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On the structure and the qualitative behavior of an economic model

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Abstract
In this paper, we build an economic model of a non-linear system of difference equations and present a qualitative study for the obtained model, where a mathematical model of a bounded rationality multiple game with an exponential demand function will be introduced, and then we obtain the equilibrium points of the model and classify if they are locally stable or not. Also, we investigate the boundedness and global convergence of solutions for the obtained system.

Keywords: difference equations; economic model; boundedness; global stability

1 Introduction

In the recent years, the study of the bounded rationality duopoly game has attracted a very high attention. In 1998 Bischi and Naimzada [1] introduced the bounded rationality duopoly game as a modification of the original model work of Cournot [2], where they proposed the duopoly game which describes a market with two players producing homogeneous goods, updating their production strategies in order to maximize their profits. Each player thinks with bounded rationality, adjusts his output according to the expected marginal profit, therefore the decision of each player depends on local information about his output. Also, they have studied the bounded rationality duopoly game with a simple case when the demand function and the cost function are linear [1]. Recently, many works of bounded rationality duopoly game have been studied [1, 3–11]. Agiza *et al.* [5] studied the complex dynamics in a bounded rationality duopoly game with a nonlinear demand function and a linear cost function. The asymptotic behavior of the economic model was investigated by El-Metwally [12].

The main aim for this paper is to analyze the dynamics of a nonlinear discrete-time map generated by a bounded rationality duopoly game with an exponential demand function. In Section 2 we present and describe a bounded rationality duopoly game with an exponential demand function. The existence of the equilibrium points of the obtained model and the studying of their local stability are given in Section 3. The boundedness of the solutions is studied in Section 4. Finally, Section 5 is concerned with the global attractivity of the solutions for the obtained system.

Now consider the following first-order system of difference equations:

$$\left.\begin{aligned} x_{n+1} &= f(x_n, y_n), \\ y_{n+1} &= g(x_n, y_n), \end{aligned}\right\} \quad n = 0, 1, 2, \ldots, \qquad (*)$$

where f and g are continuous functions on a subset $S \subset R^2$.

Definition System (∗) is competitive if $f(x;y)$ is non-decreasing in x and non-increasing in y, and $g(x;y)$ is non-increasing in x and non-decreasing in y. If both f and g are non-decreasing in x and y, System (∗) is *cooperative*. *Competitive* and *cooperative* maps are defined similarly. *Strongly competitive* systems of difference equations or *strongly competitive* maps are those for which the functions f and g are coordinate-wise strictly monotone.

Theorem A [13] *Let $T = (f,g)$ be a monotone map on a closed and bounded rectangular region $S \subset R^2$. Suppose that T has a unique fixed point $E = (\overline{x},\overline{y})$ in S. Then E is a global attractor of T on S.*

2 The model

We consider a Cournot duopoly game with q_i denoting the quantity supplied by firm $i = 1,2$. In addition, let $P(q_i + q_j)$, $i \neq j$, denote a twice differentiable and non-increasing inverse demand function and let $C_i(q_i)$ denote the twice differentiable increasing cost function. For the firm i, the profit resulting from the above Cournot game is given by

$$\Pi_i = P(q_i + q_j)q_i - C_i(q_i). \tag{1}$$

Since the information in the oligopoly market is incomplete, the bounded rational players have no complete knowledge of the market, hence they make their output decisions on a local estimate of the expected marginal profit $\frac{\partial \Pi_i}{\partial q_i}$ [14]. If the marginal profit is positive (negative), it increases (decreases) its production q_i at the next period output. Therefore the dynamical equation of the bounded rationality player i has the form

$$q_i(t+1) = q_i(t) + \nu_i q_i(t)\frac{\partial \Pi_i}{\partial q_i(t)}, \quad i = 1,2, \tag{2}$$

where ν_i is a positive parameter which represents the relative speed of adjustment. Bischi and Naimazada studied the dynamical behavior of the bounded duopoly game with a linear demand function [14].

To make the bounded rationality duopoly game more realistic, we assume that the demand function $f(Q)$ has the exponential form (see [15])

$$f(Q) = ae^{-Q} = ae^{-(q_1+q_2)}, \tag{3}$$

where a is a parameter of maximum price in the market. The exponential demand function has the good properties of non-zero or non-negative prices and finite prices when the total quantity in the market Q tends to zero. So, we think that the exponential demand function is a good alternative to the linear demand function and makes the game more realistic. Also, we consider the cost function is linear and is given by

$$c_i(q_i) = c_i q_i, \quad i = 1,2, \tag{4}$$

where c_i is the marginal cost of the ith firm. Thus the profit of the ith firm is given by

$$\Pi_i(q_1,q_2) = aq_i e^{-(q_1+q_2)} - c_i q_i. \tag{5}$$

Then marginal profit of ith firm is

$$\frac{\partial \Pi_i}{\partial q_i(t)} = a(1-q_i)e^{-(q_1+q_2)} - c_i, \quad i = 1, 2. \tag{6}$$

Thus the repeated duopoly game of bounded rationality by using Eq. (2) is given by

$$q_i(t+1) = q_i(t) + \nu_i q_i\bigl(a(1-q_i)e^{-(q_1+q_2)} - c_i\bigr). \tag{7}$$

Therefore the discrete two-dimensional map of the game has the form

$$T: \begin{cases} q_1(t+1) = q_1(t) + \nu_1 q_1(a(1-q_1)e^{-(q_1+q_2)} - c_1), \\ q_2(t+1) = q_2(t) + \nu_2 q_2(a(1-q_2)e^{-(q_1+q_2)} - c_2). \end{cases} \tag{8}$$

Now we can rewrite this system in the following form:

$$\begin{cases} x_{n+1} = (1-\alpha_1)x_n + \beta_1(1-x_n)x_n e^{-(x_n+y_n)}, \\ y_{n+1} = (1-\alpha_2)y_n + \beta_2(1-y_n)y_n e^{-(x_n+y_n)}, \end{cases} \tag{9}$$

where $x_n = q_1(t)$, $y_n = q_2(t)$, $\alpha_i = \nu_i c_i \in (0,\infty)$, and $\beta_i = \nu_i a \in (0,\infty)$, $i = 1, 2$.

