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Oscillation criteria for certain third-order delay dynamic equations

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Abstract

This paper is concerned with the oscillatory behavior of a certain class of third-order nonlinear variable delay neutral functional dynamic equations,

$$\left\{ r(t)\phi\left(\left[a(t)y^{\Delta}(t)\right]^{\Delta}\right)\right\}^{\Delta} + P(t)F\left(\phi\left(x(\delta(t))\right)\right) = 0,$$

on a time scale $\mathbf{T}$ with $\sup \mathbf{T} = +\infty$, where $y(t) = x(t) + B(t)g(x(\tau(t)))$, $\phi(u) = |u|^{\lambda-1}u$, $\lambda \geq 1$. By using the generalized Riccati transformation and a lot of inequality techniques, some new oscillation criteria for the equations are established, results are presented that not only complement and improve those related results in the literature, but also improve some known results for a third-order delay dynamic equation with a neutral term. Further, the main results improve some related results for third-order neutral differential equations. Some examples are given to illustrate the importance of our results.

MSC: 34K11; 34C10; 39A10

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1 Introduction

Consider a third-order nonlinear variable delay dynamic equation

$$\left\{ r(t)\phi\left(\left[a(t)y^{\Delta}(t)\right]^{\Delta}\right)\right\}^{\Delta} + P(t)F\left(\phi\left(x(\delta(t))\right)\right) = 0, \quad t \in \mathbf{T}, t \geq t_0, \tag{1.1}$$

where $y(t) = x(t) + B(t)g(x(\tau(t)))$, $\phi(u) = |u|^{\lambda-1}u$, $\lambda \geq 1$. Throughout this article, we assume that:

(H$_1$): $\mathbf{T}$ is an arbitrary time scale with $\sup \mathbf{T} = +\infty$, and $t_0 \in \mathbf{T}$ with $t_0 > 0$, we define the time scale interval $[t_0, +\infty)_{\mathbf{T}}$ by $[t_0, +\infty)_{\mathbf{T}} = [t_0, +\infty) \cap \mathbf{T}$. $r(t), a(t), B(t), P(t) \in C_{\mathrm{rd}}(\mathbf{T}, \mathbf{R})$, *i.e.*, $r(t), a(t), B(t), P(t) : \mathbf{T} \to \mathbf{R}$ are rd-continuous functions. $g(u), F(u) : \mathbf{R} \to \mathbf{R}$ are continuous functions with $ug(u) > 0$ ($u \neq 0$) and $uF(u) > 0$ ($u \neq 0$).

(H$_2$): $\tau(t), \delta(t) : \mathbf{T} \to \mathbf{T}$ are delay functions with $\tau(t) \leq t$, $\lim_{t \to +\infty} \tau(t) = +\infty$ and $\delta(t) \leq t$, $\lim_{t \to +\infty} \delta(t) = +\infty$.

(H$_3$): $0 \leq B(t) \leq 1$, $P(t) > 0$, $r(t) > 0$ and $r^{\Delta}(t) \geq 0$, $a(t) > 0$ and $a^{\Delta}(t) \geq 0$.

(H$_4$): There exist constants $0 < \beta \leq 1$ and $L > 0$, such that $\frac{g(u)}{u} \leq \beta$ ($u \neq 0$), $\frac{F(u)}{u} \geq L$ ($u \neq 0$).

(H$_5$): $\int_{t_0}^{+\infty} \frac{1}{a(s)} \Delta s = +\infty$, $\int_{t_0}^{+\infty} \frac{1}{[r(s)]^{1/\lambda}} \Delta s = +\infty$.

We recall that a solution $x(t)$ of equation (1.1) is said to be oscillatory on $[t_0, +\infty)_{\mathbf{T}}$ if it is neither eventually positive nor eventually negative; otherwise, the solution is said to be nonoscillatory. Equation (1.1) is said to be oscillatory if all of its solutions are oscillatory. Our attention is restricted to those solutions $x(t)$ of (1.1) where $x(t)$ is not eventually identically zero.

Note that if $\lambda = 1$, $r(t) \equiv 1$, $a(t) \equiv 1$, $F(u) = u$, $B(t) \equiv 0$, $\delta(t) = t$ in equation (1.1), then (1.1) is simplified to the equation

$$x^{\Delta\Delta\Delta}(t) + P(t)x(t) = 0, \quad t \in \mathbf{T}, t \geq t_0. \tag{1.2}$$

In equation (1.1), if $r(t) \equiv 1$, $a(t) \equiv 1$, $F(u) = u$, $B(t) \equiv 0$ and λ is the ratio of positive odd integers, then (1.1) is simplified to the Emden-Fowler type equation

$$\left[\left(x^{\Delta\Delta}(t)\right)^{\lambda}\right]^{\Delta} + P(t)x^{\lambda}\left(\delta(t)\right) = 0, \quad t \in \mathbf{T}, t \geq t_0. \tag{1.3}$$

In equation (1.1), if $\lambda = 1$, $a(t) \equiv 1$, $F(u) = u^{\gamma}$ (where γ is the ratio of positive odd integers), then (1.1) is simplified to the equation

$$\left[r(t)\left(x(t) + B(t)x\left(\tau(t)\right)\right)^{\Delta\Delta}\right]^{\Delta} + P(t)x^{\gamma}\left(\delta(t)\right) = 0, \quad t \in \mathbf{T}, t \geq t_0. \tag{1.4}$$

And it is easy to see that (1.1) can be transformed into the third-order nonlinear delay dynamic equations

$$\left\{r(t)\left[a(t)x^{\Delta}(t)\right]^{\Delta}\right\}^{\Delta} + P(t)f\left(x\left(\delta(t)\right)\right) = 0, \quad t \in \mathbf{T}, t \geq t_0 \tag{1.5}$$

and

$$\left\{r(t)\left[\left(a(t)x^{\Delta}(t)\right)^{\Delta}\right]^{\lambda}\right\}^{\Delta} + P(t)f\left(x\left(\delta(t)\right)\right) = 0, \quad t \in \mathbf{T}, t \geq t_0. \tag{1.6}$$

Equations of this type arise in a number of important applications such as problems in biological population dynamics, in neural network, in quantum theory, in computer science and in control theory. Hence, it is important and useful to study the oscillatory properties of solutions of equation (1.1). Recently, there has been an increasing interest in studying the oscillatory behavior of first and second-order dynamic equations on time scales (see [1–7]). However, there are very few results regarding the oscillation of third-order equations. Among these papers dealing with the subject, we refer in particular to [8–17], the monographs [1, 2] and the references therein. Our concern is especially motivated by several recent papers such as [9–13].

Erbe *et al.* [9] studied equation (1.2) and they established Hille and Nehari-type oscillation criteria for the equation. After that, in [10] and [11], the author discussed oscillatory criteria of equations (1.3) and (1.4), respectively, under the condition

$$\int_{t_0}^{+\infty} P(u)\left[h_2\left(\delta(u), t_0\right)\right]^{\lambda} \Delta u = +\infty \tag{1.7}$$

and got the result that every solution of equations (1.3) and (1.4) oscillates or converges to zero, where the Taylor monomials $\{h_n(t,s)\}_{n=0}^{+\infty}$ are defined as follows:

$$h_0(t,s) = 1, \qquad h_{n+1}(t,s) = \int_s^t h_n(\tau,s)\Delta\tau, \quad n \geq 0, t, s \in \mathbf{T}.$$

In [12], the author discussed oscillatory criteria of equation (1.5) under the conditions

$$a^\Delta(t) \leq 0 \quad \text{and} \quad \int_{t_0}^{+\infty} P(u)\delta(u)\Delta u = +\infty$$

and got the result that every solution of equation (1.5) oscillates or converges to zero. Moreover, in [13], the author considered the oscillation for equation (1.6) under the conditions

$$\delta \circ \sigma = \sigma \circ \delta, \qquad a^\Delta(t) \leq 0 \quad \text{and} \quad \int_{t_0}^{+\infty} P(u)\big[\delta(u)\big]^\lambda \Delta u = +\infty \tag{1.8}$$

and obtained the result that every solution of equation (1.6) oscillates or converges to zero.

