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Exact solutions and conservation laws of a (3 + 1)-dimensional B-type Kadomtsev-Petviashvili equation

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Abstract

In this paper we study a (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili (BKP) equation. This equation is an extension of the well-known Kadomtsev-Petviashvili equation, which describes weakly dispersive and small amplitude waves propagating in quasi-two-dimensional media. We first obtain exact solutions of the BKP equation using the multiple-exp function and simplest equation methods. Furthermore, the conservation laws for the BKP equation are constructed by using the multiplier method.

Keywords: (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili equation; multiple-exp function method; simplest equation method; conservation laws

1 Introduction

It is well known that many phenomena in science and engineering, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and biology, are described by the nonlinear partial differential equations (NLPDEs). Therefore the investigation of exact solutions of NLPDEs plays an important role in the study of NLPDEs. For this reason, during the last few decades, researchers have established several methods to find exact solutions to NLPDEs. Some of these methods include the inverse scattering transform method [1], the Bäcklund transformation [2], the Darboux transformation [3], the Hirota bilinear method [4], the (G'/G)-expansion method [5], the homogeneous balance method [6], the variable separation approach [7], the tri-function method [8, 9], the sine-cosine method [10], the Jacobi elliptic function expansion method [11, 12], the exp-function expansion method [13] and the Lie symmetry method [14–16].

The purpose of this paper is to study one such NLPDE, namely the (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili (BKP) equation, that is given by [17]

$$u_{xxxy} + \alpha(u_x u_y)_x + (u_x + u_y + u_z)_t - (u_{xx} + u_{zz}) = 0, \tag{1.1}$$

where α is a real-valued constant. This is a nonlinear wave equation in three spatial (x, y, z) and one temporal coordinate (t).

It is well known that the Kadomtsev-Petviashvili (KP) equation describes weakly dispersive and small amplitude waves propagating in quasi-two-dimensional media [18]. The KP hierarchy of B-type possesses many integrable structures as the KP hierarchy. The (3 + 1)-dimensional nonlinear generalized BKP equation

$$u_{yt} - u_{xxxy} - 3(u_x u_y)_x + (3u_{xx} + 3u_{zz}) = 0, \tag{1.2}$$

was studied in [18–20] by different approaches. In [17] a new form of the (3 + 1)-dimensional BKP equation given by (1.1) was investigated and it was shown, using the simplified form of the Hirota method, that one- and two-soliton solutions exist for (1.1). Also, specific constraints were developed that guarantee the existence of multiple soliton solutions for (1.1).

In this paper we employ the multiple exp-function method [21] and the simplest equation method [22, 23] to obtain some exact solutions of (1.1). In addition to this, conservation laws are constructed for (1.1) using the multiplier method [24].

2 Exact solutions of (1.1)

In this section we employ two methods of solution.

2.1 Exact solutions using the multiple exp-function method

In this subsection we employ the multiple exp-function method and obtain exact explicit one-wave and two-wave solutions of (1.1). For details of the method, the reader is referred to the paper [20], in which this method was introduced. So, following the method and using the notation of [20], for a one-wave solution, we have

$$p = A_0 + A_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t},$$
$$q = B_0 + B_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t}$$

and the resulting one-wave solution is

$$u(x, y, z, t) = \frac{p}{q},$$

with

$$A_1 = \frac{(6k_1 B_0 + \alpha A_0) B_1}{\alpha B_0},$$
$$m_1 = \theta k_1,$$
$$\omega_1 = k_1^3,$$

where θ is any root of $\theta^2 + k_1^2 \theta + k_1^2 + 1 = 0$.

Likewise, for a two-wave solution, we have

$$\begin{aligned} p &= 2k_1 e^{k_1 x + l_1 y + m_1 z - \omega_1 t} + 2k_2 e^{k_2 x + l_2 y + m_2 z - \omega_2 t} \\ &\quad + 2A_{12}(k_1 + k_2) e^{k_1 x + l_1 y + m_1 z - \omega_1 t} e^{k_2 x + l_2 y + m_2 z - \omega_2 t}, \\ q &= 1 + e^{k_1 x + l_1 y + m_1 z - \omega_1 t} + e^{k_2 x + l_2 y + m_2 z - \omega_2 t} + A_{12} e^{k_1 x + l_1 y + m_1 z - \omega_1 t} e^{k_2 x + l_2 y + m_2 z - \omega_2 t} \end{aligned}$$

and the resulting two-wave solution is

$$u(x,y,z,t) = \frac{p}{q},$$

where

$$A_{12} = -1,$$
$$k_1 = 1,$$
$$k_2 = 1,$$
$$l_1 = 1,$$
$$l_2 = 1,$$
$$\alpha = 3,$$
$$m_1 = \theta,$$
$$\omega_1 = -\frac{-12 - 6m_2 - 2m_2^2 + 4\theta m_2 + \theta m_2^2}{(2+m_2)^2},$$
$$\omega_2 = -\frac{m_2^2}{2+m_2}$$

and θ is any root of $2\theta^2 - (m_2 - 6)\theta + 2m_2^2 + 6m_2 + 12 = 0$.

