

Theorem 2.4 (see [23]) *Consider the difference equation*

$$a_{n+1} - a_n + \sum_{j=-k}^{l} q_j a_{n+j} = 0, \tag{3}$$

assume that $k, l \in \mathbf{N}$ *and* $q_j \in \mathbf{R}$ *for* $j = -k, \dots, l$. *Then the following statements are equivalent*:

(i) *Every solution of Equation* (3) *oscillates*;

(ii) *The characteristic equation* $\lambda - 1 + \sum_{j=-k}^{l} q_j \lambda^j = 0$ *has no positive roots.*

Theorem 2.5 (see [23]) *Consider the difference equation*

$$a_{n+1} - a_n + \omega a_{n-k} = 0, \tag{4}$$

where $n = 0, 1, 2, \dots$, $\omega \in \mathbf{R}$ *and* $k \in \mathbf{Z}$. *Then every solution of Equation* (4) *oscillates if and only if one of the following conditions holds*:

(i) $k = -1$ *and* $\omega \leq -1$;

(ii) $k = 0$ *and* $\omega \geq 1$;

(iii) $k \in \{\dots, -3, -2\} \cup \{1, 2, \dots\}$ *and* $\omega (k+1)^{k+1}/k^k > 1$.

Lemma 2.6 *For* $x > -1$ *and* $x \neq 0$, *we have* $\ln(1+x) > x/(1+x)$.

Lemma 2.7 *For* $x < -1$ *and* $x \neq 0$, *we have* $e^x < 1/(1-x)$.

Lemma 2.8 (see [24]) *For all* $m \geq M$,

(i) $(1 + a/(m - \theta a))^m \geq e^a$ *if and only if* $1/2 \leq \theta \leq 1$ *for* $a > 0$, $\varphi(-1) \leq \theta \leq 1$ *for* $a < 0$;

(ii) $(1 + a/(m - \theta a))^m < e^a$ *if and only if* $0 \leq \theta < 1/2$ *for* $a < 0$, $0 \leq \theta \leq \varphi(1)$ *for* $a > 0$,

where $\varphi(x) = 1/x - 1/(e^x - 1)$ *and* M *is a positive constant.*

Theorem 2.9 (see [20]) *Assume that condition* (2) *holds, then every solution of Equation* (1) *oscillates about* K *if and only if*

$$\frac{\alpha q K^\alpha}{(r + K^\alpha)^2} \tau > \frac{1}{e}, \tag{5}$$

where

$$K = \left(\frac{q}{p} - r \right)^{1/\alpha}$$

is the unique positive equilibrium point of Equation (1).

For simplicity, let

$$T = \frac{\alpha q K^\alpha}{(r + K^\alpha)^2}, \tag{6}$$

then (5) can be written as

$$T\tau > \frac{1}{e}. \tag{7}$$

3 Oscillations of numerical solutions

3.1 Transformation

We associate an initial condition of the form

$$x(t) = \psi(t), \quad -\tau \le t \le 0, \tag{8}$$

with Equation (1), where $\psi \in C([-\tau, 0], (0, \infty))$, $\psi(0) > 0$.

According to the corresponding method in [20], let us introduce an invariant oscillation transformation $x(t) = Ke^{z(t)}$, then Equation (1) can be reduced to

$$z'(t) + \frac{\alpha q K^\alpha}{(r + K^\alpha)^2} f\bigl(z(t-\tau)\bigr) = 0, \tag{9}$$

where

$$f(u) = \frac{r + K^\mu}{\alpha} \frac{e^{\alpha u} - 1}{r + K^\alpha e^{\alpha u}}.$$

Then $x(t)$ oscillates about K if and only if $z(t)$ oscillates.

3.2 The difference scheme

Applying the linear θ-method and the one-leg θ-method to Equation (9), we obtain the same recurrence relation

$$z_{n+1} = z_n - h\theta \frac{\alpha q K^\alpha}{(r + K^\alpha)^2} f(z_{n+1-m}) - h(1-\theta) \frac{\alpha q K^\alpha}{(r + K^\alpha)^2} f(z_{n-m}), \tag{10}$$

where $0 \le \theta \le 1$, $h = \tau/m$, m is a positive integer. z_{n+1} and z_{n+1-m} are approximations to $z(t)$ and $z(t-\tau)$ of Equation (9) at t_{n+1}, respectively.

Let $z_n = -\ln(x_n/K)$, then we have

$$x_{n+1} = x_n \exp\left(h\theta p \frac{K^\alpha - x_{n+1-m}^\alpha}{r + x_{n+1-m}^\alpha} + h(1-\theta) p \frac{K^\alpha - x_{n-m}^\alpha}{r + x_{n-m}^\alpha} \right). \tag{11}$$

Definition 3.1 We call the iteration formula (11) the exponential θ-method for Equation (1), where $\theta \in [0,1]$, $h = \tau/m$, $m \in \mathbf{N} = \{1, 2, \ldots\}$, x_{n+1} and x_{n+1-m} are approximations to $x(t)$ and $x(t-\tau)$ of Equation (1) at t_{n+1}, respectively.

The convergence of the exponential θ-method is given in the following theorem.