Obviously, the results in [9–13] are inapplicable for the following differential equation

$$\big\{t^2\big[\big(tx'(t)\big)'\big]^3\big\}' + t(t-1)^2\big(3t^2 - 8t + 2\big)e^{6t-18}x^9(t-2) = 0.$$

Therefore, this topic is fairly new for dynamic equations on time scales. The purpose of this article is to obtain new oscillation criteria for the oscillation of (1.1), these criteria can improve the restriction of the conditions for the equation, which promote some existing results. We should note that many of our results of this article are new for the corresponding third-order nonlinear differential and difference equations. In fact, the obtained results extend, unify and correlate many of the existing results in the literature.

2 Preliminaries

We shall employ the following lemmas.

Lemma 2.1 [1] *Suppose that $x(t)$ is delta-differentiable and eventually positive or eventually negative, then*

$$\big(\big(x(t)\big)^\lambda\big)^\Delta = \lambda \int_0^1 \big[hx^\sigma + (1-h)x\big]^{\lambda-1}x^\Delta(t)\,dh. \tag{2.1}$$

Lemma 2.2 [3] *Assume that*:
  (1) $u \in C_{\mathrm{rd}}^2(\mathbf{I}, \mathbf{R})$, *where* $\mathbf{I} = [t^*, +\infty)$, $t^* > 0$.
  (2) $u(t) > 0$, $u^\Delta(t) > 0$, $u^{\Delta\Delta}(t) \leq 0$, $t \geq t^*$.
Then, for every $k \in (0,1)$, there exists a constant $t_k \in \mathbf{T}$, $t_k > t^*$, such that $u(\sigma(t)) \leq \frac{\sigma(t)u(\delta(t))}{k\delta(t)}$ ($t \geq t_k$).

Lemma 2.3 [2] *Suppose that a and b are nonnegative real numbers, then $rab^{r-1} - a^r \leq (r-1)b^r$ for all $r > 1$, where the equality holds if and only if $a = b$.*

Lemma 2.4 [8] *Assume that* $u(t) > 0$, $u^\Delta(t) > 0$, $u^{\Delta\Delta}(t) > 0$, $u^{\Delta\Delta\Delta}(t) < 0$, *then*

$$\liminf_{t \to +\infty} \frac{tu(t)}{h_2(t, t_0)u^\Delta(t)} \geq 1.$$

Lemma 2.5 *Let* $x(t)$ *be an eventually position solution of equation* (1.1). *Then there exists* $t_1 \in [t_0, +\infty)_{\mathbf{T}}$, *for all* $t \in [t_1, +\infty)_{\mathbf{T}}$, *such that either*

(i) $y(t) > 0$, $y^\Delta(t) > 0$, $[a(t)y^\Delta(t)]^\Delta > 0$, $\{r(t)\phi([a(t)y^\Delta(t)]^\Delta)\}^\Delta < 0$,

or

(ii) $y(t) > 0$, $y^\Delta(t) < 0$, $[a(t)y^\Delta(t)]^\Delta > 0$, $\{r(t)\phi([a(t)y^\Delta(t)]^\Delta)\}^\Delta < 0$.

Proof Since $x(t)$ is an eventually position solution of (1.1), then there exists $t_1 \in [t_0, +\infty)_{\mathbf{T}}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$, thus, $y(t) > 0$. From (1.1), we obtain

$$\left\{r(t)\phi\left(\left[a(t)y^\Delta(t)\right]^\Delta\right)\right\}^\Delta = -P(t)F\left(\phi\left(x\left(\delta(t)\right)\right)\right) \leq -LP(t)\left(x\left(\delta(t)\right)\right)^\lambda < 0. \tag{2.2}$$

Hence, $r(t)\phi([a(t)y^\Delta(t)]^\Delta)$ is decreasing and, therefore, eventually of one sign, so $[a(t) \times y^\Delta(t)]^\Delta$ is either eventually positive or eventually negative. We assert that $[a(t)y^\Delta(t)]^\Delta > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$. Assume that $[a(t)y^\Delta(t)]^\Delta < 0$ eventually, then there exists $t_2 \in [t_1, +\infty)_{\mathbf{T}}$, such that $[a(t_2)y^\Delta(t_2)]^\Delta < 0$. Then for all $t \in [t_2, +\infty)_{\mathbf{T}}$, we obtain

$$r(t)\phi\left(\left[a(t)y^\Delta(t)\right]^\Delta\right) \leq r(t_2)\phi\left(\left[a(t_2)y^\Delta(t_2)\right]^\Delta\right) = -M < 0, \tag{2.3}$$

where $M = -r(t_2)\phi([a(t_2)y^\Delta(t_2)]^\Delta) = r(t_2)|[a(t_2)y^\Delta(t_2)]^\Delta|^{\lambda-1}\{-[a(t_2)y^\Delta(t_2)]^\Delta\} > 0$. By (2.3), we obtain

$$\left(-\left[a(t)y^\Delta(t)\right]^\Delta\right)^\lambda \geq \frac{M}{r(t)}, \quad \text{namely,} \quad \left[a(t)y^\Delta(t)\right]^\Delta \leq -M^{1/\lambda}\frac{1}{[r(t)]^{1/\lambda}}.$$

Integrating this inequality from t_2 to t ($t \in [t_2, +\infty)_{\mathbf{T}}$) provides

$$a(t)y^\Delta(t) \leq a(t_2)y^\Delta(t_2) - M^{1/\lambda}\int_{t_2}^t \frac{1}{[r(s)]^{1/\lambda}}\Delta s \to -\infty \quad (t \to +\infty).$$

Then there exists $t_3 \in [t_2, +\infty)_{\mathbf{T}}$, such that $a(t)y^\Delta(t) \leq a(t_3)y^\Delta(t_3) < 0$. Similarly, we can get

$$y(t) \leq y(t_3) + a(t_3)y^\Delta(t_3)\int_{t_3}^t \frac{1}{a(s)}\Delta s \to -\infty \quad (t \to +\infty),$$

which contradicts with $y(t) > 0$. So, $[a(t)y^\Delta(t)]^\Delta > 0$, this implies that $y^\Delta(t) > 0$ or $y^\Delta(t) < 0$. This completes the proof. □

Lemma 2.6 *Assume that* $x(t)$ *is a solution of equation* (1.1), *which satisfies the case* (ii) *in Lemma* 2.5, *if either*

$$\int_{t_0}^{+\infty} P(s)\Delta s = +\infty \tag{2.4}$$

or

$$\int_{t_0}^{+\infty} P(s)\Delta s < +\infty \quad and$$

$$\int_{t_0}^{+\infty} \left[\frac{1}{a(v)} \int_v^{+\infty} \left(\frac{1}{r(u)} \int_u^{+\infty} P(s)\Delta s \right)^{1/\lambda} \Delta u \right] \Delta v = +\infty$$

(2.5)

holds, then $\lim_{t\to+\infty} x(t) = 0$.

