Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

The C^0 solutions of the Feigenbaum-like functional equation

Ying Liang, Xiaopei Li* and Yuzhen Mi

*Correspondence:
lxpzhanjiang2013@163.com
Mathematics and Computational School, Zhanjiang Normal University, Zhanjiang, Guangdong 524048, China

Abstract

By using the Schauder's fixed point theorem, and constructing the special functional space and the construction operator, the existence, uniqueness, quasi-convexity (or quasi-concavity), symmetry and stability of the C^0 solutions of the Feigenbaum-like functional equations are discussed.

MSC: 39B22; 39B12; 39A10

Keywords: Feigenbaum-like functional equations; quasi-convex (or quasi-concave); symmetry; stability

1 Introduction

As early as 1978, Feigenbaum found the period-doubling bifurcation phenomenon by researching the iteration of a single-peak function class [1]. To reveal the mechanism of the Feigenbaum phenomenon, many years ago, the Feigenbaum functional equation had been researched extensively. McCarthy [2] obtained the general continuous exact bijective solutions. Epstein [3] gave a new proof of the existence of analytic, unimodal solutions by taking advantage of the normality properties of Herglotz functions and the Schauder-Tikhonov theorem. Eokmann and Wittwer [4] studied the Feigenbaum fixed point by using the computer. Thompson [5] investigated an essentially singular solution by expressing Feigenbaum's equation as a singular Schroder functional equation whose solution was obtained using a scaling ansatz, and so on. Thus, some solutions in specific cases were found.

Specifically, in 1985, to give a feasible method, the second kind of the Feigenbaum functional equation,

$$\begin{cases} f(x) = \frac{1}{\lambda} f(f(\lambda x)), & 0 < \lambda < 1, \\ f(0) = 1, \\ 0 \le f(x) \le 1, & x \in [0,1] \end{cases}$$

a kind of the equivalent equation, was given by Yang and Zhang [6]. The continuous valley-unimodal solutions were shown by using the constructive method. Recently, there have been a lots of results about the polynomial-like iterative equation. In 1987, by using Schauder's fixed point theorem to an operator defined by a linear combination of iterates of the unknown mapping f, a result on the existence of continuous solutions of the polynomial-like iterative equation was given in [7]. Furthermore, the results were given for its differentiable solutions [8]. Then the convex solutions and concave solutions [9, 10], the analytic solutions [11–13], the symmetric solutions [14], the higher-dimensional solutions [15], and the results on the unit circle [16] were obtained. In order to understand the dynamics of a second order delay differential equation with a piecewise constant argument, the derived planar mappings and their invariant curves were studied [17]. Based on the iterative root theory for monotone functions, an algorithm for computing polygonal iterative roots of increasing polygonal functions was given [18]. Else, a problem about the Hyers-Ulam stability was raised first by Ulam in 1940 and solved for Cauchy equation by Hyers [19]. Later, many papers on the Hyers-Ulam stability have been published, especially, for the polynomial-like iterative equation [20–22].

In this paper, by using Schauder's fixed point theorem, and constructing the special functional space and the construction operator, we consider the properties of the solutions of the Feigenbaum-like functional equation, which is a non-extended iterative equation,

$$\begin{cases} f(x) = \frac{1}{\lambda+1}f^2(\lambda x) + \frac{\lambda}{\lambda+1}g(x), \quad 0 < \lambda < 1, \\ a \le f(x) \le b, \quad x \in I, \end{cases} \tag{1.1}$$

where $g(x)$ is a given disturbance function, $f(x)$ is an unknown function, and $f^2(x) = f(f(x))$, $I = [a,b]$. It is clear that $a \le 0 \le b$, since $\lambda x \in I$ for all $x \in I$. We give not only the existence of continuous solutions of (1.1) but also their uniqueness, stability (the continuous dependence and the Hyers-Ulam stability), quasi-convexity (or quasi-concavity), symmetry by applying fixed point theorems. Finally, we give an example to verify those conditions given in theorems.

2 Preliminary

In this section, we give several important definitions, lemmas and notions.

Let $C^0(I,\mathbf{R}) = \{f : I \to \mathbf{R}, f \text{ is continuous}\}$. Obviously, $C^0(I,\mathbf{R})$ is a Banach space with the norm $\|\cdot\|_{c^0}$, where the norm $\|f\|_{c^0} = \max_{x\in I} |f(x)|$ for $f \in C^0(I,\mathbf{R})$.

Let $C^0(I) = \{f \in C^0(I,\mathbf{R}) : a \le f(x) \le b, a \le f(\lambda x) \le b, f \text{ is continuous}\}$. Then $C^0(I)$ is a complete metric space.