Theorem 3.2 *The exponential θ-method* (11) *is convergent with order*

$$\begin{cases} 1, & \textit{when } \theta \ne 1/2, \\ 2, & \textit{when } \theta = 1/2. \end{cases}$$

Proof By the method of steps which is introduced in [25], we can easily get this proof. □

3.3 Oscillation analysis

It is not difficult to show that x_n oscillates about K if and only if z_n is oscillatory. In order to study the oscillations of (11), we only need to consider the oscillations of (10). The following conditions which are taken from [20] will be used in the next analysis:

$$\begin{aligned} &uf(u) > 0, \quad \text{for } u \neq 0, \\ &\lim_{u\to 0} \frac{f(u)}{u} = 1. \end{aligned} \tag{12}$$

For (10), its linearized form is given by

$$z_{n+1} = z_n - h\theta \frac{\alpha q K^\alpha}{(r+K^\alpha)^2} z_{n+1-m} - h(1-\theta) \frac{\alpha q K^\alpha}{(r+K^\alpha)^2} z_{n-m}. \tag{13}$$

Then, by taking into account (6), Equation (13) is equivalent to

$$z_{n+1} - z_n + h\theta T z_{n+1-m} + h(1-\theta) T z_{n-m} = 0. \tag{14}$$

It follows from [23] that (10) oscillates if (14) oscillates under condition (12).

Definition 3.3 The iteration (11) is said to be oscillatory if all of its solutions are oscillatory.

Definition 3.4 We say that the exponential θ-method preserves the oscillations of Equation (1) if Equation (1) oscillates, then there is a $h_0 > 0$ or $h_0 = \infty$ such that (11) oscillates for $h < h_0$. Similarly, we say that the exponential θ-method preserves the non-oscillations of Equation (1) if Equation (1) non-oscillates, then there is a $h_0 > 0$ or $h_0 = \infty$ such that (11) non-oscillates for $h < h_0$.

In the following, we study whether the exponential θ-method preserves the oscillations of Equation (1). That is, when Theorem 2.9 holds, we investigate the conditions under which (11) is oscillatory.

Lemma 3.5 *The characteristic equation of* (13) *is given by*

$$\xi = R\left(-hT\xi^{-m}\right), \tag{15}$$

where the function $R(x) = 1 + x/(1-\theta x)$, θ *is a parameter in the exponential* θ*-method.*

Proof Let $z_n = \xi^n z_0$ in (13), we have

$$\xi^{n+1} z_0 = \xi^n z_0 - h\theta T \xi^{n+1-m} z_0 - h(1-\theta) T \xi^{n-m} z_0.$$

That is,

$$\xi = 1 - h\theta T \xi^{1-m} - h(1-\theta) T \xi^{-m},$$

which is equivalent to

$$\xi = \frac{1 - h(1-\theta)T\xi^{-m}}{1 + h\theta T\xi^{-m}} = 1 - \frac{hT\xi^{-m}}{1 + h\theta T\xi^{-m}}.$$

In view of [26], the stability function of the θ-method is

$$R(x) = \frac{1 + (1-\theta)x}{1 - \theta x} = 1 + \frac{x}{1 - \theta x},$$

then the characteristic equation of (13) is given by (15). This completes the present proof. □

Lemma 3.6 *If* $T\tau > 1/e$, *then the characteristic equation* (15) *has no positive roots for* $0 \le \theta \le 1/2$.

Proof Let $V(\xi) = \xi - R(-hT\xi^{-m})$. By Lemma 2.8, we know that

$$R\bigl(-hT\xi^{-m}\bigr) \le \exp\bigl(-hT\xi^{-m}\bigr), \quad \xi > 0, 0 \le \theta \le 1/2. \tag{16}$$

Now we are going to prove that $W(\xi) = \xi - \exp(-hT\xi^{-m}) > 0$ for $\xi > 0$. Suppose the opposite, that is, there exists a $\xi_0 > 0$ such that $W(\xi_0) \le 0$, then $\xi_0 \le \exp(-hT\xi_0^{-m})$, and

$$\xi_0^m \le \exp\bigl(-hmT\xi_0^{-m}\bigr) = \exp\bigl(-T\tau\xi_0^{-m}\bigr). \tag{17}$$

Multiplying both sides of the inequality (17) by $T\tau e\xi_0^{-m}$, we obtain

$$T\tau e\xi_0^{-m}\xi_0^m \le T\tau e\xi_0^{-m}\exp\bigl(-T\tau\xi_0^{-m}\bigr),$$

which gives

$$T\tau e \le T\tau\xi_0^{-m}\exp\bigl(1 - T\tau\xi_0^{-m}\bigr),$$

therefore we have the following two cases.

Case I: If $1 - T\tau\xi_0^{-m} = 0$, then $T\tau e \le 1$, we arrive at the contradiction with the condition $T\tau > 1/e$.

Case II: If $1 - T\tau\xi_0^{-m} \ne 0$, then according to Lemma 2.7, we get

$$\exp\bigl(1 - T\tau\xi_0^{-m}\bigr) < \frac{1}{1 - (1 - T\tau\xi_0^{-m})} = \frac{1}{T\tau\xi_0^{-m}},$$

that is,

$$T\tau\xi_0^{-m}\exp\bigl(1 - T\tau\xi_0^{-m}\bigr) < 1,$$

so $T\tau e < 1$, which is also a contradiction to $T\tau > 1/e$.