Proof Since $x(t)$ is a solution of equation (1.1), which satisfies the case (ii) in Lemma 2.5, *i.e.*,

$$y(t) > 0, \qquad y^\Delta(t) < 0, \qquad \left[a(t)y^\Delta(t) \right]^\Delta > 0,$$

$$\left\{ r(t)\phi\left(\left[a(t)y^\Delta(t) \right]^\Delta \right) \right\}^\Delta < 0, \quad t \in [t_1, +\infty)_{\mathbf{T}}.$$

Therefore, it follows that $\lim_{t\to+\infty} y(t) = c \geq 0$. If $c > 0$, then in view of $y(t) \leq x(t) + \beta B(t)x(\tau(t))$ and $0 \leq \beta B(t) \leq 1$, it is easy to see that there exists $t_2 \in [t_1, +\infty)_{\mathbf{T}}$ such that $x(\delta(t)) \geq \frac{c}{2}$ for all $t \in [t_2, +\infty)_{\mathbf{T}}$. Therefore, from (2.2), we obtain

$$\left\{ r(t)\phi\left(\left[a(t)y^\Delta(t) \right]^\Delta \right) \right\}^\Delta \leq -LP(t)\left(x(\delta(t)) \right)^\lambda \leq -L\left(\frac{c}{2} \right)^\lambda P(t).$$

(2.6)

If (2.4) holds, then integrating (2.6) from t_2 to t ($t \in [t_2, +\infty)_{\mathbf{T}}$) provides

$$r(t)\phi\left(\left[a(t)y^\Delta(t) \right]^\Delta \right) \leq r(t_2)\phi\left(\left[a(t_2)y^\Delta(t_2) \right]^\Delta \right) - L\left(\frac{c}{2} \right)^\lambda \int_{t_2}^t P(s)\Delta s \to -\infty \quad (t \to +\infty),$$

which contradicts with $[a(t)y^\Delta(t)]^\Delta > 0$. So $\lim_{t\to+\infty} y(t) = 0$, in view of $0 < x(t) \leq y(t)$, hence $\lim_{t\to+\infty} x(t) = 0$.

If (2.5) holds, then integrating (2.6) from t to T ($T \geq t$, $T, t \in [t_2, +\infty)_{\mathbf{T}}$) provides

$$r(T)\phi\left(\left[a(T)y^\Delta(T) \right]^\Delta \right) - r(t)\phi\left(\left[a(t)y^\Delta(t) \right]^\Delta \right) \leq -L\left(\frac{c}{2} \right)^\lambda \int_t^T P(s)\Delta s,$$

i.e., $r(t)\phi([a(t)y^\Delta(t)]^\Delta) \geq L(\frac{c}{2})^\lambda \int_t^T P(s)\Delta s$, letting $T \to +\infty$, we have $r(t)\phi([a(t)y^\Delta(t)]^\Delta) \geq L(\frac{c}{2})^\lambda \int_t^{+\infty} P(s)\Delta s$, this implies that $[a(t)y^\Delta(t)]^\Delta \geq L^{1/\lambda} \frac{c}{2} (\frac{1}{r(t)} \int_t^{+\infty} P(s)\Delta s)^{1/\lambda}$, $t \in [t_2, +\infty)_{\mathbf{T}}$. In view of $y^\Delta(t) < 0$, similarly, we can get

$$y^\Delta(t) \leq -L^{1/\lambda} \frac{c}{2} \frac{1}{a(t)} \int_t^{+\infty} \left(\frac{1}{r(u)} \int_u^{+\infty} P(s)\Delta s \right)^{1/\lambda} \Delta u.$$

Integrating the above inequality from t_2 to t ($t \in [t_2, +\infty)_{\mathbf{T}}$), we obtain

$$y(t) \leq y(t_2) - L^{\frac{1}{\lambda}} \frac{c}{2} \int_{t_2}^t \left[\frac{1}{a(v)} \int_v^{+\infty} \left(\frac{1}{r(u)} \int_u^{+\infty} P(s)\Delta s \right)^{1/\lambda} \Delta u \right] \Delta v \to -\infty \quad (t \to +\infty),$$

which contradicts with $y(t) > 0$. So, $\lim_{t\to+\infty} y(t) = 0$, further, $\lim_{t\to+\infty} x(t) = 0$. This completes the proof. □

Due to the above reasons, in the next section, we assume that either (2.4) or (2.5) holds.

Lemma 2.7 [17] (Hölder's inequality) *Let $a, b \in \mathbf{T}$ and $a < b$. For rd-continuous functions $f, g : [a, b] \to \mathbf{R}$, we have $\int_a^b |f(u)g(u)| \Delta u \leq (\int_a^b |f(u)|^p \Delta u)^{\frac{1}{p}} (\int_a^b |g(u)|^q \Delta u)^{\frac{1}{q}}$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

3 Main results

In this section, we establish some sufficient conditions which guarantee that every solution $x(t)$ of (1.1) either oscillates on $[t_0, +\infty)_{\mathbf{T}}$ or converges as $t \to +\infty$.

Theorem 3.1 *If there exists a function $\varphi(t) \in C^1_{rd}(\mathbf{T}, (0, +\infty))$ with $\varphi^\Delta(t) \geq 0$, such that*

$$\limsup_{t \to +\infty} \int_{t_0}^t \left\{ L\varphi(\sigma(s))P(s)\Psi(s) - \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \right\} \Delta s = +\infty, \quad (3.1)$$

where $\Psi(t) = \frac{k^\lambda [1 - \beta B(\delta(t))]^\lambda [h_2(\delta(t), t_0)]^\lambda}{2^\lambda [a(\delta(t))\sigma(t)]^\lambda}$, then every solution $x(t)$ of equation (1.1) is either oscillatory or $\lim_{t \to +\infty} x(t) = 0$.

Proof Suppose that equation (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, +\infty)_{\mathbf{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$, $t_1 \in [t_0, +\infty)_{\mathbf{T}}$. Then, by Lemma 2.5, we see that $x(t)$ satisfies either case (i) or case (ii).

If case (i) in Lemma 2.5 holds, then in view of $x(t) \leq y(t)$, we have

$$y(t) \leq x(t) + \beta B(t)x(\tau(t)) \leq x(t) + \beta B(t)y(\tau(t)) \leq x(t) + \beta B(t)y(t),$$

i.e.,

$$x(t) \geq [1 - \beta B(t)]y(t). \tag{3.2}$$

Now define the function $V(t)$ by

$$V(t) = \varphi(t) \frac{r(t)\phi([a(t)y^\Delta(t)]^\Delta)}{\phi(a(t)y^\Delta(t))} = \varphi(t) \frac{r(t)[(a(t)y^\Delta(t))^\Delta]^\lambda}{(a(t)y^\Delta(t))^\lambda}, \quad t \in [t_1, +\infty)_{\mathbf{T}}. \tag{3.3}$$

Then $V(t) > 0$ ($t \in [t_1, +\infty)_{\mathbf{T}}$). In view of (2.2) and (3.3), we obtain for $t \in [t_1, +\infty)_{\mathbf{T}}$,

$$V^\Delta(t) = \varphi^\Delta(t) \frac{r(t)\phi([a(t)y^\Delta(t)]^\Delta)}{\phi(a(t)y^\Delta(t))} + \varphi(\sigma(t))$$

$$\times \frac{\{r(t)[(a(t)y^\Delta(t))^\Delta]^\lambda\}^\Delta (a(t)y^\Delta(t))^\lambda - r(t)[(a(t)y^\Delta(t))^\Delta]^\lambda [(a(t)y^\Delta(t))^\lambda]^\Delta}{(a(t)y^\Delta(t))^\lambda (a(\sigma(t))y^\Delta(\sigma(t)))^\lambda}$$

$$\leq \frac{\varphi^\Delta(t)}{\varphi(t)} V(t) - \varphi(\sigma(t))$$

$$\times \left\{ \frac{LP(t)(x(\delta(t)))^\lambda}{(a(\sigma(t))y^\Delta(\sigma(t)))^\lambda} + \frac{r(t)[(a(t)y^\Delta(t))^\Delta]^\lambda [(a(t)y^\Delta(t))^\lambda]^\Delta}{(a(t)y^\Delta(t))^\lambda (a(\sigma(t))y^\Delta(\sigma(t)))^\lambda} \right\}. \tag{3.4}$$

