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On the transcendence of some power series

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Abstract

In this study, we consider some power series with rational coefficients and investigate transcendence of the values of these series for Liouville number arguments. It is proved that these values are either a Liouville number or a rational number under certain conditions.

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1 Introduction

A real number α is algebraic if it is a zero of a polynomial with integer coefficients. The real numbers which are not algebraic are known as transcendental. The theory of transcendental numbers has a long history and was originated back to Liouville in his famous paper [1] where he explicitly constructed a number and proved that it is transcendental. Later, Cantor [2] gave another proof of the existence of transcendental numbers by establishing the denumerability of the set of algebraic numbers. It follows from this that almost all real numbers are transcendental. Further, the development of the theory of transcendental numbers has proved to have a strong influence on some new studies in Diophantine equations; see [3, 4].

A classification of the set of all transcendental numbers into three disjoint classes, termed S , T and U , which was introduced by Mahler [5], proved to be of considerable value in the general development of the subject. The first classification of this kind was outlined by Maillet in [6], and others were described by Perna in [7] and Morduchai-Boltovskoj [8], but Mahler's classification receives most of the interest. Mahler described this classification in the following way.

Let $P(x) = a_n x^n + \dots + a_1 x + a_0$ be a polynomial with integer coefficients. The height $H(P)$ of the polynomial P is defined by

$$H(P) = \max_n (|a_n|, \dots, |a_0|)$$

and if the degree of P is denoted by $\deg(P)$, then $\deg(P) = n$, and for a given arbitrary complex number ξ , it can be written as

$$\omega_n(H, \xi) = \min_n \{ |P(\xi)| : \deg(P) \leq n, H(P) \leq H, P(\xi) \neq 0 \},$$

where n and H are positive integers; see [5]. Next, Mahler puts

$$\omega_n(\xi) = \overline{\lim}_{H \rightarrow \infty} \frac{-\log \omega_n(H, \xi)}{\log H}$$

and

$$\omega(\xi) = \overline{\lim}_{n \rightarrow \infty} \frac{\omega_n(\xi)}{n}.$$

The inequalities $0 \leq \omega_n(\xi) \leq \infty$ and $0 \leq \omega(\xi) \leq \infty$ hold. From $\omega_{n+1}(H, \xi) \leq \omega_n(H, \xi)$, we get $\omega_{n+1}(\xi) \geq \omega_n(\xi)$. If index $\omega_n(\xi) = \infty$, then the $\mu(\xi)$ is defined as the smallest of them; otherwise $\mu(\xi) = \infty$. Thus, $\mu(\xi)$ is uniquely determined. Furthermore, the two quantities $\mu(\xi)$ and $\omega(\xi)$ are never finite simultaneously, for the finiteness of $\mu(\xi)$ implies that there is an $n < \infty$ such that $\omega_n = \infty$, hence $\omega = \infty$.

Therefore, there are the following four possibilities for ξ : it is said to be

Class-*A* if $\omega(\xi) = 0, \mu(\xi) = \infty$,

Class-*S* if $0 < \omega(\xi) < \infty, \mu(\xi) = \infty$,

Class-*T* if $\omega(\xi) = \infty, \mu(\xi) = \infty$,

Class-*U* if $\omega(\xi) = \infty, \mu(\xi) < \infty$.

In [9], Koksma introduced an analogous classification of complex numbers. He divided the complex numbers into four classes A^*, S^*, T^* and U^* . More information may be found in [10].

In [11], Wirsing proved that both classifications are equivalent. Namely, A, S, T and U numbers are the same as A^*, S^*, T^* and U^* numbers. The class A is precisely the set of algebraic numbers. The ξ is called a U -number of degree m if $\mu(\xi) = m$. The set of U -numbers of degree m is denoted by U_m . It is obvious that for any $m \geq 1$, the U_m is a subclass of U , and U is the union of all disjoint sets U_m .

Leveque proved that U_m is not empty for any $m \geq 1$ in [12]. Later, in [13, 14], Oryan considered a class of power series with algebraic coefficients and proved that under certain conditions these series take values in the subclass U_m for algebraic arguments and in the set of Liouville numbers for Liouville number arguments, respectively.

Now, consider the infinite convergent sum $T = \sum_{n=0}^{\infty} \frac{P(n)}{Q(n)}$, where $P(x) \in \bar{\mathbb{Q}}[x]$, $Q(x) \in \mathbb{Q}[x]$ and $Q(x)$ has only simple rational zeros. Then, Saradha and Tijdeman have obtained the sufficient and necessary conditions for the transcendence of T if the degree of $Q(x)$ is 3; see [15]. Similarly, Ping and Yuan gave sufficient and necessary conditions for the transcendence of T if the degree of $Q(x)$ is 4 and $Q(x)$ is reduced; see [16].