Let $X(I;M) = \{f \in C^0(I) : |f(x) - f(y)| \le M|y - x|, \forall x,y \in I\}$, where M is a positive constant.

Let $X(I;m,M) = \{f \in X(I;M) : |f(x) - f(y)| \ge m|x_2 - x_1|, \forall x \in I, 0 < m \le M\}$, where m is a positive constant.

Let $f(\lambda x) := f^{(\lambda)}(x), f^2(\lambda x) := f(f(\lambda x)) := f^{2(\lambda)}(x), f^k(\lambda x) := f^{k(\lambda)}(x)$.

Definition 2.1 $f : I \to R$ is a quasi-convex (or quasi-concave) function [23] if for $\forall x,y \in I$ and $\lambda \in [0,1]$, we have

$$f\bigl(\lambda x + (1-\lambda)y\bigr) \le \max\bigl\{f(x),f(y)\bigr\} \qquad \bigl(\text{or} \quad f\bigl(\lambda x + (1-\lambda)y\bigr) \ge \min\bigl\{f(x),f(y)\bigr\}\bigr).$$

Let $X_\sigma(I;M)$ denote the families consisting of all quasi-convex functions or quasi-concave ones in $X(I;M)$, where $\sigma = qcv$ or $\sigma = qcc$.

The following Lemma 2.1 and Lemma 2.2 are useful, and the methods of their proofs are similar to ones in the paper [24].

Lemma 2.1 *$X(I;M)$, $X_{qcv}(I;M)$, and $X_{qcc}(I;M)$ are compact convex subsets of $C^0(I,R)$.*

Lemma 2.2 *The composition $f \circ g$ is quasi-convex (or quasi-concave) if f is increasing and g is quasi-convex (or quasi-concave). In particular, for an increasing quasi-convex (or quasi-concave) function f, f^k is also quasi-convex (or quasi-concave).*

Lemma 2.3 *If $f, h \in X(I;M)$, then*

$$\left\| f^{2(\lambda)} - h^{2(\lambda)} \right\|_{c^0} \le (M+1)\|f-h\|_{c^0}. \tag{2.1}$$

Proof Note that

$$\begin{aligned} \left\| f^{2(\lambda)} - h^{2(\lambda)} \right\|_{c^0} &= \max_{x\in I}\left|f^2(\lambda x) - h^2(\lambda x)\right| \\ &\le \max_{x\in I}\left|f\bigl(f(\lambda x)\bigr) - f\bigl(h(\lambda x)\bigr)\right| + \max_{x\in I}\left|f\bigl(h(\lambda x)\bigr) - h\bigl(h(\lambda x)\bigr)\right| \\ &\le M\left\|f^{(\lambda)} - h^{(\lambda)}\right\|_{c^0} + \|f-h\|_{c^0}. \end{aligned}$$

Let $y = \lambda x$. Then $y \in \lambda I$, and

$$\begin{aligned} \left\| f^{(\lambda)} - h^{(\lambda)} \right\|_{c^0} &= \max_{y\in \lambda I}\left|f(y) - h(y)\right| \\ &\le \max_{y\in I}\left|f(y) - h(y)\right| \\ &\le \|f-h\|_{c^0}. \end{aligned}$$

Then

$$\begin{aligned} \left\| f^{2(\lambda)} - h^{2(\lambda)} \right\|_{c^0} &\le M\|f-h\|_{c^0} + \|f-h\|_{c^0} \\ &= (M+1)\|f-h\|_{c^0}. \end{aligned}$$

Thus, (2.1) holds. □

Lemma 2.4 *Suppose that $\varphi \in X(I;m,M)$. If the positive constants m, M and λ satisfy*

$$\frac{1}{\lambda} + 1 > \frac{M^2}{m}, \tag{2.2}$$

then $L\varphi$, defined by

$$L\varphi(x) = \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi^2\bigl(\lambda\varphi^{-1}(x)\bigr), \tag{2.3}$$

is an orientation-preserving homeomorphism from I onto itself, and

$$(L\varphi)^{-1} \in X\left(I; \frac{1}{\xi_2}, \frac{1}{\xi_1}\right), \tag{2.4}$$

where

$$\xi_1 = 1 + \frac{1}{\lambda} + \frac{M^2}{m}, \qquad \xi_2 = 1 + \frac{1}{\lambda} - \frac{M^2}{m} > 0. \tag{2.5}$$