3 Local stability of the equilibrium points

In this section, we examine the existence of non-negative equilibrium points of System (9) and then give a powerful criterion for the asymptotic stability of the obtained points.

Proposition 1 (1) *When $\alpha_1 \geq \beta_1$ and $\alpha_2 \geq \beta_2$, System* (9) *has a unique equilibrium point* $E_0 = (0,0)$.

(2) *When $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$, System* (9) *has two equilibrium points $E_1 = (x^*, 0)$and $E_2 = (0, y^*)$, where x^* and y^* satisfy $\alpha_1 = \beta_1(1-x^*)e^{-x^*}$ and $\alpha_2 = \beta_2(1-y^*)e^{-y^*}$, respectively.*

(3) *When $\alpha_1 < \beta_1 e^{\frac{\alpha_2\beta_1}{\alpha_1\beta_2}-1}$, System* (9) *has a unique positive equilibrium point $E_3 = (u^*, v^*)$, where u^* satisfies $\alpha_1 = \rho_1(1-u^*)e^{-\gamma_1 u^*}$, $v^* = \frac{\alpha_2\beta_1}{\alpha_1\beta_2}u^* - \frac{\alpha_2\beta_1}{\alpha_1\beta_2} + 1$, $\gamma_1 = 1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2}$ and $\rho_1 = \beta_1 e^{\gamma_1 - 2}$.*

Proof Observe that the equilibrium points of System (9) are given by the relations

$$\overline{x} = (1-\alpha_1)\overline{x} + \beta_1\overline{x}(1-\overline{x})e^{-(\overline{x}+\overline{y})} \quad \text{and} \quad \overline{y} = (1-\alpha_2)\overline{y} + \beta_2\overline{y}(1-\overline{y})e^{-(\overline{x}+\overline{y})}.$$

Therefore

$$\overline{x} = 0, \quad \alpha_1 = \beta_1(1-\overline{x})e^{-(\overline{x}+\overline{y})}, \quad \overline{y} = 0 \quad \text{and} \quad \alpha_2 = \beta_2(1-\overline{y})e^{-(\overline{x}+\overline{y})}. \tag{10}$$

First, set $g(z) = \alpha_1 - \beta_1(1-z)e^{-z}$. Then

$$g(0) = \alpha_1 - \beta_1, \qquad \lim_{z\to\infty} g(z) = \alpha_1 \quad \text{and} \quad g'(z) = \beta_1(2-z)e^{-z}.$$

Therefore $z = 2$ is the unique critical point of g and $g(2)$ is the absolute maximum of g on $(0,\infty)$. Now we consider the following two cases.

(1) If $\alpha_1 \geq \beta_1$, then $g(z) \geq 0$ for all $z > 0$ and so $g(z)$ has no positive roots. Similarly, it is easy to show that the function $w(z) = \alpha_2 - \beta_2(1 - z)e^{-z}$ has no positive roots provided that $\alpha_2 \geq \beta_2$. Thus System (9) has the unique equilibrium point $(0, 0)$.

(2) If $\alpha_1 < \beta_1$, then $g(0) < 0$ and since $g'(z) > 0$ for all $z \in (0, 2)$, $g(z)$ has a unique positive root. Since $g(1) = \alpha_1 > 0$, the positive root of $g(z)$ lies in $(0, 1)$. So, the equation $\alpha_1 = \beta_1(1 - x^*)e^{-x^*}$ has a unique solution $x^* \in (0, 1)$. Similarly, it is easy to show that the equation $\alpha_2 = \beta_2(1 - y^*)e^{-y^*}$ has a unique solution $y^* \in (0, 1)$ provided that $\alpha_2 < \beta_2$. Therefore System (9) has the equilibrium points $(x^*, 0)$ and $(0, y^*)$ where x^* and y^* satisfy $\alpha_1 = \beta_1(1 - x^*)e^{-x^*}$ and $\alpha_2 = \beta_2(1 - y^*)e^{-y^*}$, respectively.

Second, assume that (u^*, v^*) is a solution of System (10) with $u^* > 0$ and $v^* > 0$. It follows from (10) that u^* and v^* have to be less than one and

$$e^{u^*+v^*} = \frac{\beta_1}{\alpha_1}\left(1 - u^*\right) = \frac{\beta_2}{\alpha_2}\left(1 - v^*\right),$$

which gives that $v^* = \sigma u^* - \sigma + 1$, where $\sigma = \frac{\alpha_2\beta_1}{\alpha_1\beta_2}$. Now set $h(\mu) = \alpha_1 - \rho_1(1 - \mu)e^{-\gamma_1\mu}$, where $\sigma_1 = 1 + \frac{\alpha_2\beta_1}{\alpha_1\beta_2}$ and $\rho_1 = \beta_1 e^{\sigma_1 - 2}$. Similarly to above, one can easily see that h has no positive roots if $\alpha_1 \geq \rho_1$ and it has a unique positive root which lies in $(0, 1)$ whenever $\alpha_1 < \rho_1$. Therefore System (9) has the unique positive equilibrium point (u^*, v^*) where u^* satisfies $\alpha_1 = \rho_1(1 - u^*)e^{-\sigma_1 u*}$ and $v^* = \frac{\alpha_2\beta_1}{\alpha_1\beta_2}u^* - \frac{\alpha_2\beta_1}{\alpha_1\beta_2} + 1$. □

Recall that E_0, E_1 and E_2 are called boundary equilibrium points of System (9) and E_3 is called a Nash equilibrium point of System (9). See [3].