Consequently, for $\xi > 0$,

$$V(\xi) = \xi - R\bigl(-hT\xi^{-m}\bigr) \ge \xi - \exp\bigl(-hT\xi^{-m}\bigr) = W(\xi) > 0,$$

which implies that the characteristic equation (15) has no positive roots. The proof is completed. □

Without loss of generality, in the case of $1/2 < \theta \leq 1$, we assume that $m > 1$.

Lemma 3.7 *If* $T\tau > 1/e$ *and* $1/2 < \theta \leq 1$, *then the characteristic equation* (15) *has no positive roots for* $h < h_0$, *where*

$$h_0 = \begin{cases} \infty, & \text{if } T\tau \geq 1, \\ \tau(1 + \ln T\tau), & \text{if } T\tau < 1. \end{cases}$$

Proof Since $R(-hT\xi^{-m})$ is an increasing function of θ when $\xi > 0$, then for $\xi > 0$ and $1/2 < \theta \leq 1$,

$$R\bigl(-hT\xi^{-m}\bigr) = \frac{1 - h(1-\theta)T\xi^{-m}}{1 + h\theta T\xi^{-m}} \leq \frac{1}{1 + hT\xi^{-m}}.$$

In the following, we prove that the inequality

$$\xi - \frac{1}{1 + hT\xi^{-m}} > 0, \quad \xi > 0, \tag{18}$$

holds under certain conditions.

From (18), it follows that

$$\xi - \frac{1}{1 + hT\xi^{-m}} = \frac{\xi^{1-m}}{1 + hT\xi^{-m}} \rho(\xi),$$

where

$$\rho(\xi) = \xi^m - \xi^{m-1} + hT,$$

so we only need to prove $\rho(\xi) > 0$ for $\xi > 0$. It is easy to know that $\rho(\xi)$ is the characteristic polynomial of the following difference scheme:

$$z_{n+1} - z_n + hTz_{n+1-m} = 0.$$

According to Theorems 2.4 and 2.5, we have that $\rho(\xi)$ has no positive roots if and only if

$$hT \frac{m^m}{(m-1)^{m-1}} > 1,$$

or, equivalently,

$$\ln T\tau + (m-1) \ln \frac{m}{m-1} > 0. \tag{19}$$

We examine two cases depending on the position of $T\tau$: Either $T\tau \geq 1$ or $T\tau < 1$.

Case I: If $T\tau \geq 1$, in view of $m > 1$, then (19) holds true.

Case II: If $T\tau < 1$ and $h < \tau(1 + \ln T\tau)$, then by Lemma 2.6 we obtain

$$\begin{aligned}\ln T\tau + (m-1)\ln\frac{m}{m-1} &= \ln T\tau + (m-1)\ln\left(1 + \frac{1}{m-1}\right)\\ &> \ln T\tau + (m-1)\frac{\frac{1}{m-1}}{1+\frac{1}{m-1}}\\ &= \ln T\tau + \frac{m-1}{m} > 0.\end{aligned}$$

Therefore the inequality (18) holds for $h < h_0$, where

$$h_0 = \begin{cases}\infty, & \text{if } T\tau \geq 1,\\ \tau(1+\ln T\tau), & \text{if } T\tau < 1.\end{cases}$$

So we arrive at

$$V(\xi) = \xi - R\left(-hT\xi^{-m}\right) \geq \xi - \frac{1}{1+hT\xi^{-m}} > 0$$

holds for $h < h_0$ and $\xi > 0$, which implies that the characteristic equation (15) has no positive roots. This completes the proof. □

Remark 3.8 Since $T\tau \in (1/e, 1)$, then $h_0 = \tau(1 + \ln T\tau) > 0$, thus h_0 is meaningful.

In view of (12), Lemmas 3.6, 3.7 and Theorem 2.4, we have the first main theorem of this paper.

Theorem 3.9 *If $T\tau > 1/e$, then* (11) *is oscillatory for*

$$h < \begin{cases}\infty, & \textit{when } 0 \leq \theta \leq 1/2,\\ h_0, & \textit{when } 1/2 < \theta \leq 1,\end{cases}$$

where h_0 is defined in Lemma 3.7.

4 Asymptotic behavior of non-oscillatory solutions

In this section, we investigate the asymptotic behavior of non-oscillatory solutions of (11). The following lemma is a useful result on asymptotic behavior of Equation (1).

Lemma 4.1 (see [20]) *Let $x(t)$ be a positive solution of Equation* (1), *which does not oscillate about K. Then* $\lim_{t\to\infty} x(t) = K$.

From the relationship between Equations (1) and (9), we know that the non-oscillatory solution of Equation (9) satisfies $\lim_{t\to\infty} z(t) = 0$ if Lemma 4.1 holds. Next, we will prove that the numerical solution of Equation (1) can inherit this property.

Lemma 4.2 *Let z_n be a non-oscillatory solution of* (10), *then* $\lim_{n\to\infty} z_n = 0$.