Proof Because $\varphi \in X(I; m, M)$, by the paper [7], $\varphi^{-1} \in X(I; \frac{1}{M}, \frac{1}{m})$. Thus, for any $x_1 \neq x_2 \in I$, by (2.3) and (2.5)

$$\begin{aligned}
\left|L\varphi(x_2) - L\varphi(x_1)\right| &= \left|\left(1 + \frac{1}{\lambda}\right)(x_2 - x_1) + \frac{1}{\lambda}\left(\varphi^2\left(\lambda\varphi^{-1}(x_2)\right) - \varphi^2\left(\lambda\varphi^{-1}(x_1)\right)\right)\right| \\
&\geq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| - M^2\left|\varphi^{-1}(x_2) - \varphi^{-1}(x_1)\right| \\
&\geq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| - \frac{M^2}{m}|x_2 - x_1| \\
&\geq \xi_2|x_2 - x_1| > 0.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|L\varphi(x_2) - L\varphi(x_1)\right| &= \left|\left(1 + \frac{1}{\lambda}\right)(x_2 - x_1) + \frac{1}{\lambda}\left(\varphi^2\left(\lambda\varphi^{-1}(x_2)\right) - \varphi^2\left(\lambda\varphi^{-1}(x_1)\right)\right)\right| \\
&\leq \left(1 + \frac{1}{\lambda}\right)|x_2 - x_1| + \frac{M^2}{m}|x_2 - x_1| \\
&\leq \xi_1|x_2 - x_1|.
\end{aligned}$$

Therefore, $L\varphi \in X(I; \xi_1, \xi_2)$. This implies that $L\varphi$ is strictly increasing and invertible on I, and $(L\varphi)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$. □

Lemma 2.5 *Suppose that* $g \in X(I; m_1, M_1)$ *and* $\varphi_0 \in X(I; m, M)$. *If*

$$1 + \frac{1}{\lambda} > \max\left\{\frac{M^2}{m}, \frac{1}{2}\left(\frac{M_1}{M} + \frac{m_1}{m}\right)\right\}, \tag{2.6}$$

then

$$\varphi_k := (L\varphi_{k-1})^{-1} \circ g \tag{2.7}$$

and

$$L\varphi_{k-1}(x) := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_{k-1}^2\left(\lambda\varphi_{k-1}^{-1}(x)\right) \tag{2.8}$$

are well defined, and $\varphi_k \in X(I; m, M)$, $k = 1, 2, \ldots$.

Proof Let

$$L\varphi_0 := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_0^2\left(\lambda\varphi_0^{-1}(x)\right). \tag{2.9}$$

From Lemma 2.4, $L\varphi_0$ is well defined and is an orientation-preserving homeomorphism from I onto itself, and $(L\varphi_0)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$. Then $\varphi_1(x) := (L\varphi_0)^{-1} \circ g(x)$ is well defined and $\varphi_1 \in X(I; \frac{m_1}{\xi_2}, \frac{M_1}{\xi_1}) \subset X(I; m, M)$ by (2.6) and Lemma 2.4. If

$$L\varphi_k := \left(1 + \frac{1}{\lambda}\right)x - \frac{1}{\lambda}\varphi_k^2\left(\lambda\varphi_k^{-1}(x)\right) \tag{2.10}$$

is well defined and is an orientation-preserving homeomorphism from I onto itself, then $(L\varphi_k)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$. We similarly see that

$$\varphi_{k+1}(x) := (L\varphi_k)^{-1} \circ g(x) \tag{2.11}$$

is well defined, and

$$\varphi_{k+1} \in X\left(I; \frac{m_k}{\xi_2}, \frac{M_k}{\xi_1}\right) \subset X(I; m, M). \tag{2.12}$$

This implies that the results in Lemma 2.5 are also true for $k + 1$, which completes the proof. □

3 Main results

In this section, we give several important theorems on the existence, uniqueness, quasi-convex (quasi-concave), symmetry and stability of equation (1.1).

Theorem 3.1 (Existence) *Given a positive constant M_1 and $g(x) \in X(I; M_1)$. If there exist constants M and λ such that*

$$\frac{\lambda}{\lambda + 1}\left(M^2 + M_1\right) \le M, \tag{3.1}$$

then equation (1.1) *has a solution f in $X(I; M)$.*

Proof Define $T : X(I; M) \to C^0(I)$ by

$$Tf(x) = \frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x), \quad \forall x \in I. \tag{3.2}$$