We also note that a transcendental function is an analytic function having a single value or many values, and to calculate the values, we need a limiting process. However, an analytic function is transcendental if and only if its Riemann surface is non-compact; see [17].

In the present work, we consider certain power series with rational coefficients and show that these series take values of either the set of Liouville numbers or rational numbers under certain conditions. Thus, we give a new result for obtaining U_1 -numbers.

2 Preliminaries

In this paper, $|x|$ means the absolute value of x and the least common multiple of x_1, x_2, \dots, x_n is denoted by $[x_1, x_2, \dots, x_n]$.

Definition 2.1 A real number ξ is called Liouville number if and only if for every positive integer n , there exist integers p_n, q_n ($q_n > 1$) with

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^n}.$$

The set of all Liouville numbers is identical with the U_1 subclass. More information about Liouville numbers may be found in [18, 19] and [20].

Now, in order to prove our main theorem, we need the following lemma which was proved in [21].

Lemma 2.2 Let ξ be a real number. If there exist a sequence $\omega_1, \omega_2, \omega_3, \dots$ of real numbers tending to infinity and a sequence $\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}, \dots$ of rational numbers satisfying

$$0 < \left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{\omega_n}} \quad (q_n > 1, n = 1, 2, 3, \dots),$$

then ξ is a Liouville number.

3 Main theorem

Theorem 3.1 Let

$$g(x) = \sum_{n=0}^{\infty} \frac{f_n}{e_n} x^n$$

be a power series with non-zero rational coefficients $\frac{f_n}{e_n}$ ($e_n, f_n \in \mathbb{Z}, e_n > 1$) which satisfies the following conditions:

$$\liminf_{n \rightarrow \infty} \frac{\log e_{n+1}}{\log e_n} = \eta > 1, \tag{3.1}$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log e_{n+1}}{\log e_n} = \infty, \tag{3.2}$$

$$\overline{\lim}_{n \rightarrow \infty} \frac{\log |f_n|}{\log e_n} = \mu < 1. \tag{3.3}$$

Further, let ξ be a Liouville number and satisfy the following two properties:

- (1) The ξ has an approximation with rational numbers $\frac{p_n}{q_n}$ ($q_n > 1$) so that the following inequality holds for sufficiently large n :

$$\left| \xi - \frac{p_n}{q_n} \right| < \frac{1}{q_n^{ns_n}} \quad \left(\lim_{n \rightarrow \infty} s_n = +\infty \right). \tag{3.4}$$

(2) There exist two positive real numbers γ_1 and γ_2 with $\frac{\eta}{\eta-1} < \gamma_1 < \gamma_2$ and

$$e_n^{\gamma_1} \leq q_n^n \leq e_n^{\gamma_2} \tag{3.5}$$

for sufficiently large n .

Then $g(\xi)$ is either a Liouville number or a rational number.

Proof It follows from (3.1) that

$$\log e_{n+1} > \eta_1 \log e_n \tag{3.6}$$

for sufficiently large n , where $\eta_1 = \eta - \varepsilon_1$ and ε_1 is chosen as $0 < \varepsilon_1 < \eta - \frac{\eta}{\gamma_1-1}$. Then it follows from (3.6) that the sequence $\{e_n\}$ is strictly increasing, thus we have

$$\lim_{n \rightarrow \infty} e_n = +\infty, \tag{3.7}$$

$$\lim_{n \rightarrow \infty} \frac{\log e_n}{n} = \infty \quad \text{and} \quad \lim_{n \rightarrow \infty} \frac{\log e_n}{n^2} = \infty. \tag{3.8}$$

Now, by using equation (3.6), we get

$$e_n < e_{n+1}^{\frac{1}{\eta_1}}. \tag{3.9}$$

Let $E_n = [e_0, e_1, \dots, e_n]$. Then on using (3.9), we obtain

$$e_n \leq E_n \leq e_n^{\varepsilon_2 + \frac{\eta_1}{\eta_1-1}}, \tag{3.10}$$

where ε_2 is chosen as $0 < \varepsilon_2 < \gamma_1 - \frac{\eta}{\eta_1-1}$.