Now, let $u(t) = a(t)y^{\Delta}(t)$, then from (i) in Lemma 2.5, we have $u(t) > 0$, $u^{\Delta}(t) > 0$. In view of that

$$\left\{r(t)\left[a(t)y^{\Delta}(t)\right]^{\Delta}\right\}^{\Delta} = \left\{r(t)u^{\Delta}(t)\right\}^{\Delta} = r^{\Delta}(t)u^{\Delta}\left(\sigma(t)\right) + r(t)u^{\Delta\Delta}(t) < 0$$

and $r^{\Delta}(t) \geq 0$, it is not difficult to see that $u^{\Delta\Delta}(t) < 0$. Therefore, by Lemma 2.2, $\forall k \in (0,1)$, $\exists t_2 \in [t_1, +\infty)_{\mathbf{T}}$ with $t_2 \geq \max\{t_k, t_1\}$, such that $u(\sigma(t)) \leq \frac{\sigma(t)u(\delta(t))}{k\delta(t)} \leq \frac{\sigma(t)u(t)}{k\delta(t)}$ for all $t \in [t_2, +\infty)_{\mathbf{T}}$, this implies that

$$a\left(\sigma(t)\right)y^{\Delta}\left(\sigma(t)\right) \leq \frac{\sigma(t)a(\delta(t))y^{\Delta}(\delta(t))}{k\delta(t)} \leq \frac{\sigma(t)a(t)y^{\Delta}(t)}{k\delta(t)}. \tag{3.5}$$

From (2.1), we get $[(u(t))^{\lambda}]^{\Delta} \geq \lambda \int_0^1 [hu + (1-h)u]^{\lambda-1} u^{\Delta}(t)\, dh = \lambda(u(t))^{\lambda-1}u^{\Delta}(t)$, therefore,

$$\left[\left(a(t)y^{\Delta}(t)\right)^{\lambda}\right]^{\Delta} \geq \lambda\left(a(t)y^{\Delta}(t)\right)^{\lambda-1}\left(a(t)y^{\Delta}(t)\right)^{\Delta}. \tag{3.6}$$

Using (3.2), (3.5), and (3.6) in (3.4), we find

$$V^{\Delta}(t) \leq \frac{\varphi^{\Delta}(t)}{\varphi(t)}V(t) - \varphi\left(\sigma(t)\right)$$

$$\times \left\{\frac{LP(t)[1 - \beta B(\delta(t))]^{\lambda}(y(\delta(t)))^{\lambda}}{(a(\sigma(t))y^{\Delta}(\sigma(t)))^{\lambda}} + \frac{\lambda(k\delta(t))^{\lambda}r(t)[(a(t)y^{\Delta}(t))^{\Delta}]^{\lambda+1}}{(\sigma(t))^{\lambda}[a(t)y^{\Delta}(t)]^{\lambda+1}}\right\}$$

$$= \frac{\varphi^{\Delta}(t)}{\varphi(t)}V(t) - \varphi\left(\sigma(t)\right)\frac{LP(t)[1 - \beta B(\delta(t))]^{\lambda}(y(\delta(t)))^{\lambda}}{(a(\sigma(t))y^{\Delta}(\sigma(t)))^{\lambda}}$$

$$- \frac{\lambda k^{\lambda}\varphi(\sigma(t))r(t)\delta^{\lambda}(t)[V(t)]^{\frac{\lambda+1}{\lambda}}}{[r(t)\varphi(t)]^{(\lambda+1)/\lambda}(\sigma(t))^{\lambda}}$$

$$\leq \frac{\varphi^{\Delta}(t)}{\varphi(t)}V(t) - \varphi\left(\sigma(t)\right)\frac{LP(t)[1 - \beta B(\delta(t))]^{\lambda}(y(\delta(t)))^{\lambda}}{(a(\sigma(t))y^{\Delta}(\sigma(t)))^{\lambda}}$$

$$- \frac{\lambda k^{\lambda}\delta^{\lambda}(t)[V(t)]^{\frac{\lambda+1}{\lambda}}}{[r(t)\varphi(t)]^{1/\lambda}\sigma^{\lambda}(t)}. \tag{3.7}$$

On the other hand, let $U(t) = \int_{t_1}^{t} a(s)y^{\Delta}(s)\Delta s$ $(t \in (t_1, +\infty)_{\mathbf{T}})$, similarly, it is easy to see that $U(t) > 0$, $U^{\Delta}(t) > 0$, $U^{\Delta\Delta}(t) > 0$, $U^{\Delta\Delta\Delta}(t) < 0$. Therefore, by Lemma 2.4, $\exists t_{1/2} \in (t_1, +\infty)_{\mathbf{T}}$, such that $\frac{tU(t)}{h_2(t,t_0)U^{\Delta}(t)} \geq \frac{1}{2}$ for all $t \in [t_{1/2}, +\infty)_{\mathbf{T}}$. Then we see that

$$\frac{\int_{t_1}^{t} a(s)y^{\Delta}(s)\Delta s}{a(t)y^{\Delta}(t)} \geq \frac{h_2(t, t_0)}{2t}, \quad t \in [t_{1/2}, +\infty)_{\mathbf{T}}. \tag{3.8}$$

From $\int_{t_1}^{t} a(s)y^{\Delta}(s)\Delta s = a(t)y(t) - a(t_1)y(t_1) - \int_{t_1}^{t} a^{\Delta}(s)y(\sigma(s))\Delta s$, we get $a(t)y(t) \geq \int_{t_1}^{t} a(s) \times y^{\Delta}(s)\Delta s$. In view of (3.8), we then obtain

$$\frac{y(t)}{y^{\Delta}(t)} = \frac{a(t)y(t)}{a(t)y^{\Delta}(t)} \geq \frac{\int_{t_1}^{t} a(s)y^{\Delta}(s)\Delta s}{a(t)y^{\Delta}(t)} \geq \frac{h_2(t, t_0)}{2t}, \quad t \in [t_{1/2}, +\infty)_{\mathbf{T}}. \tag{3.9}$$