In the following, we deal with the local stability of the equilibrium points of System (9). Now rewrite System (9) as follows:

$$\left.\begin{aligned} x_{n+1} &= F(x_n, y_n) = (1 - \alpha_1)x_n + \beta_1 x_n(1 - x_n)e^{-(x_n+y_n)}, \\ y_{n+1} &= G(x_n, y_n) = (1 - \alpha_2)y_n + \beta_2 y_n(1 - y_n)e^{-(x_n+y_n)}, \end{aligned}\right\} \tag{11}$$

where $F(x, y) = (1 - \alpha_1)x + \beta_1 x(1 - x)e^{-(x+y)}$ and $G(x, y) = (1 - \alpha_2)y + \beta_2 y(1 - y)e^{-(x+y)}$ are continuous functions. Then we obtain

$$\left.\begin{aligned} \tfrac{\partial F(x,y)}{\partial x} &= 1 - \alpha_1 + \beta_1(x^2 - 3x + 1)e^{-(x+y)}, \\ \tfrac{\partial F(x,y)}{\partial y} &= -\beta_1 x(1 - x)e^{-(x+y)}, \\ \tfrac{\partial G(x,y)}{\partial x} &= -\beta_2 y(1 - y)e^{-(x+y)}, \\ \tfrac{\partial G(x,y)}{\partial y} &= 1 - \alpha_2 + \beta_2(y^2 - 3y + 1)e^{-(x+y)}. \end{aligned}\right\} \tag{12}$$

Proposition 2 *The equilibrium point E_0 of System* (9) *is locally asymptotically stable if $\beta_i < \alpha_i < 2 + \beta_i$ for $i = 1, 2$ and it is unstable elsewhere.*

Proof The Jacobian matrix of System (9) about the equilibrium point $E_0(0, 0)$ has the form

$$J(E_0) = \begin{bmatrix} 1 - \alpha_1 + \beta_1 & 0 \\ 0 & 1 - \alpha_2 + \beta_2 \end{bmatrix}.$$

Therefore the eigenvalues of $J(E_0)$ are given by

$$\lambda_1 = 1 - \alpha_1 + \beta_1 \quad \text{and} \quad \lambda_2 = 1 - \alpha_2 + \beta_2.$$

It is well known that the equilibrium point E_0 of System (9) is locally asymptotically stable if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$ are satisfied if $\beta_1 < \alpha_1 < 2 + \beta_1$ and $\beta_2 < \alpha_2 < 2 + \beta_2$. The proof is completed. □

Proposition 3 *The equilibrium points E_1 and E_2 of System* (9) *are saddle points.*

Proof The Jacobian matrix of System (9) about the equilibrium point $E_1(x^*, 0)$ has the form

$$J(E_1) = \begin{bmatrix} 1 - \alpha_1 + \beta_1(x^{*2} - 3x^* + 1)e^{-x^*} & -\beta_1 x^*(1 - x)e^{-x^*} \\ 0 & 1 - \alpha_2 + \beta_2 e^{-x^*} \end{bmatrix}.$$

Thus $J(E_1)$ has the eigenvalues

$$\lambda_1 = 1 - \alpha_1 + \beta_1\bigl(x^{*2} - 3x^* + 1\bigr)e^{-x^*} \quad \text{and} \quad \lambda_2 = 1 - \alpha_2 + \beta_2 e^{-x^*}.$$

Note that

$$\lambda_1 = 1 - \alpha_1 + \beta_1\bigl(x^{*2} - 3x^* + 1\bigr)e^{-x^*} < 1 - \beta_1 x^*\bigl(2 - x^*\bigr)e^{-x^*} < 1$$

and

$$\lambda_2 = 1 - \alpha_2 + \frac{\alpha_2}{1 - x^*} = 1 + \frac{\alpha_2 x^*}{1 - x^*} > 1.$$

Thus it follows that the equilibrium point $E_1(x^*, 0)$ of System (9) is a saddle point. Similarly, one can easily prove that the equilibrium point $E_2(0, y^*)$ of System (9) is also a saddle point. □

Proposition 4 *The Nash equilibrium point E_3 of System* (9) *is asymptotically stable if* $2 < \frac{\alpha_1 u^*(2-u^*)}{1-u^*} + \frac{\alpha_2 v^*(2-v^*)}{1-v^*} < 1 + \frac{\alpha_1\alpha_2 u^* v^*(3-u-v)}{(1-u^*)(1-v^*)}$ *and it is unstable elsewhere.*

Proof The Jacobian matrix of System (9) about the equilibrium point $E_3(u^*, v^*)$ is

$$\begin{aligned} J(E_3) &= \begin{bmatrix} 1 - \alpha_1 + \frac{\alpha_1(u^{*2}-3u^*+1)}{1-u^*} & -\alpha_1 u^* \\ -\alpha_2 v^* & 1 - \alpha_2 + \frac{\alpha_2(v^{*2}-3v^*+1)}{1-v^*} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \frac{\alpha_1 u^*(2-u^*)}{1-u^*} & -\alpha_1 u^* \\ -\alpha_2 v^* & 1 - \frac{\alpha_2 v^*(2-v^*)}{1-v^*} \end{bmatrix}. \end{aligned}$$