Now, if we consider the following polynomials:

$$g_n(x) = \sum_{v=0}^n \frac{f_v}{e_v} x^v \quad (n = 1, 2, 3, \dots),$$

then we have

$$g_n(\xi) - g_n\left(\frac{p_n}{q_n}\right) = \sum_{v=1}^n \frac{f_v}{e_v} \left(\xi - \frac{p_n}{q_n}\right) \left(\xi^{v-1} + \xi^{v-2} \frac{p_n}{q_n} + \dots + \frac{p_n^{v-1}}{q_n^{v-1}}\right), \tag{3.11}$$

and from (3.4) it follows that

$$\left| \frac{p_n}{q_n} \right| \leq |\xi| + 1. \tag{3.12}$$

Further, on using equations (3.4), (3.11) and (3.12), we get

$$\left| g_n(\xi) - g_n\left(\frac{p_n}{q_n}\right) \right| \leq \frac{1}{q_n^{ns_n}} \sum_{v=1}^n \frac{|f_v|}{e_v} v (|\xi| + 1)^{v-1}. \tag{3.13}$$

Now, define

$$F_n = \max_n (|f_0|, |f_1|, \dots, |f_n|).$$

Then we obtain

$$\sum_{\nu=1}^n \frac{|f_\nu|}{e_\nu} \nu (|\xi| + 1)^{\nu-1} \leq n^2 F_n (|\xi| + 1)^{n-1} \tag{3.14}$$

for sufficiently large n .

On the other hand, we can easily deduce from (3.3) and $\mu < \frac{\mu+1}{2}$ that

$$\frac{\log |f_n|}{\log e_n} < \frac{\mu + 1}{2}, \tag{3.15}$$

and since the sequence $\{e_n\}$ is strictly increasing, then it follows that

$$\frac{\log F_n}{\log e_n} < \frac{\mu + 1}{2}. \tag{3.16}$$

On using $0 < \varepsilon_2 < \gamma_1 - \frac{\eta_1}{n-1}$, and from (3.7), (3.8) and (3.16), we get

$$n^2 F_n (|\xi| + 1)^{n-1} \leq \frac{1}{2} e_n^{(\gamma_1 - \varepsilon_2 - \frac{\eta_1}{n-1}) \frac{\eta_1}{2}}.$$

That is, on using (3.5), (3.10) and (3.14),

$$\sum_{\gamma=1}^n \frac{|f_\gamma|}{e_\gamma} \gamma (|\xi| + 1)^{\gamma-1} \leq \frac{(q_n^n)^{\frac{\eta_1}{2}}}{2(E_n)^{\frac{\eta_1}{2}}}$$

for sufficiently large n . Then on using (3.13), we have

$$\left| g_n(\xi) - g_n\left(\frac{p_n}{q_n}\right) \right| \leq \frac{1}{2(E_n q_n^n)^{\frac{\eta_1}{2}}} \tag{3.17}$$

for sufficiently large n .

Moreover, the following inequality holds:

$$|g(\xi) - g_n(\xi)| \leq \sum_{i=1}^{\infty} \frac{|f_{n+i}|}{e_{n+i}} |\xi|^{n+i}. \tag{3.18}$$

It follows from (3.15) that

$$\frac{|f_n|}{e_n} < \frac{1}{e_n^{(\frac{1-\mu}{2})}}$$

for sufficiently large n . We get from here and (3.18)

$$|g(\xi) - g_n(\xi)| \leq \frac{|\xi|^{n+1}}{e_{n+1}^{(\frac{1-\mu}{2})}} \left[1 + \left(\frac{e_{n+1}}{e_{n+2}}\right)^{\frac{1-\mu}{2}} |\xi| + \left(\frac{e_{n+1}}{e_{n+3}}\right)^{\frac{1-\mu}{2}} |\xi|^2 + \dots \right].$$

Thus, we can deduce from (3.9) that

$$0 < \frac{e_{n+1}}{e_{n+2}} < \frac{1}{e_{n+2}^{(1-\frac{1}{\eta_1})}}$$

Since (3.7) holds, we then obtain

$$\lim_{n \rightarrow \infty} \frac{e_{n+1}}{e_{n+2}} = 0.$$

Similarly, since $\frac{1-\mu}{2} > 0$, we have

$$\left(\frac{e_{n+1}}{e_{n+1+k}}\right)^{\frac{1-\mu}{2}} |\xi|^k < \left(\frac{1}{2}\right)^k \quad (k = 1, 2, 3, \dots),$$

and therefore

$$|g(\xi) - g_n(\xi)| \leq \frac{2|\xi|^{n+1}}{e_{n+1}^{(\frac{1-\mu}{2})}}$$

On the other hand, from (3.8) we get

$$2|\xi|^{n+1} \leq e_{n+1}^{(\frac{1-\mu}{4})}$$

for sufficiently large n . From here we obtain

$$|g(\xi) - g_n(\xi)| \leq \frac{1}{e_{n+1}^{(\frac{1-\mu}{4})}} \tag{3.19}$$

for sufficiently large n . Now, if further we define

$$r_n = \frac{\log e_{n+1}}{\log e_n},$$

then we have

$$|g(\xi) - g_n(\xi)| \leq \frac{1}{e_n^{r_n(\frac{1-\mu}{4})}}$$

Using (3.2) then it follows that there exists a subsequence $\{r_{n_k}\}$ of $\{r_n\}$ such that