Therefore, by (3.5) and (3.9), $\exists T_0 \in [t_0, +\infty)_{\mathbf{T}}$ with $T_0 \geq \max\{t_2, t_{1/2}\}$, such that

$$\frac{y(\delta(t))}{a(\sigma(t))y^\Delta(\sigma(t))} = \frac{1}{a(\sigma(t))} \frac{y(\delta(t))}{y^\Delta(\delta(t))} \frac{y^\Delta(\delta(t))}{y^\Delta(\sigma(t))}$$

$$\geq \frac{1}{a(\sigma(t))} \frac{h_2(\delta(t), t_0)}{2\delta(t)} \frac{k\delta(t)a(\sigma(t))}{\sigma(t)a(\delta(t))} = \frac{kh_2(\delta(t), t_0)}{2a(\delta(t))\sigma(t)}$$

for all $t \in [T_0, +\infty)_{\mathbf{T}}$. Using the above inequality in (3.7), we obtain for all $t \in [T_0, +\infty)_{\mathbf{T}}$,

$$V^\Delta(t) \leq -\varphi(\sigma(t)) \frac{LP(t)[1 - \beta B(\delta(t))]^\lambda [kh_2(\delta(t), t_0)]^\lambda}{[2a(\delta(t))\sigma(t)]^\lambda} + \frac{\varphi^\Delta(t)}{\varphi(t)} V(t)$$

$$- \frac{\lambda k^\lambda \delta^\lambda(t)}{[r(t)\varphi(t)]^{1/\lambda}\sigma^\lambda(t)} \big[V(t)\big]^{\frac{\lambda+1}{\lambda}},$$

i.e.,

$$L\varphi(\sigma(t))P(t)\Psi(t) \leq -V^\Delta(t) + \frac{\varphi^\Delta(t)}{\varphi(t)} V(t) - \frac{\lambda k^\lambda \delta^\lambda(t)}{[r(t)\varphi(t)]^{1/\lambda}\sigma^\lambda(t)} \big[V(t)\big]^{\frac{\lambda+1}{\lambda}}. \tag{3.10}$$

In Lemma 2.3, we let

$$r = \frac{\lambda + 1}{\lambda}, \qquad a = \lambda^{\frac{\lambda}{\lambda+1}} \left[\frac{k\delta(t)}{(r(t)\varphi(t))^{1/\lambda^2}\sigma(t)}\right]^{\frac{\lambda^2}{\lambda+1}} V(t),$$

$$b = \frac{\lambda^{\frac{\lambda}{\lambda+1}}}{(\lambda+1)^\lambda} \left[\frac{\varphi^\Delta(t)}{\varphi(t)}\right]^\lambda \left[\frac{k\delta(t)}{(r(t)\varphi(t))^{1/\lambda^2}\sigma(t)}\right]^{\frac{-\lambda^3}{\lambda+1}}.$$

From Lemma 2.3, we then obtain

$$\frac{\varphi^\Delta(t)}{\varphi(t)} V(t) - \frac{\lambda k^\lambda \delta^\lambda(t)}{[r(t)\varphi(t)]^{1/\lambda}\sigma^\lambda(t)} \big[V(t)\big]^{\frac{\lambda+1}{\lambda}} \leq \frac{\varphi(t)r(t)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(t)}{\varphi(t)}\right]^{\lambda+1} \left[\frac{\sigma(t)}{\delta(t)}\right]^{\lambda^2}.$$

Using the above inequality in (3.10), we find

$$L\varphi(\sigma(t))P(t)\Psi(t) \leq -V^\Delta(t) + \frac{\varphi(t)r(t)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(t)}{\varphi(t)}\right]^{\lambda+1} \left[\frac{\sigma(t)}{\delta(t)}\right]^{\lambda^2}. \tag{3.11}$$

Integrating inequality (3.11) from T_0 to $t \in [T_0, +\infty)_{\mathbf{T}}$ provides

$$\int_{T_0}^t L\varphi(\sigma(s))P(s)\Psi(s)\Delta s \leq -V(t) + V(T_0) + \int_{T_0}^t \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)}\right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)}\right]^{\lambda^2}\Delta s.$$

Consequently,

$$\int_{T_0}^t \left\{ L\varphi(\sigma(s))P(s)\Psi(s) - \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)}\right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)}\right]^{\lambda^2} \right\}\Delta s$$

$$\leq -V(t) + V(T_0) \leq V(T_0).$$

Taking limsup on both sides of the above inequality as $t \to +\infty$, we obtain a contradiction to condition (3.1).

If case (ii) in Lemma 2.5 holds, then by Lemma 2.6, we have $\lim_{t\to+\infty} x(t) = 0$. This completes the proof. □

Remark 3.1 From Theorem 3.1, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $\varphi(t)$. For example, $\varphi(t) = M$ (where M is a constant) or $\varphi(t) = t$ ($k = 1/2$). Then we have the following results respectively.

Corollary 3.2 *If* $\limsup_{t\to+\infty} \int_{t_0}^{t} \frac{[1-\beta B(\delta(s))]^\lambda [h_2(\delta(s),t_0)]^\lambda}{[a(\delta(s))\sigma(s)]^\lambda} P(s)\Delta s = +\infty$, *then every solution* $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t\to+\infty} x(t) = 0$.

Corollary 3.3 *If* $\limsup_{t\to+\infty} \int_{t_0}^{t} \{ \frac{L[1-\beta B(\delta(s))]^\lambda [h_2(\delta(s),t_0)]^\lambda}{[4\sigma(s)a(\delta(s))]^\lambda} \sigma(s)P(s) - \frac{r(s)[\frac{2\sigma(s)}{\delta(s)}]^{\lambda^2}}{(\lambda+1)^{\lambda+1}s^\lambda} \}\Delta s = +\infty$, *then every solution* $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t\to+\infty} x(t) = 0$.

Theorem 3.4 *If there exists a function* $\varphi(t) \in C_{rd}^1(\mathbf{T}, (0, +\infty))$ *and constant* $m \geq 1$, *such that*

$$\limsup_{t\to+\infty} \frac{1}{t^m} \int_{T_0}^{t} (t-s)^m \Big\{ L\varphi(\sigma(s))P(s)\Psi(s)$$

$$- \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \Big[\frac{\varphi^\Delta(s)}{\varphi(s)} \Big]^{\lambda+1} \Big[\frac{\sigma(s)}{\delta(s)} \Big]^{\lambda^2} \Big\}\Delta s = +\infty \tag{3.12}$$

for some constant $T_0 \geq t_0$, *where the function* $\Psi(s)$ *is defined as in Theorem* 3.1, *then every solution* $x(t)$ *of equation* (1.1) *is either oscillatory or* $\lim_{t\to+\infty} x(t) = 0$.

Proof Suppose that equation (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, +\infty)_\mathbf{T}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_\mathbf{T}$, $t_1 \in [t_0, +\infty)_\mathbf{T}$. By Lemma 2.5 there are two possible cases. If case (ii) in Lemma 2.5 holds, then clearly $\lim_{t\to+\infty} x(t) = 0$. If case (i) in Lemma 2.5 holds, we proceed as in the proof of Theorem 3.1 to obtain (3.10). Then from (3.10), we have

$$L\varphi(\sigma(s))P(s)\Psi(s) \leq -V^\Delta(s) + \frac{\varphi^\Delta(s)}{\varphi(s)}V(s) - \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda}\sigma^\lambda(s)}\big[V(s)\big]^{\frac{\lambda+1}{\lambda}}$$

for all $s \in [T_0, +\infty)_\mathbf{T}$. Multiplying both sides of the above inequality by $(t-s)^m$, and integrating with respect to s from T_0 to t ($t \geq T_0$), we can obtain

$$\int_{T_0}^{t} L(t-s)^m \varphi(\sigma(s))P(s)\Psi(s)\Delta s$$

$$\leq -\int_{T_0}^{t} (t-s)^m V^\Delta(s)\Delta s$$

$$+ \int_{T_0}^{t} (t-s)^m \frac{\varphi^\Delta(s)}{\varphi(s)}V(s)\Delta s - \int_{T_0}^{t} \frac{\lambda k^\lambda (t-s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda}\sigma^\lambda(s)}\big[V(s)\big]^{\frac{\lambda+1}{\lambda}}\Delta s$$

$$= -\big[(t-s)^m V(s)\big]\big|_{T_0}^{t} + \int_{T_0}^{t} \big[(t-s)^m\big]_s^\Delta V(\sigma(s))\Delta s$$

$$+ \int_{T_0}^{t} (t-s)^m \frac{\varphi^\Delta(s)}{\varphi(s)}V(s)\Delta s - \int_{T_0}^{t} \frac{\lambda k^\lambda (t-s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda}\sigma^\lambda(s)}\big[V(s)\big]^{\frac{\lambda+1}{\lambda}}\Delta s$$

$$\leq (t - T_0)^m V(T_0) + \int_{T_0}^t (t - s)^m \frac{\varphi^\Delta(s)}{\varphi(s)} V(s) \Delta s$$

$$- \int_{T_0}^t \frac{\lambda k^\lambda (t - s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s. \tag{3.13}$$