By some simple computations, we obtain that

$$\begin{aligned} \operatorname{Tr}\bigl(J(E_3)\bigr) &= 2 - \frac{\alpha_1 u^*(2 - u^*)}{1 - u^*} - \frac{\alpha_2 v^*(2 - v^*)}{1 - v^*}, \\ \operatorname{Det}\bigl(J(E_3)\bigr) &= 1 - \frac{\alpha_1 u^*(2 - u^*)}{1 - u^*} - \frac{\alpha_2 v^*(2 - v^*)}{1 - v^*} + \frac{\alpha_1\alpha_2 u^* v^*(3 - u^* - v^*)}{(1 - u^*)(1 - v^*)}. \end{aligned}$$

It is well known that the Nash equilibrium point E_3 of System (9) is asymptotically stable if $\operatorname{Tr}(J(E_3)) < 0$ and $\operatorname{Det}(J(E_3)) > 0$, *i.e.*, the following condition is satisfied:

$$2 < \frac{\alpha_1 u^*(2 - u^*)}{1 - u^*} + \frac{\alpha_2 v^*(2 - v^*)}{1 - v^*} < 1 + \frac{\alpha_1\alpha_2 u^* v^*(3 - u - v)}{(1 - u^*)(1 - v^*)}.$$

This completes the proof. □

4 Boundedness and invariant

In this section we concern ourselves with the boundedness character of the solutions for System (9). Under appropriate conditions, we give some bounded results related to System (9).

Theorem 5 *Assume that* $\alpha_i + \frac{\beta_i}{e^2} < 1$, $i = 1, 2$. *Then every solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *of System* (9), *with* $x_0 > 0$ *and* $y_0 > 0$, *satisfies that* $x_n > 0$ *and* $y_n > 0$ *for all* $n > 0$.

Proof Let $H_i(x, y)$, $i = 1, 2$, be continuous functions defined by

$$H_i(x, y) = 1 - \alpha_i + \beta_i(1 - x)e^{-(x+y)}, \quad i = 1, 2.$$

Then System (9) can be rewritten in the form

$$x_{n+1} = x_n H_1(x_n, y_n),$$
$$y_{n+1} = y_n H_2(x_n, y_n).$$

Now assume that $\{(x_n, y_n)\}_{n=0}^{\infty}$ is a solution of System (9) with positive initial values. Then it suffices to show that $H_i(x, y)$, $i = 1, 2$, are positive for all $x > 0$, $y > 0$. Observe that

$$\frac{\partial H_i(x, y)}{\partial x} = \beta_i(x - 2)e^{-(x+y)} \quad \text{and} \quad \frac{\partial H_i(x, y)}{\partial y} = -\beta_i(1 - x)e^{-(x+y)}, \quad i = 1, 2.$$

Therefore H_1 and H_2 have no positive critical points. Let a and b be arbitrary positive numbers and consider the domain

$$D = \{(x, y) : 0 \le x \le a, 0 \le y \le b\}.$$

Then for $i = 1, 2$, we see that

$$H_i(0, y) = 1 - \alpha_i + \beta_i e^{-y}, \quad 0 \le y \le b,$$
$$H_i(x, 0) = 1 - \alpha_i + \beta_i(1 - x)e^{-x}, \quad 0 \le x \le a,$$
$$H_i(x, b) = 1 - \alpha_i + \beta_i(1 - x)e^{-(x+b)}, \quad 0 \le x \le a,$$
$$H_i(a, y) = 1 - \alpha_i + \beta_i(1 - a)e^{-(a+y)}, \quad 0 \le y \le b.$$

Using elementary differential calculus, we obtain that the absolute minimum of each one of the above functions is $1 - \alpha_i - \frac{\beta_i}{e^2}$. Therefore $H_i(x, y) \ge 1 - \alpha_i - \frac{\beta_i}{e^2} > 0$ for all $(x, y) \in D$. Since a and b are arbitrary positive numbers, we can conclude that $H_i(x, y) > 0$ for $i = 1, 2$ and for all $(x, y) \in (0, \infty)^2$. □

Theorem 6 *Let* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *be a solution of System* (9) *with* $(x_{n_0}, y_{n_0}) \in (0, 1]^2$ *for some* $n_0 \ge 0$ *and assume, for* $i = 1, 2$, *that one of the following statements is true*:

(i) $\beta_i \le e(1 - \alpha_i)$.
(ii) $e(1 - \alpha_i) < \beta_i \le e$.
(iii) $(\sqrt{\beta_i} - 1)^2 \le \alpha_i$.

Then $(x_n, y_n) \in (0, 1]^2$ *for all* $n \ge n_0$.