$$\lim_{k \rightarrow \infty} r_{n_k} = \infty.$$

Therefore,

$$|g(\xi) - g_{n_k}(\xi)| \leq \frac{1}{e_{n_k}^{r_{n_k}(\frac{1-\mu}{4})}} \tag{3.20}$$

for sufficiently large n_k . On using (3.5), (3.7) and (3.10), we deduce that there exists a suitable sequence $\{r'_{n_k}\}$ with $\lim_{k \rightarrow \infty} r'_{n_k} = +\infty$. Then from (3.20) for r'_{n_k} we obtain

$$\frac{1}{e_{n_k}^{\frac{1-\mu}{4}}} \leq \frac{1}{2(E_{n_k} q_{n_k}^{n_k})^{r'_{n_k}}},$$

and therefore

$$|g(\xi) - g_{n_k}(\xi)| \leq \frac{1}{2(E_{n_k} q_{n_k}^{n_k})^{r'_{n_k}}} \tag{3.21}$$

for sufficiently large n_k . On the other hand, using (3.17) we get

$$\left| g_{n_k}(\xi) - g_{n_k}\left(\frac{p_{n_k}}{q_{n_k}}\right) \right| \leq \frac{1}{2(E_{n_k} q_{n_k}^{n_k})^{\frac{s_{n_k}}{2}}} \tag{3.22}$$

for sufficiently large n_k .

Let

$$\omega_{n_k} = \min\left(r'_{n_k}, \frac{s_{n_k}}{2}\right).$$

It follows from (3.21) and (3.22) that

$$\left| g(\xi) - g_{n_k}\left(\frac{p_{n_k}}{q_{n_k}}\right) \right| \leq \frac{1}{(E_{n_k} q_{n_k}^{n_k})^{\omega_{n_k}}}, \tag{3.23}$$

where $\lim_{k \rightarrow \infty} \omega_{n_k} = \infty$. Moreover,

$$g_{n_k}\left(\frac{p_{n_k}}{q_{n_k}}\right) = \frac{z_{n_k}}{E_{n_k} q_{n_k}^{n_k}}$$

are rational numbers with z_{n_k} integers. It follows from (3.23) that

$$\lim_{k \rightarrow \infty} g_{n_k}\left(\frac{p_{n_k}}{q_{n_k}}\right) = g(\xi).$$

Thus, if the sequence $\{g_{n_k}(\frac{p_{n_k}}{q_{n_k}})\}$ is constant, then $g(\xi)$ is a rational number. Otherwise, using Lemma 2.2 we get from (3.23) that $g(\xi)$ is a Liouville number. \square

Corollary 3.2 *If $f_n > 0$ ($n = 0, 1, 2, \dots$) and $\xi > 0$ in Theorem 3.1, then $f(\xi)$ is a Liouville number.*

Proof Since $\xi > 0$, it is possible to choose a subsequence $\{\frac{p_{n_{k_t}}}{q_{n_{k_t}}}\}$ of $\{\frac{p_{n_k}}{q_{n_k}}\}$ so that the terms $\frac{p_{n_{k_t}}}{q_{n_{k_t}}}$ are positive and strictly increasing or decreasing. Let us assume that $\{\frac{p_{n_{k_t}}}{q_{n_{k_t}}}\}$ is strictly increasing. From $\frac{f_{n_t}}{e_{n_t}} > 0$ it follows

$$g_{n_{k_{t+1}}}\left(\frac{p_{n_{k_{t+1}}}}{q_{n_{k_{t+1}}}}\right) - g_{n_{k_t}}\left(\frac{p_{n_{k_t}}}{q_{n_{k_t}}}\right) > 0. \tag{3.24}$$

From (3.24) we deduce that the sequence $\{g_{n_k}(\frac{p_{n_k}}{q_{n_k}})\}$ is not constant. Thus, $g(\xi)$ is a Liouville number. In the case of strictly decreasing $\{\frac{p_{n_k t}}{q_{n_k t}}\}$, the proof follows similarly. \square

4 Conclusion

In this paper, the series with rational coefficients are treated and it is shown that under certain conditions these series take values belonging to either the set of Liouville numbers or the rational number field for Liouville number arguments.

The similar results can be proved for the power series which are defined in the p -adic field Q_p and in the field of formal Laurent series.

Competing interests

The authors declare that they do not have any competing interest.

Authors' contributions

All the authors contributed equally.

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