Now, in Lemma 2.3, we let

$$r = \frac{\lambda + 1}{\lambda}, \qquad a = \lambda^{\frac{\lambda}{\lambda+1}} \left\{ \frac{k\delta(s)}{[r(s)\varphi(t)]^{1/\lambda^2} \sigma(s)} \right\}^{\frac{\lambda^2}{\lambda+1}} V(s) \quad \text{and}$$

$$b = \frac{\lambda^{\frac{\lambda}{\lambda+1}}}{(\lambda+1)^\lambda} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^\lambda \left\{ \frac{k\delta(s)}{[r(s)\varphi(s)]^{1/\lambda^2} \sigma(s)} \right\}^{\frac{-\lambda^3}{\lambda+1}}.$$

From Lemma 2.3, we then obtain

$$\frac{\varphi^\Delta(s)}{\varphi(s)} V(s) - \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \leq \frac{\varphi(s) r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2}.$$

Hence, (3.13) implies

$$\int_{T_0}^t L(t - s)^m \varphi(\sigma(s)) P(s) \Psi(s) \Delta s$$

$$\leq (t - T_0)^m V(T_0) + \int_{T_0}^t \frac{(t - s)^m \varphi(s) r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \Delta s$$

and, therefore,

$$\frac{1}{t^m} \int_{T_0}^t (t - s)^m \left\{ L\varphi(\sigma(s)) P(s) \Psi(s) - \frac{\varphi(s) r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \right\} \Delta s$$

$$\leq \left(1 - \frac{T_0}{t} \right)^m V(T_0). \tag{3.14}$$

Taking the limit superior as $t \to +\infty$ in the above inequality, we find

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{T_0}^t (t - s)^m \left\{ L\varphi(\sigma(s)) P(s) \Psi(s) - \frac{\varphi(s) r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \right\} \Delta s$$

$$< +\infty,$$

contradicting (3.12). This completes the proof. □

Remark 3.2 From Theorem 3.4, we can obtain different conditions for oscillation of all solutions of (1.1) with different choices of $\varphi(t)$. For example, $\varphi(t) = M$ (where M is a constant) or $\varphi(t) = t$ ($k = 1/2$), then we have the following results, respectively.

Corollary 3.5 *If there exists a constant $m \geq 1$, such that*

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{T_0}^t (t - s)^m \frac{P(s)[1 - \beta B(\delta(s))]^\lambda [h_2(\delta(s), t_0)]^\lambda}{[\sigma(s) a(\delta(s))]^\lambda} \Delta s < +\infty$$

for some constant $T_0 \geq t_0$, then every solution $x(t)$ of equation (1.1) *is either oscillatory or* $\lim_{t \to +\infty} x(t) = 0$.

Corollary 3.6 *If there exists a constant $m \geq 1$, such that*

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \left\{ \frac{L[1 - \beta B(\delta(s))]^\lambda [h_2(\delta(s), t_0)]^\lambda}{[4\sigma(s)a(\delta(s))]^\lambda} \sigma(s)P(s) - \frac{r(s)[\frac{2\sigma(s)}{\delta(s)}]^{\lambda^2}}{(\lambda+1)^{\lambda+1}s^\lambda} \right\} \Delta s$$

$$= +\infty$$

for some constant $T_0 \geq t_0$, then every solution $x(t)$ of equation (1.1) *is either oscillatory or* $\lim_{t \to +\infty} x(t) = 0$.

Remark 3.3 Clearly, Kamenev-type oscillation criteria for second-order linear differential equation was extended to third-order nonlinear variable delay dynamic equations on time scales. One can easily see that the recent results cannot be applied in equation (1.1), so our results are new ones.

If (3.12) does not hold, then we have the following result.

Theorem 3.7 *If there exist functions $\varphi(t) \in C^1_{\mathrm{rd}}(\mathbf{T}, (0, +\infty))$, $\xi(t) \in C_{\mathrm{rd}}(\mathbf{T}, \mathbf{R})$ and a constant $m \geq 1$, such that*

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{T_0}^{t} (t-s)^m \varphi(s)r(s) \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \Delta s < +\infty, \tag{3.15}$$

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{u}^{t} (t-s)^m \left\{ L\varphi(\sigma(s))P(s)\Psi(s) - \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \right\} \Delta s$$

$$\geq \xi(u), \quad u \geq T_0, \tag{3.16}$$

$$\int_{T_0}^{+\infty} \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda}\sigma^\lambda(s)} [\xi(s)]^{\frac{\lambda+1}{\lambda}} \Delta s = +\infty \tag{3.17}$$

for some constant $T_0 \geq t_0$, where $\xi_+(t) = \max\{\xi(t), 0\}$, $\Psi(s)$ is defined as in Theorem 3.1, *then every solution $x(t)$ of equation* (1.1) *is either oscillatory or* $\lim_{t \to +\infty} x(t) = 0$.

Proof Suppose that equation (1.1) has a nonoscillatory solution $x(t)$ on $[t_0, +\infty)_{\mathbf{T}}$. We may assume without loss of generality that $x(t) > 0$ and $x(\tau(t)) > 0$, $x(\delta(t)) > 0$ for all $t \in [t_1, +\infty)_{\mathbf{T}}$, $t_1 \in [t_0, +\infty)_{\mathbf{T}}$. By Lemma 2.5, there are two possible cases. If case (ii) in Lemma 2.5 holds, then clearly $\lim_{t \to +\infty} x(t) = 0$. If case (i) in Lemma 2.5 holds, we proceed as in the proof of Theorem 3.4 to obtain (3.13) and (3.14). Then from (3.14), for $t \geq u \geq T_0$, $t, u \in [T_0, +\infty)_{\mathbf{T}}$, we have

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{u}^{t} (t-s)^m \left\{ L\varphi(\sigma(s))P(s)\Psi(s) - \frac{\varphi(s)r(s)}{k^{\lambda^2}(\lambda+1)^{\lambda+1}} \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \right\} \Delta s$$

$$\leq V(u).$$

In view of (3.16), we can get

$$\xi(u) \le V(u), \quad u \ge T_0;$$

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{T_0}^t L(t-s)^m \varphi(\sigma(s)) P(s) \Psi(s) \Delta s \ge \xi(T_0).$$

(3.18)

From (3.13), we obtain for all $t \in [T_0, +\infty)_{\mathbf{T}}$,

$$\frac{1}{t^m} \int_{T_0}^t L(t-s)^m \varphi(\sigma(s)) P(s) \Psi(s) \Delta s$$

$$\le \left(1 - \frac{T_0}{t}\right)^m V(T_0) + \frac{1}{t^m} \int_{T_0}^t (t-s)^m \frac{\varphi^\Delta(s)}{\varphi(s)} V(s) \Delta s$$

$$- \frac{1}{t^m} \int_{T_0}^t \frac{\lambda k^\lambda (t-s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s.$$