Proof Without loss of generality, we may assume that $z_n > 0$ for sufficiently large n. Then by condition (12) we know that $f(z_i) > 0$ for sufficiently large i. Moreover, it can be seen from (10) that

$$z_{n+1} - z_n + h\theta Tf(z_{n+1-m}) + h(1-\theta)Tf(z_{n-m}) = 0, \tag{20}$$

which gives

$$z_{n+1} - z_n = -h\theta Tf(z_{n+1-m}) - h(1-\theta)Tf(z_{n-m}) < 0,$$

hence $z_{n+1} - z_n < 0$, then $\{z_n\}$ is decreasing. So there exists an $\eta \geq 0$ such that

$$\lim_{n\to\infty} z_n = \eta. \tag{21}$$

Now we are going to prove that $\eta = 0$. If this is not the case, that is, if $\eta > 0$, then there exists $N \in \mathbf{N}$ and $\varepsilon > 0$ such that for $n - m > N$, $0 < \eta - \varepsilon < z_n < \eta + \varepsilon$. Hence $\eta - \varepsilon < z_{n-m}$ and $\eta - \varepsilon < z_{n-1+m}$. So (20) yields

$$\begin{aligned} z_{n+1} - z_n &= -h\theta Tf(z_{n+1-m}) - h(1-\theta)Tf(z_{n-m}) \\ &< -h\theta Tf(\eta-\varepsilon) - h(1-\theta)Tf(\eta-\varepsilon) \\ &= -hTf(\eta-\varepsilon) < 0, \end{aligned}$$

which implies that $z_{n+1} - z_n < A < 0$, where

$$A = \frac{hT(r+K^\alpha)}{\alpha}\frac{1-e^{\alpha(\eta-\varepsilon)}}{r+K^\alpha e^{\alpha(\eta-\varepsilon)}}.$$

Thus $z_n \to -\infty$ as $n \to \infty$, which is a contradiction to (21). Hence, we finish the proof. □

Therefore, the second main theorem of this paper is as follows.

Theorem 4.3 *Let x_n be a positive solution of* (11), *which does not oscillate about K, then* $\lim_{n\to\infty} x_n = K$.

5 Numerical experiments

In this section, we give some numerical examples to illustrate our results, consider the nonlinear DDEs [22]

$$x'(t) + x(t) - \frac{2x(t)}{1+x^2(t-\tau)} = 0, \quad t \geq 0. \tag{22}$$

Obviously, the parameters are $p = 1$, $q = 2$, $r = 1$, $\alpha = 2$ and $q/p = 2 > 1$ in Equation (1) and the positive equilibrium is $K = 1$. In the following, we give three different values of τ and discuss the accuracy of the numerical solution and the oscillatory behavior of Equation (22).

First of all, we consider the equation

$$x'(t) + x(t) - \frac{2x(t)}{1+x^2(t-1.4)} = 0, \quad t \geq 0, \tag{23}$$

Table 1 Comparisons of errors between the exponential θ-method and the Euler method

	Exponential θ-method						Euler method	
	$\theta = 0.2$		$\theta = 0.5$		$\theta = 0.7$			
	AE	RE	AE	RE	AE	RE	AE	RE
$m = 7$	0.0144	0.0112	0.0030	0.0023	0.0174	0.0135	0.0229	0.0177
$m = 14$	0.0085	0.0066	0.0010	7.8091e–4	0.0080	0.0062	0.0140	0.0109
$m = 28$	0.0044	0.0034	5.1373e–4	3.9864e–4	0.0040	0.0031	0.0075	0.0058
$m = 35$	0.0035	0.0027	4.5459e–4	3.5275e–4	0.0032	0.0025	0.0060	0.0047
$m = 70$	0.0016	0.0013	3.7574e–4	2.9156e–4	0.0017	0.0013	0.0029	0.0023

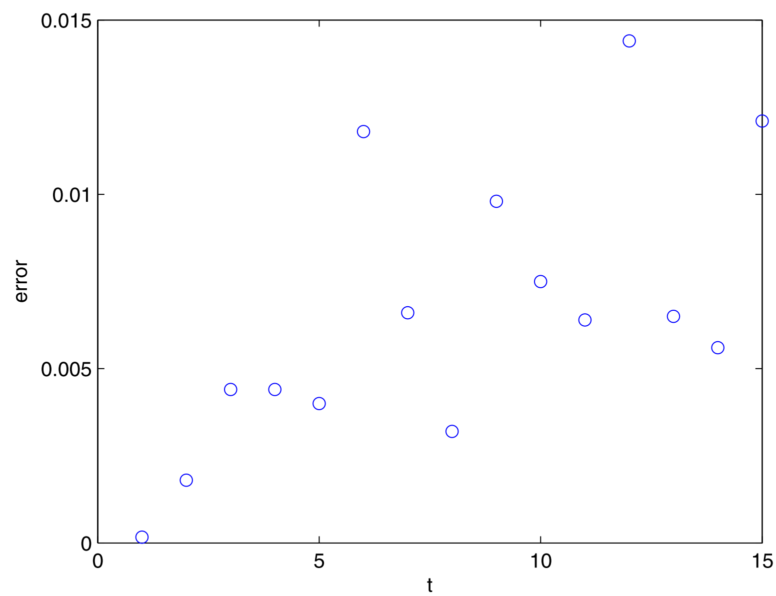

Figure 1 The error as a function of time.