Because $f, f(\lambda x)$ and g are continuous for all $x \in I$, we obtain that Tf is continuous for all $x \in I$, and $Tf \in C^0(I)$. By (3.1), for any x, y in I,

$$\begin{aligned}
\left|Tf(x) - Tf(y)\right| &= \left|\frac{1}{\lambda + 1}f^2(\lambda x) + \frac{\lambda}{\lambda + 1}g(x) - \frac{1}{\lambda + 1}f^2(\lambda y) - \frac{\lambda}{\lambda + 1}g(y)\right| \\
&\le \frac{1}{\lambda + 1}\left|f^2(\lambda x) - f^2(\lambda y)\right| + \frac{\lambda}{\lambda + 1}\left|g(x) - g(y)\right| \\
&\le \frac{M}{\lambda + 1}\left|f(\lambda x) - f(\lambda y)\right| + \frac{\lambda M_1}{\lambda + 1}|x - y| \\
&\le \frac{\lambda M^2}{\lambda + 1}|x - y| + \frac{\lambda M_1}{\lambda + 1}|x - y| \\
&= \frac{\lambda}{\lambda + 1}\left(M^2 + M_1\right)|x - y| \\
&\le M|x - y|.
\end{aligned}$$

Thus, $T_f \in X(I,M)$. Now we prove the continuity of T under the norm $\|\cdot\|_{c^0}$. For arbitrary $f_1, f_2 \in X(I,M)$,

$$\begin{aligned}\|Tf_1 - Tf_2\|_{c^0} &= \max_{x\in I}\left|Tf_1(x) - Tf_2(x)\right| \\ &= \max_{x\in I}\left|\frac{1}{\lambda+1}f_1^2(\lambda x) - \frac{1}{\lambda+1}f_2^2(\lambda x)\right| \\ &= \frac{1}{\lambda+1}\max_{x\in I}\left|f_1^2(\lambda x) - f_2^2(\lambda x)\right| \\ &= \frac{1}{\lambda+1}\left\|f_1^{2(\lambda)} - f_2^{2(\lambda)}\right\| \\ &\le E\|f_1 - f_2\|_{C_0},\end{aligned}$$

where Lemma 2.3 is used and

$$E = \frac{M+1}{\lambda+1}. \tag{3.3}$$

Thus, T is continuous under the norm $\|\cdot\|_{c^0}$. Summarizing all the above, we see that T is a continuous mapping from the compact convex subset $X(I,M)$ of the Banach space $C^0(I,R)$ into itself. By Schauder's fixed point theorem, we assert that there is a mapping $f \in X(I,M)$ such that

$$f(x) = Tf(x) = \frac{1}{\lambda+1}f^2(\lambda x) + \frac{\lambda}{\lambda+1}g(x), \quad \forall x \in I. \tag{3.4}$$

This completes the proof. □

Theorem 3.2 (Uniqueness) *Suppose that* (3.1) *is satisfied and*

$$M < \lambda. \tag{3.5}$$

For any function $g \in X(I,M_1)$, equation (1.1) *has a unique solution $f \in X(I,M)$.*

Proof The existence of equation (1.1) in $X(I;M)$ is given by Theorem 3.1. Note that $X(I;M)$ is a closed subset of $C^0(I)$. By (3.3) and (3.5), we see that $T: X(I;M) \to X(I;M)$ is a contraction mapping. Therefore T has a unique fixed point $f(x)$ in $X(I;M)$, that is, equation (1.1) has a unique solution $f(x)$ in $X(I;M)$. □

Below, we discuss the quasi-convex (or quasi-concave) solutions of equation (1.1).

Definition 3.1 Suppose that Γ is a Lie group of all linear transformations on R. A mapping $f: R \to R$, is said to be Γ-equivariant [25] if $f(\gamma x) = \gamma f(x)$, $\forall \gamma \in \Gamma$, $\forall x \in R$.

This implies that f^i is the Γ-equivariant. Let $G_\Gamma(I) = \{g \in C^0(I) \mid g(\gamma x) = \gamma g(x), \forall \gamma \in \Gamma, \forall x \in I\}$ and $G_\Gamma(I;M) = G_\Gamma(I) \cap X(I;M)$.

Lemma 3.1 *$G_\Gamma(I)$ is a closed convex subset of $C^0(I)$, and $G_\Gamma(I;M)$ is a compact convex subset of $C^0(I)$.*

The methods of the proofs are similar to the paper [24].

Theorem 3.3 (Quasi-convexity (Quasi-concavity)) *If* $g \in X_\sigma(I;M_1)$, (3.1) *and* (3.5) *are satisfied, then* (1.1) *has a solution* $f \in X_\sigma(I;M)$.