Next, in the above inequality, we set

$$\varphi_1(t) = \frac{1}{t^m} \int_{T_0}^t (t-s)^m \frac{\varphi^\Delta(s)}{\varphi(s)} V(s) \Delta s \quad \text{and}$$

$$\varphi_2(t) = \frac{1}{t^m} \int_{T_0}^t \frac{\lambda k^\lambda (t-s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s,$$

in view of the second inequality in (3.18), then, it is not difficult to see that

$$\liminf_{t \to +\infty} [\varphi_2(t) - \varphi_1(t)]$$

$$\le V(T_0) - \limsup_{t \to +\infty} \frac{1}{t^m} \int_{T_0}^t L(t-s)^m \varphi(\sigma(s)) P(s) \Psi(s) \Delta s$$

$$\le V(T_0) - \xi(T_0) < +\infty.$$

Now we claim that

$$\int_{T_0}^{+\infty} \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s < +\infty.$$

(3.19)

Suppose to the contrary that

$$\int_{T_0}^{+\infty} \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s = +\infty.$$

(3.20)

Let $M > 0$ be arbitrary. Then it follows from the above formula that there exists $t_2 \in [T_0, +\infty)_{\mathbf{T}}$ such that

$$\int_{T_0}^t \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} [V(s)]^{\frac{\lambda+1}{\lambda}} \Delta s > \frac{M 2^m}{\lambda k^\lambda} \quad \text{for } t \in [t_2, +\infty)_{\mathbf{T}}.$$

Therefore (use the integration by parts formula $\int_a^b f(s)g^\Delta(s)\Delta s = [f(s)g(s)]_a^b - \int_a^b f^\Delta(s) \times g(\sigma(s))\Delta s$),

$$
\begin{aligned}
\varphi_2(t) &= \frac{1}{t^m}\int_{T_0}^t \frac{\lambda k^\lambda (t-s)^m \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda}\sigma^\lambda(s)}\big[V(s)\big]^{\frac{\lambda+1}{\lambda}}\Delta s \\
&= \frac{1}{t^m}\int_{T_0}^t \left\{\lambda k^\lambda (t-s)^m \left(\int_{T_0}^s \frac{\delta^\lambda(u)}{[r(u)\varphi(u)]^{1/\lambda}\sigma^\lambda(u)}\big[V(u)\big]^{\frac{\lambda+1}{\lambda}}\Delta u\right)_s^\Delta\right\}\Delta s \\
&= \frac{1}{t^m}\int_{T_0}^t \left\{\lambda k^\lambda \big[-(t-s)^m\big]_s^\Delta \int_{T_0}^{\sigma(s)} \frac{\delta^\lambda(u)}{[r(u)\varphi(u)]^{1/\lambda}\sigma^\lambda(u)}\big[V(u)\big]^{\frac{\lambda+1}{\lambda}}\Delta u\right\}\Delta s \\
&\geq \frac{1}{t^m}\int_{t_2}^t \left\{\lambda k^\lambda \big[-(t-s)^m\big]_s^\Delta \int_{T_0}^s \frac{\delta^\lambda(u)}{[r(u)\varphi(u)]^{1/\lambda}\sigma^\lambda(u)}\big[V(u)\big]^{\frac{\lambda+1}{\lambda}}\Delta u\right\}\Delta s \\
&> \frac{1}{t^m}\int_{t_2}^t \lambda k^\lambda \big[-(t-s)^m\big]_s^\Delta \frac{M2^m}{\lambda k^\lambda}\Delta s = \frac{M2^m}{t^m}\int_{t_2}^t \big[-(t-s)^m\big]_s^\Delta \Delta s = M2^m\left(1-\frac{t_2}{t}\right)^m.
\end{aligned}
$$

Now, taking $t_3 = 2t_2$, then for $t \in [t_3, +\infty)_{\mathbf{T}}$, $\varphi_2(t) > M2^m(1-\frac{t_2}{t})^m \geq M2^m(1-\frac{t_2}{2t_2})^m = M$. Since M is arbitrary,

$$\lim_{t\to+\infty}\varphi_2(t) = +\infty. \tag{3.21}$$

Next, we consider a sequence $\{T_n\}_{n=1}^{+\infty} : T_n \in [T_0, +\infty)_{\mathbf{T}}$ with $\lim_{n\to+\infty} T_n = +\infty$ satisfying

$$\lim_{n\to+\infty}\big[\varphi_2(T_n) - \varphi_1(T_n)\big] = \liminf_{t\to+\infty}\big[\varphi_2(t) - \varphi_1(t)\big] < +\infty.$$

Then there exists a constant $M_0 > 0$ such that

$$\varphi_2(T_n) - \varphi_1(T_n) \leq M_0 \tag{3.22}$$

for all sufficiently large positive integer n. Since (3.21) ensures that

$$\lim_{n\to+\infty}\varphi_2(T_n) = +\infty, \tag{3.23}$$

(3.22) implies that

$$\lim_{n\to+\infty}\varphi_1(T_n) = +\infty. \tag{3.24}$$

Furthermore, (3.22) and (3.23) lead to the inequality

$$\frac{\varphi_1(T_n)}{\varphi_2(T_n)} - 1 \geq -\frac{M_0}{\varphi_2(T_n)} > -\frac{M_0}{2M_0} = -\frac{1}{2}, \quad i.e. \ \frac{\varphi_1(T_n)}{\varphi_2(T_n)} > \frac{1}{2}$$

for large enough positive integer n, which together with (3.24) implies

$$\lim_{n\to+\infty}\frac{[\varphi_1(T_n)]^{\lambda+1}}{[\varphi_2(T_n)]^\lambda} = \lim_{n\to+\infty}\left[\frac{\varphi_1(T_n)}{\varphi_2(T_n)}\right]^\lambda \varphi_1(T_n) = +\infty. \tag{3.25}$$

On the other hand, by Lemma 2.7, we find that

$$
\begin{aligned}
\varphi_1(T_n) &= \frac{1}{T_n^m} \int_{T_0}^{T_n} (T_n - s)^m \frac{\varphi^\Delta(s)}{\varphi(s)} V(s) \Delta s \\
&= \int_{T_0}^{T_n} \left(\left[\frac{(T_n - s)^m}{T_n^m} \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \right]^{\frac{\lambda}{\lambda+1}} V(s) \right) \\
&\quad \times \left(\frac{(T_n - s)^m}{T_n^m} \frac{\varphi^\Delta(s)}{\varphi(s)} \left[\frac{(T_n - s)^m}{T_n^m} \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \right]^{\frac{-\lambda}{\lambda+1}} \right) \Delta s \\
&\leq \left\{ \int_{T_0}^{T_n} \frac{(T_n - s)^m}{T_n^m} \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \left[V(s) \right]^{\frac{\lambda+1}{\lambda}} \Delta s \right\}^{\frac{\lambda}{\lambda+1}} \\
&\quad \times \left\{ \int_{T_0}^{T_n} \left[\frac{(T_n - s)^m}{T_n^m} \frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{(T_n - s)^m}{T_n^m} \frac{\lambda k^\lambda \delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \right]^{-\lambda} \Delta s \right\}^{\frac{1}{\lambda+1}} \\
&= \left[\varphi_2(T_n) \right]^{\frac{\lambda}{\lambda+1}} \left\{ \frac{(\lambda k^\lambda)^{-\lambda}}{T_n^m} \int_{T_0}^{T_n} (T_n - s)^m r(s) \varphi(s) \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \Delta s \right\}^{\frac{1}{\lambda+1}}
\end{aligned}
$$

and accordingly,

$$
\frac{[\varphi_1(T_n)]^{\lambda+1}}{[\varphi_2(T_n)]^\lambda} \leq \frac{(\lambda k^\lambda)^{-\lambda}}{T_n^m} \int_{T_0}^{T_n} (T_n - s)^m r(s) \varphi(s) \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \Delta s.
$$