2.2 The simplest equation method

In this subsection we use the simplest equation method and obtain exact solutions of (1.1). This method was introduced by Kudryashov [22] and modified by Vitanov [23]. The simplest equations we use in this paper are the Bernoulli and Riccati equations. Their solutions can be written in elementary functions. For details, see, for example, [25].

Making use of the wave variable

$$\nu = k_1 x + k_2 y + k_3 z + k_4 t + k_5,$$

where k_i, $i = 1, \dots, 5$ are constants, the $(3+1)$-dimensional generalized B-type Kadomtsev-Petviashvili (1.1) transforms to a fourth-order nonlinear ordinary differential equation (ODE)

$$\begin{aligned} &k_2 k_1^3 F''''(\nu) - k_1^2 F''(\nu) + k_4 k_1 F''(\nu) - k_3^2 F''(\nu) \\ &\quad + k_2 k_4 F''(\nu) + k_3 k_4 F''(\nu) + 2\alpha k_2 k_1^2 F'(\nu) F''(\nu) = 0. \end{aligned} \tag{2.1}$$

Let us consider the solutions of ODE (2.1) in the form

$$F(\nu) = \sum_{i=0}^{M} A_i \big(G(\nu)\big)^i, \tag{2.2}$$

where $G(\nu)$ satisfies the Bernoulli and Riccati equations, M is a positive integer that can be determined by balancing procedure as in [23] and $A_0, \dots, A_M$ are parameters to be determined.

2.2.1 Solutions of (1.1) *using the Bernoulli equation as the simplest equation*

The balancing procedure yields $M = 1$ so the solutions of (2.1) are of the form

$$F(\nu) = A_0 + A_1 G. \tag{2.3}$$

Substituting (2.3) into ODE (2.1) and making use of the Bernoulli equation and then equating the coefficients of the functions G^i to zero, we obtain an algebraic system of equations. Solving this system with the aid of Mathematica, we obtain

$$\alpha = -\frac{6k_1 b}{A_1},$$
$$k_2 = \frac{k_1^2 + k_3^2 - k_1k_4 - k_3k_4}{k_1^3 a^2 + k_4}.$$

As a result, a solution of (1.1) is

$$u(x,y,z,t) = A_0 + A_1 a\left\{\frac{\cosh[a(\nu + C)] + \sinh[a(\nu + C)]}{1 - b\cosh[a(\nu + C)] - b\sinh[a(\nu + C)]}\right\},$$

where $\nu = k_1x + k_2y + k_3z + k_4t + k_5$ and C is a constant of integration.

2.2.2 Solutions of (1.1) *using the Riccati equation as the simplest equation*

The balancing procedure yields $M = 1$, so the solutions of (2.1) are of the form

$$F(\nu) = A_0 + A_1 G. \tag{2.4}$$

Substituting (2.4) into ODE (2.1) and making use of the Riccati equation, we obtain an algebraic system of equations by equating all coefficients of the functions G^i to zero. Solving the algebraic equations, one obtains

$$\alpha = -\frac{6k_1 a}{A_1},$$
$$c = \frac{k_2k_1^3b^2 + k_1k_4 - k_3^2 + k_3k_4 + k_2k_4 - k_1^2}{4k_2k_1^3 a}.$$

Hence solutions of (1.1) are

$$u(x,y,z,t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{1}{2}\theta(\nu + C)\right]\right\}$$

and

$$u(x,y,z,t) = A_0 + A_1\left\{-\frac{b}{2a} - \frac{\theta}{2a}\tanh\left(\frac{1}{2}\theta\nu\right) + \frac{\operatorname{sech}(\frac{\theta\nu}{2})}{C\cosh(\frac{\theta\nu}{2}) - \frac{2a}{\theta}\sinh(\frac{\theta\nu}{2})}\right\},$$

where $\nu = k_1x + k_2y + k_3z + k_4t + k_5$ and C is a constant of integration.

3 Conservation laws

In this section we construct conservation laws for (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili equation (1.1). The multiplier method will be used [15, 24, 26]. First we recall some results that will be used in the computation of conserved vectors.