Proof Let $n_0 \geq 0$ be such that $x_{n_0} \in (0,1]$. It follows from System (9) that

$$x_{n_0+1} \leq (1-\alpha_1)x_{n_0} + \beta_1(1-x_{n_0})x_{n_0}e^{-x_{n_0}}, \tag{13}$$

$$x_{n_0+1} \leq (1-\alpha_1)x_{n_0} + \beta_1(1-x_{n_0})e^{-1} = \left(1-\alpha_1-\frac{\beta_1}{e}\right)x_{n_0} + \frac{\beta_1}{e} \tag{14}$$

and

$$x_{n_0+1} \leq (1-\alpha_1)x_{n_0} + \beta_1(1-x_{n_0})x_{n_0}. \tag{15}$$

Set $w(x) = (1-\alpha_1)x + \beta_1(1-x)xe^{-x}$ for $x \leq 1$. Then it follows from (13) that $x_{n_0+1} \leq w(x_{n_0})$. Also, we obtain that

$$w'(x) = (1-\alpha_1) + \beta_1\left(x^2-3x+1\right)e^{-x}$$

and

$$\begin{aligned} w''(x) &= \beta_1\left(-x^2+5x-4\right)e^{-x} = -\beta_1(x-1)(x-4)e^{-x} \\ &\leq 0 \quad \text{for all } x \in (0,1]. \end{aligned}$$

Then $w'(x) \geq w'(1) = 1-\alpha_1-\frac{\beta_1}{e}$. If (i) holds, then $w'(1) \geq 0$ and hence $w(x)$ is increasing on $(0,1]$. Therefore $x_{n_0+1} \leq w(1) < 1$. If (ii) holds, then (14) yields $x_{n_0+1} \leq \frac{\beta_1}{e} < 1$.

Now suppose that (iii) holds. In this case, it follows from (15) that $x_{n_0+1} \leq p(x_{n_0})$, where $p(x) = (1-\alpha_1)x + \beta_1 x(1-x)$ for all $x \in (0,1]$. It is not difficult to see that $p(x_*)$ is the absolute maximum of $p(x)$ on $(0,1]$ where $x_* = \frac{1-\alpha_1+\beta_1}{2\beta_1}$. According to (iii) and since $p(x_*) = \frac{(1-\alpha_1+\beta_1)^2}{4\beta_1} \leq 1$, $x_{n_0+1} \leq p(x_*) \leq 1$. That is, in all cases we obtain that whenever $x_{n_0} \leq 1$ yields $x_{n_0+1} \leq 1$. So it is easy to prove by induction that $x_n \in (0,1]$ for all $n \geq 1$. The proof of y_n is similar and so will be omitted. This completes the proof. □

Theorem 7 *For every solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *of System* (9), *the following statements hold*:

(i) $x_n \leq x_{n_0}(1-\alpha_1)^{n-n_0} + \frac{\beta_1}{e\alpha_1}(1-(1-\alpha_1)^{n-n_0})$, $n \geq n_0 \geq 0$.

(ii) $y_n \leq y_{n_0}(1-\alpha_2)^{n-n_0} + \frac{\beta_2}{e\alpha_2}(1-(1-\alpha_2)^{n-n_0})$, $n \geq n_0 \geq 0$.

Proof We obtain, for $n_0 \geq 0$, from System (9) that

$$\begin{aligned} x_{n+1} &\leq (1-\alpha_1)x_n + \beta_1 x_n e^{-x_n} \\ &\leq (1-\alpha_1)x_n + \frac{\beta_1}{e} \quad \text{for all } n \geq n_0. \end{aligned}$$

Then it follows by Theorem 5 and Theorem 6 that Case (i) is true. The proof of Case (ii) is similar and will be omitted. □

The following corollaries are coming immediately from Theorem 7.

Corollary 8 *Assume that* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is a positive solution of System* (9) *with* $(x_{n_0}, y_{n_0}) \in (0, \frac{\beta_1}{\alpha_1 e}] \times (0, \frac{\beta_2}{\alpha_2 e}]$ *for some* $n_0 \geq 0$. *Then* $(x_n, y_n) \in (0, \frac{\beta_1}{\alpha_1 e}] \times (0, \frac{\beta_2}{\alpha_2 e}]$ *for all* $n \geq n_0$.

Corollary 9 *Every positive solution* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *of System* (9) *is bounded. Moreover,*

$$\limsup_{n\to\infty} x_n \le \frac{\beta_1}{\alpha_1 e}$$

and

$$\limsup_{n\to\infty} y_n \le \frac{\beta_2}{\alpha_2 e}.$$

Theorem 10 *Assume that* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is a positive solution of System* (9) *and assume, for* $i = 1, 2$, *that one of the following conditions is true*:

(i) $\beta_i < \alpha_i e$.

(ii) $2 + \nu_i^2 + 2e^{\nu_i} - 4\nu_i - \nu_i e^{\nu_i} > 0$, $1 - \alpha_i + \beta_i e^{-\nu_i}[1 - \nu_i(2e^{-\nu_i} + 1) + \nu_i^2 e^{-\nu_i}]$ *and* $(1 - \alpha)\nu_i + \beta_i \nu_i e^{-\nu_i} - \beta \nu_i^2 e^{-2\nu_i} < 1$, *where* $\nu_i = \frac{\beta_i}{\alpha_i e}$.

Then there exists $n_0 \ge 0$ *such that* $(x_n, y_n) \in (0, 1]^2$ *for all* $n \ge n_0$.

Proof The proof of the theorem, when (i) holds, follows by Corollary 9. Now consider that (ii) is true. Then it follows from Corollary 9 that for every constant $\varepsilon_1 > 0$, there exists $n_0 \ge 0$ such that $x_n \le \frac{\beta_1}{\alpha_1 e} + \varepsilon_1 = \gamma_1$, $n \ge n_0$. Set $\delta_1 = e^{-\gamma_1}$. Since $\delta_1 \to e^{-\nu_1}$ when $\varepsilon_1 \to 0$ and the inequalities in (ii) hold, depending on the continuity in ν_1 of the left-hand side of each inequality in (ii), one can choose ε_1 so small that

$$2 + \gamma_1^2 + \frac{2 - \gamma_1}{\delta_1} - 4\gamma_1 \ge 0, \tag{16}$$

$$1 - \alpha_1 + \beta_1 \delta_1 \left[1 - (2\delta_1 + 1)\gamma_1 + \delta_1 \gamma_1^2\right] \ge 0 \tag{17}$$

and

$$(1 - \alpha_1)\gamma_1 + \beta_1 \gamma_1 \delta_1 - \beta_1 \gamma_1^2 \delta_1^2 \le 1. \tag{18}$$