So, because of (3.25), we can obtain

$$
\lim_{n \to +\infty} \frac{1}{T_n^m} \int_{T_0}^{T_n} (T_n - s)^m r(s) \varphi(s) \left[\frac{\varphi^\Delta(s)}{\varphi(s)} \right]^{\lambda+1} \left[\frac{\sigma(s)}{\delta(s)} \right]^{\lambda^2} \Delta s = +\infty,
$$

contradicting (3.15). Therefore, (3.19) holds. Now, from the first formula of (3.18) and (3.19), we get

$$
\int_{T_0}^{+\infty} \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \left[\xi(s) \right]^{\frac{\lambda+1}{\lambda}} \Delta s \leq \int_{T_0}^{+\infty} \frac{\delta^\lambda(s)}{[r(s)\varphi(s)]^{1/\lambda} \sigma^\lambda(s)} \left[V(s) \right]^{\frac{\lambda+1}{\lambda}} \Delta s < +\infty,
$$

which contradicts (3.17). This completes the proof. □

Obviously, our results in this paper not only extend and improve some known results, and show some results of [3–8, 10, 14, 15] to be special examples of our results, but also unify the oscillation of the third-order nonlinear variable delay differential equations and the third-order nonlinear variable delay difference equations with a nonlinear neutral term. The theorems in this paper are new even for the cases $\mathbf{T} = \mathbf{R}$ and $\mathbf{T} = \mathbf{Z}$.

4 Applications and examples
In this section, we give some examples to illustrate our main results.

Example 4.1 Consider third-order delay differential equation

$$
\left\{ t^2 \left[\left(t x'(t) \right)' \right]^3 \right\}' + t(t-1)^2 (3t^2 - 8t + 2) e^{6t-18} x^9(t-2) = 0, \quad t \geq 2. \tag{4.1}
$$

It is easy to verify that all conditions of Corollary 3.3 are satisfied. Hence, every solution of equation (4.1) is oscillatory or tends to zero as $t \to +\infty$. For example, it is not difficult to verify that $x(t) = e^{-t}$ is a solution of equation (4.1). The important point to note here is that the recent results due to [9–13, 15] do not apply to equation (4.1) for the condition (1.7) or (1.8) can be a restrictive condition.

Example 4.2 Consider third-order variable delay dynamic equations on time scales

$$\left\{ t^{\frac{2}{5}} \phi\left(\left[t\big(x(t) + B(t)g\big(x(\tau(t))\big)\big)^{\Delta} \right]^{\Delta} \right) \right\}^{\Delta} + P(t)F\big(\phi\big(x(\delta(t))\big)\big) = 0, \quad t \in 2^{\mathbf{Z}}, t \geq 2. \quad (4.2)$$

This is a third-order 2-difference equation, here $t_0 = 2$, $r(t) = t^{\frac{2}{5}}$, $a(t) = t$. Now, pick $\lambda = \frac{3}{7}$, $B(t) = \frac{1}{\sqrt{2}} - \frac{1}{t}$, $\tau(t) = \delta(t) = \frac{t}{2}$, $P(t) = \frac{2^{\frac{3}{14}}}{t^{4/7}(t+\sqrt{2})^{3/7}} \left[\frac{\sigma(t)}{h_2(\delta(t),t_0)} \right]^{3/7}$, $g(u) = \frac{u}{\sqrt{2+\sin^4 u}}$, $f(u) = u[6 + \ln(1+u^2)]$, then

$$1 - \mu(t)\frac{b(t)}{r(t)} = 1 - t \cdot \frac{t^{-2}}{t^{2/5}} = 1 - \frac{1}{t^{7/5}} > 0 \quad (t \geq 2), \ i.e., \ -b/r \in \Re^{+},$$

$$e_{-b/r}(t,t_0) \geq 1 - \int_2^t \frac{b(s)}{r(s)} \Delta s = 1 - \int_2^t s^{-\frac{12}{5}} \Delta s = 1 - \frac{t^{-\frac{7}{5}} - 2^{-\frac{7}{5}}}{2^{-\frac{7}{5}} - 1} = \frac{t^{-\frac{7}{5}} + 1 - 2^{-\frac{2}{5}}}{1 - 2^{-\frac{7}{5}}}$$

$$> 1 - 2^{-\frac{2}{5}} > \frac{1}{5}$$

and so

$$\int_{t_0}^t \left[\frac{e_{-b/r}(s,t_0)}{r(s)} \right]^{1/\lambda} \Delta s \geq \int_2^t \left[\frac{1}{5s^{2/5}} \right]^{7/3} \Delta s = \frac{1}{5^{7/3}} \int_2^t s^{-\frac{14}{15}} \Delta s$$

$$= \frac{1}{5^{7/3}} \frac{t^{1/15} - 2^{1/15}}{2^{1/15} - 1} \to +\infty \quad (t \to +\infty).$$

Hence, conditions (H$_1$)-(H$_5$) are clearly satisfied. Let $m = 2$, in view of $\beta = 1/\sqrt{2}$, $L = 6$, then we have

$$\limsup_{t\to+\infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \frac{LP(s)[1 - \beta B(\delta(s))]^{\lambda} [h_2(\delta(s),t_0)]^{\lambda}}{[4a(\delta(s))\sigma(s)]^{\lambda}} \Delta s$$

$$= \frac{6}{4^{3/7}} \limsup_{t\to+\infty} \frac{1}{t^2} \int_2^t \frac{(t-s)^2}{s} \Delta s$$

$$= \frac{6}{4^{3/7}} \limsup_{t\to+\infty} \left\{ [\log_2(t) - 1] + \frac{1}{t^2}\left(\frac{t^2 - 2^2}{2^2 - 1} \right) - \frac{2}{t}(t-2) \right\} = +\infty,$$

$$\limsup_{t\to+\infty} \frac{1}{t^m} \int_{t_0}^t (t-s)^m \frac{[b(s)]^{\lambda+1}}{[r(s)]^{\lambda}} \Delta s$$

$$= \limsup_{t\to+\infty} \frac{1}{t^2} \int_2^t \frac{(t-s)^2}{s^{\frac{106}{35}}} \Delta s \leq \limsup_{t\to+\infty} \frac{1}{t^2} \int_2^t \frac{(t-s)^2}{s^3} \Delta s$$

$$= \limsup_{t\to+\infty} \frac{1}{t^2} \left[\frac{t^2(t^{-2} - 2^{-2})}{2^{-2} - 1} - \frac{2t(t^{-1} - 2^{-1})}{2^{-1} - 1} + [\log_2(t) - 1] \right] < +\infty.$$

This implies

$$\limsup_{t \to +\infty} \frac{1}{t^m} \int_{t_0}^{t} (t-s)^m \left\{ \frac{LP(s)[1 - \beta B(\delta(s))]^{\lambda}[h_2(\delta(s), t_0)]^{\lambda}}{[4\sigma(s)a(\delta(s))]^{\lambda}} - \frac{[b(s)]^{\lambda+1}}{(\lambda+1)^{\lambda+1}[r(s)]^{\lambda}} \right\} \Delta s$$

$$= +\infty$$

and so conditions of Corollary 3.6 are satisfied as well. Altogether, by Corollary 3.6, we have that every solution of equation (4.2) is oscillatory or tends to zero as $t \to +\infty$. But the results in [9–16] are inapplicable for equation (4.2).

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