3.1 Preliminaries

Consider a kth-order system of PDEs given by

$$E_\alpha(x, u, u_{(1)}, \dots, u_{(k)}) = 0, \quad \alpha = 1, \dots, m, \tag{3.1}$$

with n independent variables $x = (x^1, x^2, \dots, x^n)$ and m dependent variables $u = (u^1, u^2, \dots, u^m)$. Here $u_{(1)}, u_{(2)}, \dots, u_{(k)}$ denote the collections of all first, second, …, kth-order partial derivatives. That is, $u_i^\alpha = D_i(u^\alpha), u_{ij}^\alpha = D_j D_i(u^\alpha), \dots$, respectively, where the total derivative operator with respect to x^i is given by

$$D_i = \frac{\partial}{\partial x^i} + u_i^\alpha \frac{\partial}{\partial u^\alpha} + u_{ij}^\alpha \frac{\partial}{\partial u_j^\alpha} + \cdots, \quad i = 1, \dots, n. \tag{3.2}$$

The n-tuple $T = (T^1, T^2, \dots, T^n)$, $T^j \in \mathcal{A}$, $j = 1, \dots, n$, where $\mathcal{A}$ is the space of differential functions, is a conserved vector of (3.1) if T^i satisfies

$$D_i T^i|_{(3.1)} = 0 \tag{3.3}$$

and equation (3.3) defines a local conservation law of system (3.1).

The Euler-Lagrange operator, for each α, is defined as

$$\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} + \sum_{s \geq 1} (-1)^s D_{i_1} \cdots D_{i_s} \frac{\partial}{\partial u^\alpha_{i_1 i_2 \cdots i_s}}, \quad \alpha = 1, \dots, m. \tag{3.4}$$

A multiplier $\Lambda_\alpha(x, u, u_{(1)}, \dots)$ has the property that

$$\Lambda_\alpha E_\alpha = D_i T^i \tag{3.5}$$

hold identically. The right-hand side of (3.5) is a divergence expression. The determining equation for the multiplier Λ_α is given by

$$\frac{\delta(\Lambda_\alpha E_\alpha)}{\delta u^\alpha} = 0. \tag{3.6}$$

After obtaining the multipliers, we can calculate the conserved vectors by using a homotopy formula [24].

3.2 Construction of conservation laws for (1.1)

We now construct conservation laws for $(3 + 1)$-dimensional nonlinear BKP equation (1.1). We obtain a multiplier of the form

$$\Lambda = C u_x + f(t, y, z),$$

where C is an arbitrary constant and f is any solution of $f_{zz} - f_{tz} - f_{ty} = 0$. Corresponding to the above multiplier, we obtain the following conserved vectors:

$$T_1^t = \frac{1}{2}\left(-u_{xz}u - u_{xy}u + u_x^2\right),$$

$$T_1^x = \frac{1}{2}\left(-u_{zz}u + u_{tz}u + u_{ty}u + \alpha u_x^2 u_y + 2u_x u_{xxy} - u_x^2\right),$$

$$T_1^y = \frac{1}{6}\left(3u_tu_x + \alpha u_x^3 - 3u_{xx}^2\right),$$

$$T_1^z = \frac{1}{2}\left(u_{xz}u + u_tu_x + u_x(-u_z)\right)$$

and

$$T_2^t = \frac{1}{2}\left(f_z(-u) - f_yu + u_xf + u_zf + u_yf\right),$$

$$T_2^x = \frac{1}{4}(-\alpha f_yu_xu + 3\alpha u_xu_yf - \alpha u_{xy}fu - 2f_tu - 4u_xf + 3u_{xxy}f + 2u_tf - f_yu_{xx}),$$

$$T_2^y = \frac{1}{4}\left(\alpha u_x^2f + \alpha u_{xx}fu - 2f_tu + u_{xxx}f + 2u_tf\right),$$

$$T_2^z = \frac{1}{2}(2f_zu - f_tu - 2u_zf + u_tf).$$

Remark 1 Due to the presence of the arbitrary function f in the multiplier, one can obtain infinitely many conservation laws.

4 Concluding remarks

In this paper we studied (3 + 1)-dimensional generalized B-type Kadomtsev-Petviashvili equation (1.1). Exact solutions of the BKP equation were found using two distinct methods, namely the multiple-exp function method and the simplest equation method. Also, the conservation laws for the BKP equation were derived by using the multiplier method.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MA and CMK worked together in the derivation of the mathematical results. All authors read and approved the final manuscript.

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