Now we obtain from System (9) that

$$\begin{aligned} x_{n+1} &= (1 - \alpha_1)x_n + \beta_1 x_n e^{-(x_n + y_n)} - \beta_1 x_n^2 e^{-(x_n + y_n)} \\ &\le (1 - \alpha_1)x_n + \beta_1 x_n e^{-x_n} - \beta_1 \delta_1 x_n^2 e^{-x_n} \\ &= K(x_n), \quad n \ge n_0, \end{aligned}$$

where $K(x) = (1 - \alpha_1)x + \beta_1 e^{-x}(x - \delta_1 x^2)$, $x \le \gamma_1$ and then

$$K'(x) = 1 - \alpha_1 + \beta_1 e^{-x}\left[\delta_1 x^2 - (2\delta_1 + 1)x + 1\right]$$

and

$$K''(x) = -\beta_1 e^{-x}\left[\delta_1 x^2 - (4\delta_1 + 1)x + 2(\delta_1 + 1)\right].$$

On the other hand, the equation

$$\delta_1 x^2 - (4\delta_1 + 1)x + 2(\delta_1 + 1) = 0$$

has the positive roots

$$x_1 = \frac{4\delta_1 + 1 + \sqrt{8\delta_1^2 + 1}}{2\delta_1} \quad \text{and} \quad x_2 = \frac{4\delta_1 + 1 - \sqrt{8\delta_1^2 + 1}}{2\delta_1}.$$

Observe that $x_2 = 2 + \frac{1}{2\delta_1} - \sqrt{2 + \frac{1}{4\delta_1^2}} \geq \gamma_1$ if and only if $(2 + \frac{1}{2\delta_1} - \gamma_1)^2 \geq 2 + \frac{1}{4\delta_1^2}$ which holds by (16). Therefore $x_1 \geq x_2 \geq \gamma_1$. Consequently, $K''(x) < 0$ for all $x \leq \gamma_1$, which yields by (17) that $K'(x) > K'(\gamma_1) \geq 0$. Using the increasing property of $K(x)$ on $(0, \gamma_1)$ and inequality (18), we see that $K(x) \leq K(\gamma_1) \leq 1$. Since $x_n \leq \gamma_1$, it follows that

$$x_{n+1} \leq K(x_n) \leq K(\gamma_1) \leq 1 \quad \text{for all } n \geq n_0.$$

Similarly, one can easily prove that $y_n \in (0, 1]$. This completes the proof. □

Theorem 11 *Assume that* $\{(x_n, y_n)\}_{n=0}^{\infty}$ *is a positive solution of System* (9). *If either*

$$(1 - \alpha_i + \beta_i)^2 < 4\beta_i e^{-2\nu_i}$$

or

$$\frac{\beta_i e^{\nu_i}}{4} + \beta_i(1 - \alpha_i) < 1,$$

where $\nu_i = \frac{\beta_i}{\alpha_i e}$ *for* $i = 1, 2$, *then there exists* $n_0 \geq 0$ *such that* $(x_n, y_n) \in (0, 1]^2$ *for all* $n \geq n_0$.

Proof Assume that γ_1, δ_1 and the function $K(x_n)$ are defined as in the previous proof. Then

$$K(x_n) = (1 - \alpha_1 + \beta_1)x_n - \beta_1 x_n^2 \delta_1^2 = \overline{K}(x_n),$$

where $\overline{K}(x) = (1 - \alpha_1 + \beta_1)x - \beta_1 x^2 \delta_1^2$, $x \leq \gamma_1$. Thus

$$\overline{K}'(x) = 1 - \alpha_1 + \beta_1 - 2\beta_1 x \delta_1^2.$$

Hence, $\overline{K}(x)$ attains its maximum value at $x = \frac{1-\alpha_1+\beta_1}{2\beta_1\delta_1^2}$, that is,

$$\overline{K}(x) \leq \overline{K}\left(\frac{1 -_1 \alpha_1 + \beta_1}{2\beta_1\delta_1^2}\right) = \frac{(1 - \alpha_1 + \beta_1)^2}{4\beta_1\delta_1^2}.$$

Also,

$$\begin{aligned} K(x_n) &= -\beta_1\delta_1 e^{-x_n}\left(x_n - \frac{1}{2\delta_1}\right)^2 + \frac{\beta_1 e^{-x_n}}{4\delta_1} + \beta_1(1 - \alpha_1) \\ &< \frac{\beta_1}{4\delta_1} + \beta_1(1 - \alpha_1), \quad n \geq n_0. \end{aligned}$$

Similarly to the proof of Theorem 10, we can choose ε_1 so small that our assumptions imply

$$\frac{(1 - \alpha_1 + \beta_1)^2}{2\beta_1\delta_1^2} \leq 1 \quad \text{and} \quad \frac{\beta_1}{4\delta_1} + \beta_1(1 - \alpha_1) \leq 1.$$

Therefore we have either

$$x_{n+1} \leq \overline{K}(x_n) \leq \frac{(1-\alpha_1+\beta_1)^2}{2\beta_1\delta_1^2} \leq 1, \quad n \geq n_0$$

or

$$x_{n+1} \leq K(x_n) \leq \frac{\beta_1}{4\delta_1} + \beta_1(1-\alpha_1) \leq 1, \quad n \geq n_0,$$

which is our desired conclusion for x_n. Similarly, one can accomplish the same conclusion for y_n. So, the proof is complete. □

5 Global stability analysis

In this section we are interested in driving conditions under which the equilibrium points of System (9) are attractors of the solutions for System (9).