Proof Define a mapping $T : X_{qcv}(I;M) \to C^0(I)$ as in (3.2). Note that each $f \in X_{qcv}(I;M)$ is an increasing function. In fact, if $x < y$ in I, there exists $t_0 \in (0,1)$ such that $x = t_0 a + (1-t_0)y$, and by the quasi-convexity, we get

$$f(x) \le \max\bigl(f(a), f(y)\bigr) = f(y). \tag{3.6}$$

Thus, for $f \in X_{qcv}(I,M)$, $x, y \in I$, and $t \in [0,1]$, we get

$$\begin{aligned} Tf\bigl(tx + (1-t)y\bigr) &= \frac{1}{\lambda+1} f^2\bigl(\lambda\bigl(tx + (1-t)y\bigr)\bigr) + \frac{\lambda}{\lambda+1} g\bigl(tx + (1-t)y\bigr) \\ &\le \frac{1}{\lambda+1} f\bigl(\max\bigl\{f(\lambda x), f(\lambda y)\bigr\}\bigr) + \frac{\lambda}{\lambda+1} \max\bigl\{g(x), g(y)\bigr\} \\ &\le \frac{1}{\lambda+1} \max\bigl\{f^2(\lambda x), f^2(\lambda y)\bigr\} + \frac{\lambda}{\lambda+1} \max\bigl\{g(x), g(y)\bigr\} \\ &\le \max\bigl\{Tf(x), Tf(y)\bigr\}. \end{aligned}$$

Thus, T maps $X_{qcv}(I,M)$ into itself. Similarly, we can prove that T is continuous. By Lemma 2.1, $X_{qcv}(I,M)$ is a compact convex subset of the Banach space $C^0(I)$. Then Schauder's fixed point theorem guarantees the existence of a fixed point f of T in $X_{qcv}(I;M)$. In the same way, the proof of the quasi-concave solution of equation (1.1) is similarly obtained. □

Now, we study the symmetric solutions of (1.1).

Theorem 3.4 (Symmetry) *If* (3.1) *and* (3.5) *are satisfied, and* $g \in G_\Gamma(I;M_1)$, *then equation* (1.1) *has a unique* Γ*-equivariant solution* $f \in G_\Gamma(I;M_2)$.

Proof By Lemma 3.1 and (3.2), we have

$$\begin{aligned} Tf(\gamma x) &= \frac{1}{\lambda+1} f^2(\gamma\lambda x) + \frac{\lambda}{\lambda+1} g(\gamma x) \\ &= \gamma \frac{1}{\lambda+1} f^2(\lambda x) + \gamma \frac{\lambda}{\lambda+1} g(x) \\ &= \gamma Tf(x). \end{aligned}$$

From Theorem 3.1, we can find that T is a contraction mapping in $G_\Gamma(I;M_2)$. Since $G_\Gamma(I;M_2)$ is a compact convex subset of $C^0(I)$, by Banach's fixed point theorem, we assert that there is a unique fixed point $f \in G_\Gamma(I;M_2)$. □

In the following, we give the conditions to guarantee two kinds of stability: the continuous dependence and the Hyers-Ulam stability [21].

Theorem 3.5 (Continuous dependence) *If* (3.1) *and* (3.5) *are satisfied, the solutions of* (1.1) *in* $X(I;M)$ *is continuously dependent on the given function* $g(x)$ *in* $X(I;M_1)$.

Proof For $g_1, g_2 \in X(I; M_1)$, Theorem 3.1 implies that there are functions $f_1, f_2 \in X(I; M_1)$ such that

$$f_1(x) = \frac{1}{\lambda+1} f_1^2(\lambda x) + \frac{\lambda}{\lambda+1} g_1(x), \tag{3.7}$$

$$f_2(x) = \frac{1}{\lambda+1} f_2^2(\lambda x) + \frac{\lambda}{\lambda+1} g_2(x). \tag{3.8}$$

Thus, by Lemma 2.3

$$\begin{aligned}
\|f_1 - f_2\|_{c^0} &= \max_{x\in I} \left|f_1(x) - f_2(x)\right| \\
&\le \frac{1}{\lambda+1} \max_{x\in I} \left|f_1^2(\lambda x) - f_2^2(\lambda x)\right| + \frac{\lambda}{\lambda+1} \max_{x\in I} \left|g_1(x) - g_2(x)\right| \\
&= \frac{1}{\lambda+1} \left\|f_1^{2(\lambda)} - f_2^{2(\lambda)}\right\|_{c^0} + \frac{\lambda}{\lambda+1} \|g_1 - g_2\|_{c^0} \\
&\le \frac{M+1}{\lambda+1} \|f_1 - f_2\|_{c^0} + \frac{\lambda}{\lambda+1} \|g_1 - g_2\|_{c^0},
\end{aligned}$$

which implies

$$\left(1 - \frac{M+1}{\lambda+1}\right) \|f_1 - f_2\|_{c^0} \le \frac{\lambda}{\lambda+1} \|g_1 - g_2\|_{c^0},$$

so

$$\|f_1 - f_2\|_{c^0} \le \frac{\lambda}{\lambda - M} \|g_1 - g_2\|_{c^0}. \tag{3.9}$$