with initial value $x(t) = 1.5$ for $t \leq 0$. Let the stepsize $h = 1.4/m$, we shall use the exponential θ-method with different θ and the Euler method to get the numerical solution at $t = 5$. On the other hand, the exact solution is $x(5) \approx 1.2887$. In Table 1 we have listed the absolute errors (AE) and the relative errors (RE) at $t = 5$. We can see from this table that the errors of the Euler method are larger than those of the exponential θ-method. Therefore, compared with the classical Euler method, the exponential θ-method has higher accuracy. Furthermore, in Figures 1 and 2, the plots of the error as a function of time and as a function of the stepsize for a sequence of stepsizes are presented. The two figures also show that the effect of approximation of the numerical solution is good.

In addition, it is easy to see that condition (5) holds true. That is, the analytic solutions of Equation (23) are oscillatory. In Figures 3-5, we draw the figures of the analytic solutions and the numerical solutions of Equation (23), respectively. In Figure 4, $m = 28$, $\theta = 1/5 \in [0, 1/2]$ and $T\tau = 1.4 > 1/e$. Simultaneously, in Figure 5, $m = 70$, $\theta = 4/5 \in (1/2, 1]$ and $T\tau = 1.4 \geq 1 > 1/e$. From the two figures, we can see that the numerical solutions of Equation (23) oscillate about $K = 1$, which is in agreement with Theorem 3.9.

Next, we consider the following equation:

$$x'(t) + x(t) - \frac{2x(t)}{1 + x^2(t - 0.8)} = 0, \quad t \geq 0, \tag{24}$$

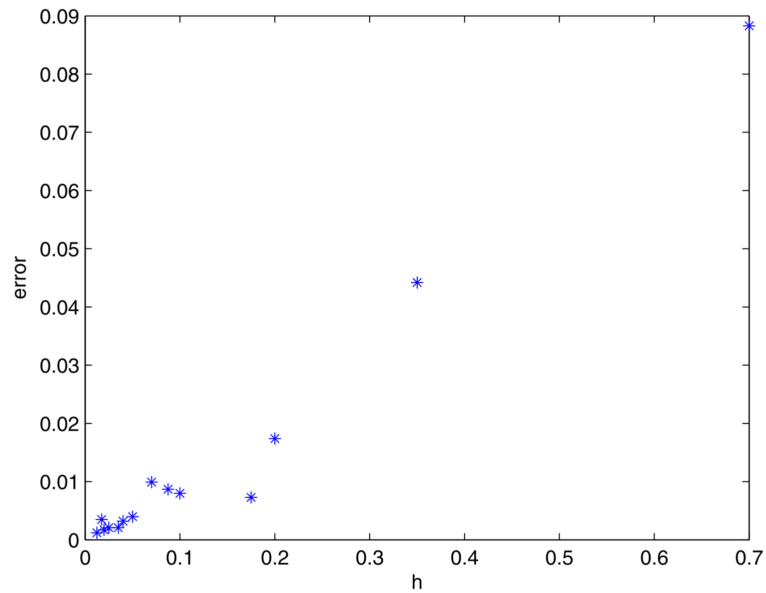

Figure 2 The error as a function of the stepsize.

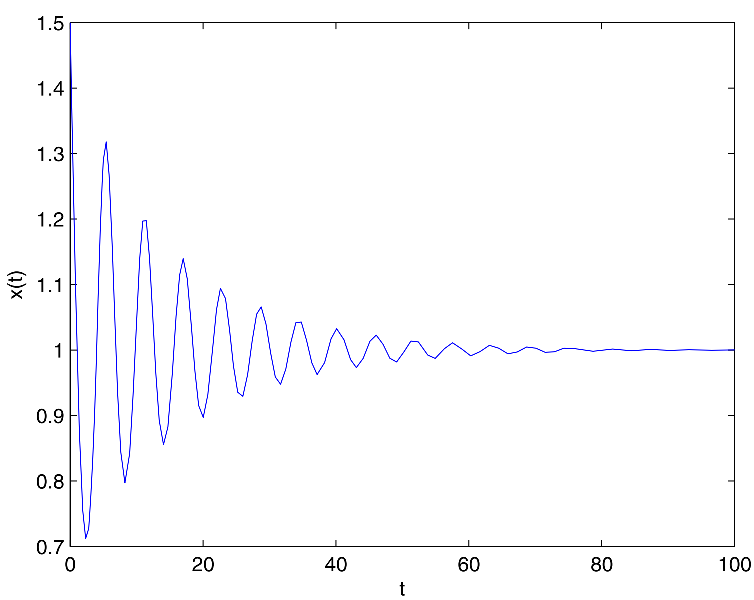

Figure 3 The analytic solution of (23).

with initial value $x(t) = 1.5$ for $t \leq 0$. In Equation (24), it is not difficult to see that condition (5) is fulfilled. That is, the analytic solutions of Equation (24) are oscillatory. In Figures 6-8, we draw the figures of the analytic solutions and the numerical solutions of Equation (24), respectively. In Figure 7, $m = 40$, $\theta = 2/5 \in [0, 1/2]$ and $T\tau = 0.8 > 1/e$. Further, in Figure 8, $m = 20$, $\theta = 3/4 \in (1/2, 1]$. By simple calculation, we have $1/e < T\tau = 0.8 < 1$ and $h = \tau/m = 0.04 < h_0 = 0.6215$. We can see from the three figures that the numerical