In the following theorem, we investigate the global attractivity of the equilibrium point $(0,0)$ of System (9).

Theorem 12 *Assume that* $\alpha_i \geq \beta_i$, $i = 1,2$. *Then* $(0,0)$ *is a global attractor of all positive solutions of System* (9).

Proof Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution of System (1). It follows from System (1) that

$$x_{n+1} = (1-\alpha_1)x_n + \beta_1 x_n(1-x_n)e^{-(x_n+y_n)} < (1-\alpha_1+\beta_1)x_n < x_n$$

and

$$y_{n+1} = (1-\alpha_2)y_n + \beta_2 y_n(1-y_n)e^{-(x_n+y_n)} < (1-\alpha_2+\beta_2)y_n < y_n.$$

Then there exist $x \geq 0$ and $y \geq 0$such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Since the only possible values of (x,y) in the present case are $(0,0)$, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} y_n = 0$. This completes the proof. □

In the following theorems, we investigate the global attractivity of the positive equilibrium point $(\overline{x};\overline{y})$ of System (9), where $\overline{x}$ and $\overline{y}$ are given by $\alpha_1 = \beta_1(1-\overline{x})e^{-(\overline{x}+\overline{y})}$ and $\alpha_2 = \beta_2(1-\overline{y})e^{-(\overline{x}+\overline{y})}$, respectively.

Theorem 13 *Assume that* $\alpha_i + \beta_i e^{-2} < 1$, $i = 1,2$. *Then the unique positive equilibrium point* $(\overline{x};\overline{y})$ *of System* (9) *is a global attractor of all positive solutions of System* (9).

Proof Let $\{(x_n,y_n)\}_{n=0}^{\infty}$ be a solution of System (9) and let $x_n \leq \overline{x}$ (the case whenever $x_n \geq \overline{x}$ is similar and it will be left to the reader). Since $x_n \leq \overline{x}$, then $h(x_n) \leq 0$, where $h(x_n) = \alpha_1 - \beta_1(1-x_n)e^{-(x_n+y_n)}$. Thus $\alpha_1 \leq \beta_1(1-x_n)e^{-(x_n+y_n)}$. Therefore we obtain from System (9) that

$$\begin{aligned} x_{n+1} &= (1-\alpha_1)x_n + \beta_1 x_n(1-x_n)e^{-(x_n+y_n)} \\ &\geq (1-\alpha_1)x_n + \alpha_1 x_n = x_n. \end{aligned}$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is increasing and since it was shown that it is bounded above, then it converges to the unique positive equilibrium point $\overline{x}$. Similarly, assume that $y_n \le \overline{y}$ (the case whenever $y_n \ge \overline{y}$ is similar and it will be left to the reader). Since $y_n \le \overline{y}$, then $g(y_n) \le 0$, where $g(y_n) = \alpha_2 - \beta_2(1 - y_n)e^{-(x_n+y_n)}$. Thus $\alpha_2 \le \beta_2(1 - y_n)e^{-(x_n+y_n)}$. Therefore we obtain from System (9) that

$$\begin{aligned} y_{n+1} &= (1-\alpha_2)y_n + \beta_2 y_n(1-y_n)e^{-(x_n+y_n)} \\ &\ge (1-\alpha_2)y_n + \alpha_2 y_n = y_n. \end{aligned}$$

Then, again, the sequence $\{y_n\}_{n=0}^{\infty}$ is increasing, and since it was shown that it is bounded above, then it converges to the unique positive equilibrium point $\overline{y}$. Thus $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to $(\overline{x}; \overline{y})$. □

Theorem 14 *Consider* $\alpha_1 = \alpha_2 = \alpha$ *and* $\beta_1 = \beta_2 = \beta$ *and assume that* $\beta(\alpha e - \beta) \ge \alpha^2 e^3$. *Then the unique positive equilibrium point* $(\overline{x}; \overline{y})$ *of System* (9) *is a global attractor of all positive solutions of System* (9).

Proof Let $\{(x_n, y_n)\}_{n=0}^{\infty}$ be a solution of System (9). It follows from System (9) that

$$\begin{aligned} x_{n+1} &= (1-\alpha)x_n + \beta x_n(1-x_n)e^{-(x_n+y_n)} \\ &\ge (1-\alpha)x_n + \beta x_n(1-x_n)e^{-2}. \end{aligned}$$

Thus we see from Corollary 9 that

$$x_{n+1} \ge \left[1 - \alpha + \beta\left(1 - \frac{\beta}{\alpha e}\right)e^{-2}\right]x_n \ge x_n.$$

Then the sequence $\{x_n\}_{n=0}^{\infty}$ is increasing and since it is bounded, then it converges to the unique positive equilibrium point $\overline{x}$. Similarly, it is easy to show that the sequence $\{y_n\}_{n=0}^{\infty}$ is also convergent to the unique positive equilibrium point $\overline{y} = \overline{x}$: Therefore $\{(x_n, y_n)\}_{n=0}^{\infty}$ converges to $(\overline{x}, \overline{y})$ and then the proof is complete. □

Theorem 15 *Consider* $\alpha_1 = \alpha_2 = \alpha$ *and* $\beta_1 = \beta_2 = \beta$ *and assume that one of the following conditions holds*:

(I) $5\beta \le 4e^2(1-\alpha)$.

(II) $\alpha + \beta < 1$.