Inequality (3.5) yields that the solution of (1.1) in $X(I; M)$ is continuously dependent on the given function g in $X(I; M_1)$. □

Definition 3.2 The functional equation

$$E_1(\varphi) = E_2(\varphi) \tag{3.10}$$

has the Hyers-Ulam stability [26] if for any approximate solution φ_s such as $\|E_1(\varphi_s) = E_2(\varphi_s)\| \le \delta$, there exist a solution φ of equation (3.10) such as $\|\varphi - \varphi_s\| \le \varepsilon$, where $\delta \ge 0$, $\varepsilon > 0$, and a constant ε dependent only on δ.

Theorem 3.6 (Hyers-Ulam stability) *Suppose that $g \in X(I; m_1, M_1)$, and $\varphi_s \in X(I; m, M)$ satisfy*

$$\left| g(x) - \left(1 + \frac{1}{\lambda}\right) \varphi_s(x) + \frac{1}{\lambda} \varphi_s^2(\lambda x) \right| \le \delta, \quad \forall x \in I, \tag{3.11}$$

where $\delta > 0$ is a positive constant. If (2.6), (3.1) *and* (3.5) *are satisfied, there exists a unique continuous solution $\varphi \in X(I; m, M)$ of* (1.1) *such that*

$$\left|\varphi(x) - \varphi_s(x)\right| \le \zeta \delta, \quad \forall x \in I, \tag{3.12}$$

where $\zeta = \frac{1}{\xi_1(1-\eta)}$, $\eta = \frac{M+1}{\lambda \xi_1} + \frac{M^2}{m_1} < 1$.

Proof Construct a sequence $\{\varphi_k\}$ of functions as follows. Take $\varphi_0 = \varphi_s$, and then define φ_k as in (2.7) and $L\varphi_k$ as in (2.8). By Lemma 2.4, both φ_k and $L\varphi_k$ are well defined for any integer $k \geq 1$. Lemma 2.4 and Lemma 2.5 also imply that φ_k or $L\varphi_k$ is an orientation-preserving homeomorphism from I into itself with $(L\varphi_k)^{-1} \in X(I; \frac{1}{\xi_2}, \frac{1}{\xi_1})$, where ξ_1 and ξ_2 are given in (2.5).

Now we claim that

$$\|g - L\varphi_{k-1} \circ \varphi_{k-1}\| \leq \eta^{k-1}\delta, \tag{3.13}$$

$$\|\varphi_k - \varphi_{k-1}\| \leq \eta^{k-1}\frac{\delta}{\xi_1} \tag{3.14}$$

for all $x \in I$ and $k = 1, 2, \ldots$.

The case $k = 1$ is trivial. Assume that (3.13) and (3.14) hold for k. Since

$$\begin{aligned}
\left\|g(x) - L\varphi_k \circ \varphi_k\right\| &\leq \|L\varphi_{k-1} \circ \varphi_k - L\varphi_k \circ \varphi_k\| \\
&\leq \frac{1}{\lambda}\left\|-\varphi_{k-1}^2\left(\lambda\varphi_{k-1}^{-1}(\varphi_k)\right) + \varphi_k^2\left(\lambda\varphi_k^{-1}(\varphi_k)\right)\right\| \\
&\leq \frac{1}{\lambda}\left\|\varphi_k^2\left(\lambda\varphi_k^{-1}\right) - \varphi_{k-1}^2\left(\lambda\varphi_{k-1}^{-1}\right)\right\| \\
&\leq \frac{1}{\lambda}\left(\left\|\varphi_k^2\left(\lambda\varphi_k^{-1}\right) - \varphi_k^2\left(\lambda\varphi_{k-1}^{-1}\right)\right\| + \left\|\varphi_k^2\left(\lambda\varphi_{k-1}^{-1}\right) - \varphi_{k-1}^2\left(\lambda\varphi_{k-1}^{-1}\right)\right\|\right) \\
&\leq \frac{1}{\lambda}\left(\lambda M^2\left\|\varphi_k^{-1} - \varphi_{k-1}^{-1}\right\| + (M+1)\|\varphi_k - \varphi_{k-1}\|\right),
\end{aligned}$$