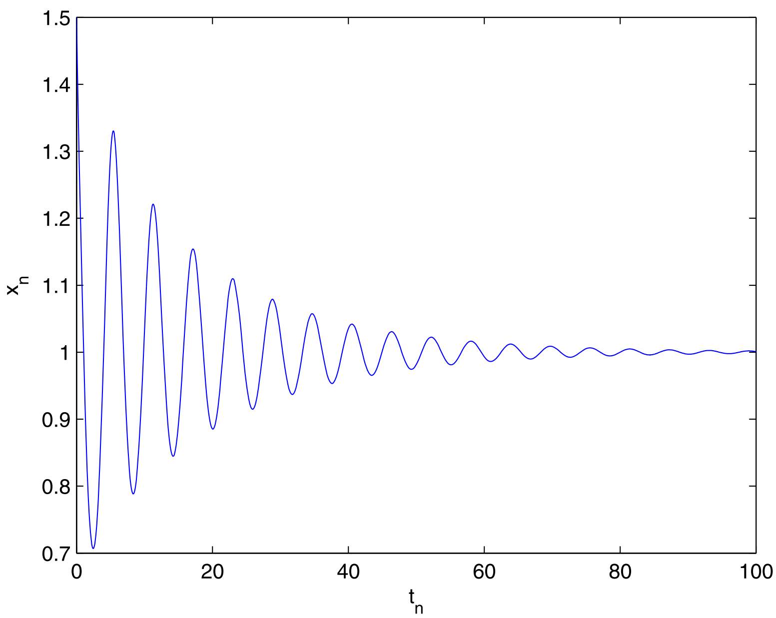

Figure 4 The numerical solution of (23) with $m = 28$ and $\theta = 0.2$.

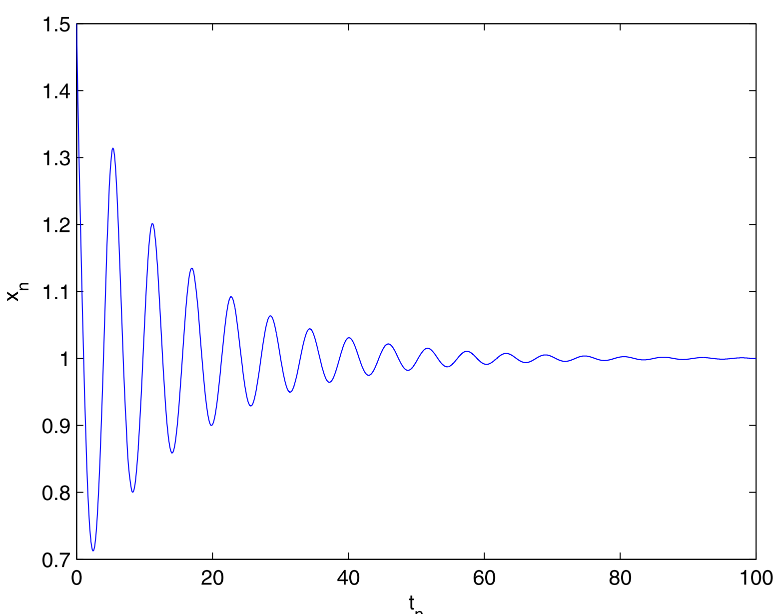

Figure 5 The numerical solution of (23) with $m = 70$ and $\theta = 0.8$.

solutions of Equation (24) oscillate about $K = 1$, which are consistent with Theorem 3.9. On the other hand, we notice that $h = 0.04 \leq 0.4 < h_0 = 0.6215$ under the assumption $1 < m \in \mathbf{N}$, so the stepsize h_0 is not optimal.

Moreover, we consider another equation

$$x'(t) + x(t) - \frac{2x(t)}{1 + x^2(t - 0.3)} = 0, \quad t \geq 0, \tag{25}$$

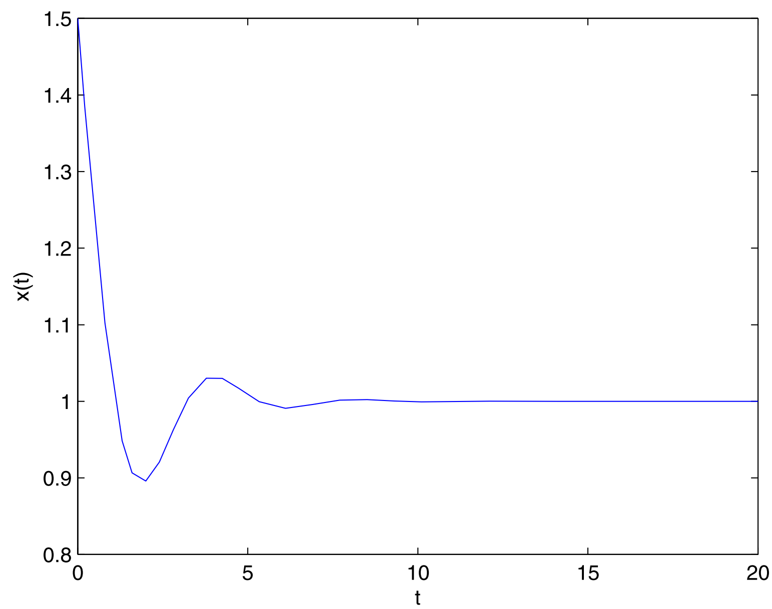

Figure 6 The analytic solution of (24).