Then the unique positive equilibrium point $(\overline{x}, \overline{x})$ *of System* (9) *is a global attractor of all positive solutions of System* (9).

Proof Rewrite System (9) as follows:

$$\left.\begin{aligned} x_{n+1} &= F(x_n, y_n) = (1-\alpha)x_n + \beta(1-x_n)x_n e^{-(x_n+y_n)}, \\ y_{n+1} &= G(x_n, y_n) = (1-\alpha)y_n + \beta(1-y_n)y_n e^{-(x_n+y_n)}, \end{aligned}\right\} \quad n = 0, 1, \ldots,$$

where $F(x,y) = (1-\alpha)x + \beta(1-x)xe^{-(x+y)}$ and $G(x,y) = (1-\alpha)y + \beta(1-y)ye^{-(x+y)}$ are continuous functions. Now consider the system

$$\left.\begin{array}{ll} m_1 = F(m_1, M_2), & M_1 = F(M_1, m_2), \\ m_2 = G(M_1, m_2), & M_2 = F(m_1, M_2). \end{array}\right\}$$

Then

$$\begin{aligned} m_1 &= (1-\alpha)m_1 + \beta m_1(1-m_1)e^{-(m_1+M_2)}, \\ M_1 &= (1-\alpha)M_1 + \beta M_1(1-M_1)e^{-(M_1+m_2)}, \\ m_2 &= (1-\alpha)m_2 + \beta m_2(1-m_2)e^{-(m_2+M_1)}, \\ M_2 &= (1-\alpha)M_2 + \beta M_2(1-M_2)e^{-(M_2+m_1)}. \end{aligned}$$

Thus either $m_1 = M_2 = m_2 = M_2$ or

$$\begin{aligned} \alpha &= \beta(1-m_1)e^{-(m_1+M_2)}, \\ \alpha &= \beta(1-M_1)e^{-(M_1+m_2)}, \\ \alpha &= \beta(1-m_2)e^{-(m_2+M_1)}, \\ \alpha &= \beta(1-M_2)e^{-(M_2+m_1)}. \end{aligned}$$

Then $m_1 = M_2$, $m_2 = M_2$ and $(1-m_1)e^{-2m_1} = (1-M_1)e^{-2M_1} = (1-m_2)e^{-2m_2} = (1-M_2)e^{-2M_2}$. Now since $(1-m_1)e^{-2m_1} = (1-M_1)e^{-2M_1}$, then $e^{2(M_1-m_1)} = \frac{1-M_1}{1-m_1}$, that is,

$$2(M_1 - m_1) = \log(1-M_1) - \log(1-m_1). \tag{19}$$

We claim that $M_1 = m_1$; otherwise, for the sake of contradiction, assume that $M_1 > m_1$ (the case where $M_1 \leq m_1$ is similar and it will be left to the reader). Then $\log(1-M_1) - \log(1-m_1) > 0 \Rightarrow \log(1-M_1) > \log(1-m_1) \Rightarrow M_1 < m_1$, which is a contradiction.

Now it is easy to see that

$$\left.\begin{array}{l} \frac{\partial F(x,y)}{\partial x} = 1-\alpha + \beta(x^2 - 3x + 1)e^{-(x+y)}, \\ \frac{\partial F(x,y)}{\partial y} = -x\beta(1-x)e^{-(x+y)}, \\ \frac{\partial G(x,y)}{\partial x} = -\beta y(1-y)e^{-(x+y)}, \\ \frac{\partial G(x,y)}{\partial y} = 1-\alpha + \beta(y^2 - 3y + 1)e^{-(x+y)}. \end{array}\right\}$$

Thus

$$\begin{aligned} \frac{\partial F(x,y)}{\partial x} &= 1-\alpha + \beta\left(x^2 - 3x + 1\right)e^{-(x+y)} \\ &\geq 1-\alpha + \beta\left(x^2 - 3x + 1\right)e^{-2} \\ &= \beta e^{-2}x^2 - 3\beta e^{-2}x + \beta e^{-2} + 1 - \alpha. \end{aligned}$$

Now, there are two cases to consider:

Case 1: Suppose that $5\beta \le 4e^2(1-\alpha)$. Therefore the function $w(x) = \beta e^{-2}x^2 - 3\beta e^{-2}x + \beta e^{-2} + 1 - \alpha$ has no real roots. Thus $\frac{\partial F(x,y)}{\partial x} \ge 0$. Similarly, it is easy to prove that $\frac{\partial G(x,y)}{\partial x} \ge 0$. Then it follows by Theorem A that the equilibrium point $(\overline{x}, \overline{y}) = (\overline{x}, \overline{x})$ of System (9) is a global attractor of all positive solutions of System (9).

Case 2: Suppose that $\alpha + 2\beta < 1$. Since $0 \le x \le 1$, $3 \ge 3 - x \ge x(3-x) = 3x - x^2$, or $2 \ge 3x - x^2 - 1$, and since $\alpha + 2\beta < 1$, then $1 - \alpha > 2\beta > 2\beta e^{-2} \ge 2\beta e^{-(x+y)} \ge \beta(3x - x^2 - 1)e^{-(x+y)}$. Thus $\frac{\partial F(x,y)}{\partial x} \ge 0$. Similarly, it is easy to prove that $\frac{\partial G(x,y)}{\partial x} \ge 0$. Then it follows again by Theorem A that the equilibrium point $(\overline{x}, \overline{y}) = (\overline{x}, \overline{x})$ of System (9) is a global attractor of all positive solutions of System (9). Thus the proof is now completed. □

Competing interests
The author declares that they have no competing interests.

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