where Lemma 2.3 is applied. From the hypothesis of induction, it follows that

$$\begin{aligned}
\left\|g(x) - L\varphi_k \circ \varphi_k\right\| &\leq M^2\left\|\varphi_k^{-1} - \varphi_{k-1}^{-1}\right\| + \frac{(M+1)}{\lambda}\|\varphi_k - \varphi_{k-1}\| \\
&\leq \left(\frac{M^2}{m_1} + \frac{M+1}{\lambda\xi_1}\right)^k \delta = \eta^k\delta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\|\varphi_{k+1} - \varphi_k\| &= \left\|(L\varphi_k)^{-1} \circ g - (L\varphi_k)^{-1} \circ L\varphi_k \circ \varphi_k\right\| \\
&\leq \frac{1}{\xi_1}\|g - L\varphi_k \circ \varphi_k\| \\
&\leq \eta^k\frac{\delta}{\xi_1}.
\end{aligned}$$

Thus, (3.13) and (3.14) hold.

On the other hand, for any positive integers k and l with $k > l$, by (3.14)

$$\begin{aligned}
\|\varphi_k - \varphi_l\| &\leq \|\varphi_k - \varphi_{k-1}\| + \|\varphi_{k-1} - \varphi_{k-2}\| + \cdots + \|\varphi_{l+1} - \varphi_l\| \\
&\leq \eta^{k-1}\frac{\delta}{\xi_1} + \eta^{k-2}\frac{\delta}{\xi_1} + \cdots + \eta^l\frac{\delta}{\xi_1} \\
&\leq \frac{\eta^k - \eta^l}{1-\eta}\frac{\delta}{\xi_1}.
\end{aligned} \tag{3.15}$$

Note that $\eta < 1$, so from (3.15), it follows that

$$\|\varphi_k - \varphi_l\| \to 0 \quad \text{as } k, l \to +\infty.$$

This implies that $\{\varphi_k(x)\}$ is a Cauchy sequence. Hence, $\{\varphi_k(x)\}$ uniformly converges in the Banach space $C^0(I)$. Let $\varphi = \lim_{k\to+\infty} \varphi_k(x)$. Clearly, $\varphi \in X(I; m, M)$. From (3.13),

$$\|g - L\varphi \circ \varphi\| = \lim_{k\to+\infty} \|g - L\varphi_k \circ \varphi_k\| \le \lim_{k\to+\infty} \eta^k \delta = 0, \tag{3.16}$$

i.e., φ is a solution of (1.1). Furthermore, from (3.14)

$$\begin{aligned}
\|\varphi - \varphi_s\| &= \lim_{k\to+\infty} \|\varphi_k - \varphi_0\| \\
&\le \lim_{k\to+\infty} \bigl(\|\varphi_k - \varphi_{k-1}\| + \|\varphi_{k-1} - \varphi_{k-2}\| + \cdots + \|\varphi_1 - \varphi_0\|\bigr) \\
&\le \lim_{k\to+\infty} \left(\eta^{k-1}\frac{\delta}{\xi_1} + \eta^{k-2}\frac{\delta}{\xi_1} + \cdots + \frac{\delta}{\xi_1}\right) \\
&\le \frac{\frac{\delta}{\xi_1}}{1-\eta} = \frac{1}{\xi_1(1-\eta)}\delta.
\end{aligned}$$

Thus, $\|\varphi - \varphi_s\| < \zeta\delta$. Then (3.12) holds.

Concerning the uniqueness, we assume that there is another continuous solution $\phi \in X(I; m, M)$ ($\phi \neq \varphi$), such that