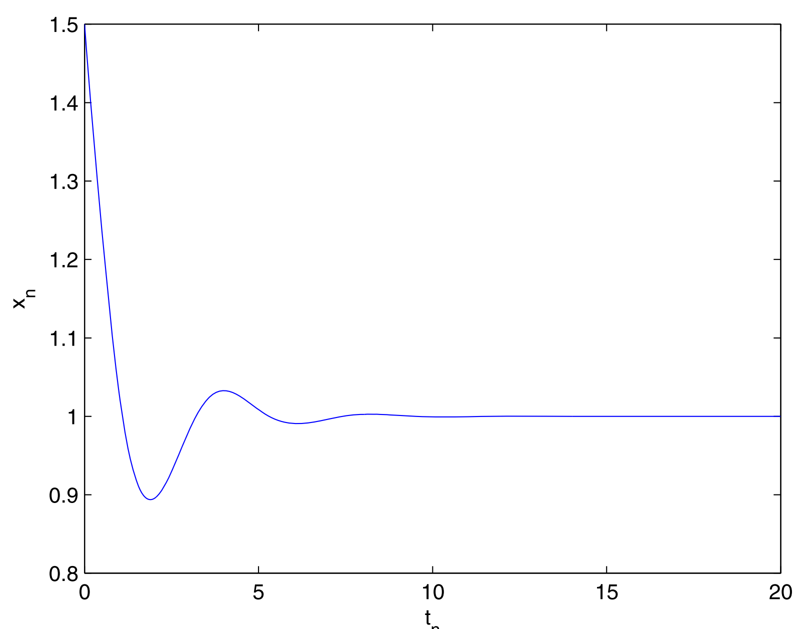

Figure 7 The numerical solution of (24) with $m = 40$ and $\theta = 0.4$.

with initial value $x(t) = 1.5$ for $t \leq 0$. For Equation (25), it is easy to see that $T\tau = 0.3 < 1/e$, so condition (5) is not satisfied. That is, the analytic solutions of Equation (25) are non-oscillatory. In Figures 9-11, we draw the figures of the analytic solutions and the numerical solutions of Equation (25), respectively. In Figure 9, we can see that $x(t) \to K = 1$ as $t \to \infty$. From Figures 10 and 11, we can also see that the numerical solutions of Equation (25)

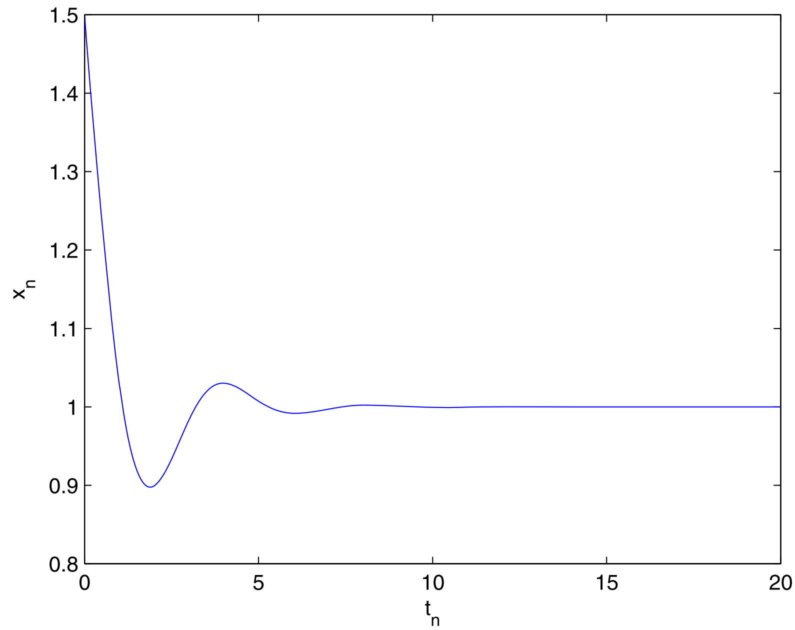

Figure 8 The numerical solution of (24) with $m = 20$ and $\theta = 0.75$.

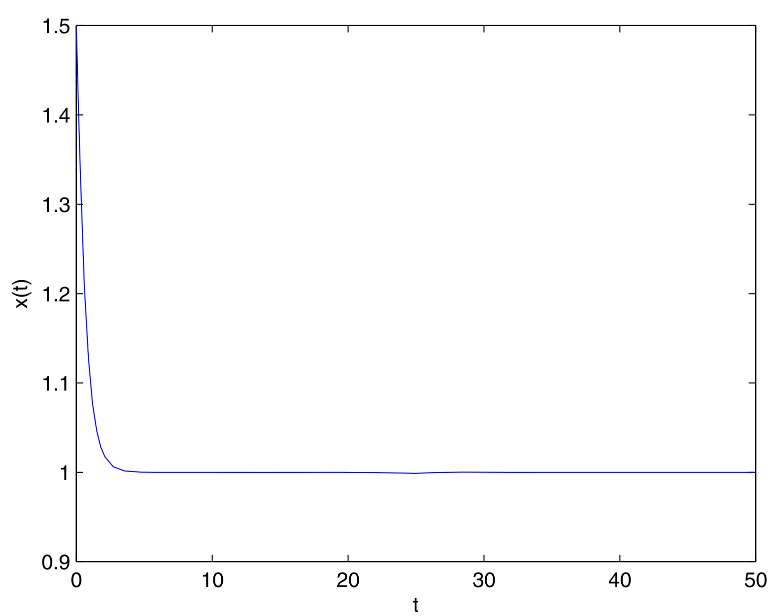

Figure 9 The analytic solution of (25).

satisfy $x_n \to K = 1$ as $n \to \infty$. That is, the numerical method preserves the asymptotic behavior of non-oscillatory solutions of Equation (25), which coincides with Theorem 4.3.

Finally, by Definition 3.4, we can see from these figures that the exponential θ-method preserves the oscillations of Equations (23) and (24) and the non-oscillations of Equation (25), respectively.

All the above numerical examples are in agreement with the main results in this paper.

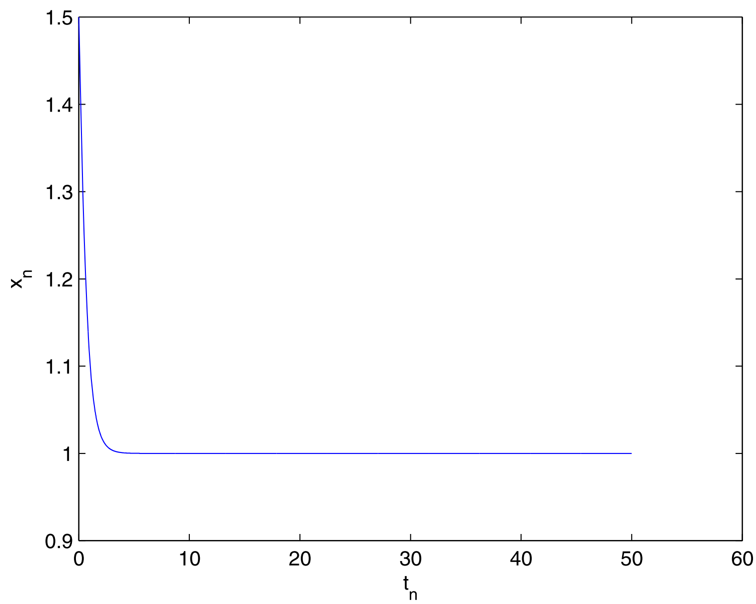

Figure 10 The numerical solution of (25) with $m = 60$ and $\theta = 0.1$.

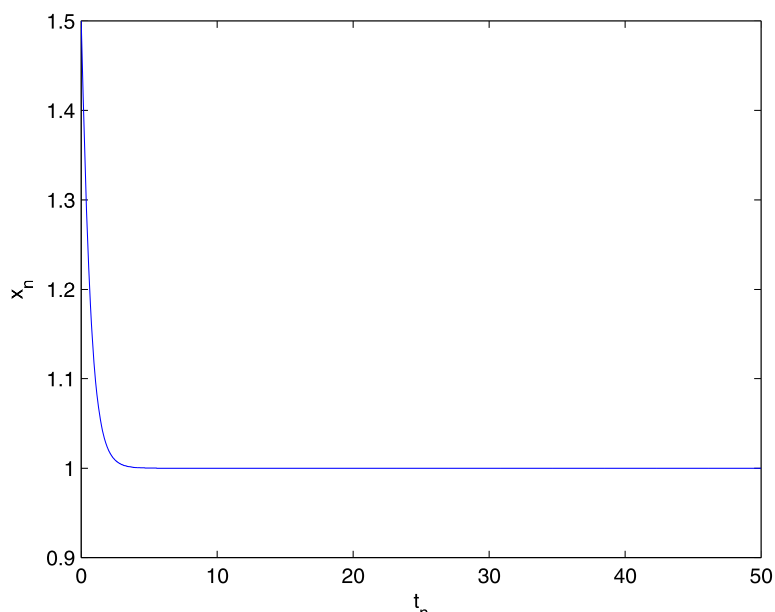

Figure 11 The numerical solution of (25) with $m = 30$ and $\theta = 0.6$.

6 Conclusions

In this paper, we discuss the oscillations of the numerical solutions of a nonlinear

DDEs in a hematopoiesis model. The convergent exponential θ-method, namely the linear θ-method and the one-leg θ-method in an exponential form, is constructed. We obtain some conditions under which the numerical solutions oscillate in the case of oscillations of the analytic solutions. We also prove that non-oscillatory numerical solutions can preserve the corresponding properties of the analytic solutions. It is pointed out that the stepsize h_0 in Lemma 3.7 is not optimal, which gives us the further working direction.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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