$$\bigl|\phi(x) - \varphi_s(x)\bigr| \le \varepsilon,$$

where $\varepsilon > 0$ only depends on δ. Then

$$\begin{aligned}
\|\varphi - \phi\| &= \bigl\|(L\varphi)^{-1} \circ g - (L\phi)^{-1} \circ g\bigr\| \\
&\le \bigl\|(L\varphi)^{-1} - (L\phi)^{-1}\bigr\| \\
&\le \bigl\|(L\varphi)^{-1} - (L\varphi)^{-1} \circ (L\varphi) \circ (L\phi)^{-1}\bigr\| \\
&\le \frac{1}{\xi_1}\bigl\|(L\phi) \circ (L\phi)^{-1} - (L\varphi) \circ (L\phi)^{-1}\bigr\| \\
&\le \frac{1}{\xi_1}\|L\varphi - L\phi\| \\
&\le \frac{1}{\lambda\xi_1}\bigl(\bigl\|\varphi^2(\lambda\varphi^{-1}) - \phi^2(\lambda\varphi^{-1})\bigr\| + \bigl\|\phi^2(\lambda\varphi^{-1}) - \phi^2(\lambda\phi^{-1})\bigr\|\bigr) \\
&\le \frac{M+1}{\lambda\xi_1}\|\varphi - \phi\| + \frac{M^2}{\xi_1}\bigl\|\varphi^{-1} - \phi^{-1}\bigr\| \\
&\le \frac{M+1}{\lambda\xi_1}\|\varphi - \phi\| + \frac{M^2}{\xi_1}\bigl\|\varphi^{-1} - \varphi^{-1} \circ \varphi \circ \phi^{-1}\bigr\| \\
&\le \frac{M+1}{\lambda\xi_1}\|\varphi - \phi\| + \frac{M^2}{m\xi_1}\bigl\|\phi \circ \phi^{-1} - \varphi \circ \phi^{-1}\bigr\| \\
&\le \frac{M+1}{\lambda\xi_1}\|\varphi - \phi\| + \frac{M^2}{m\xi_1}\|\phi - \varphi\|,
\end{aligned}$$

that is,

$$\left(1-\frac{M+1}{\lambda\xi_1}-\frac{M^2}{m\xi_1}\right)\|\varphi-\phi\|\leq 0. \tag{3.17}$$

The assumption of Theorem 3.6 yields $\|\varphi-\phi\|=0$, *i.e.*, $\varphi=\phi$, which contradicts with the assumption. The proof is completed. □

4 Example

Example 1 Consider the equation

$$f(x)=\frac{2}{3}f^2\left(\frac{1}{2}x\right)+\frac{1}{3}g(x), \tag{4.1}$$

where $x\in[0,1]$ and $\lambda=\frac{1}{2}$. Let

$$g(x)=\begin{cases}\frac{1}{8}x, & 0\leq x\leq\frac{1}{4},\\ \frac{1}{4}x-\frac{1}{32}, & \frac{1}{4}<x\leq\frac{1}{2},\\ \frac{1}{8}x+\frac{1}{32}, & \frac{1}{2}<x\leq\frac{3}{4},\\ \frac{1}{2}x-\frac{1}{4}, & \frac{3}{4}<x\leq 1.\end{cases} \tag{4.2}$$

Then $g(x)$ is quasi-convex and nonconvex (see Figure 1). Note that, for $x\in[0,\frac{1}{4}]$ and $y\in[\frac{3}{4},1]$

$$\begin{aligned}\left|f(x)-f(y)\right|&=\left|\frac{1}{2}y-\frac{1}{4}-\frac{1}{8}x\right|=\left|\frac{1}{8}(y-x)+\frac{3}{8}\left(y-\frac{2}{3}\right)\right|\\&\leq\frac{1}{8}|y-x|+\frac{3}{8}|y-x|=\frac{1}{2}|y-x|.\end{aligned}$$

Similarly, we can show that for any $x,y\in[0,1]$, $|f(x)-f(y)|\leq\frac{1}{2}|y-x|$. Thus, $M_1=\frac{1}{2}$.

For $x\in[\frac{1}{4},\frac{1}{2}]$ and $y\in[\frac{3}{4},1]$, we have

$$\left|f(x)-f(y)\right|=\left|\frac{1}{2}y-\frac{1}{4}-\frac{1}{4}x+\frac{1}{32}\right|\geq\frac{1}{4}|y-x|-\frac{1}{4}\left|y-\frac{7}{8}\right|.$$

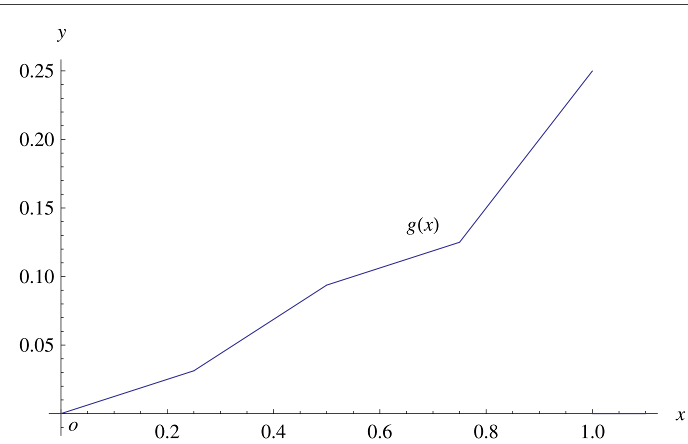

Figure 1 The graph of *g*(*x*).

If $1 \geq y \geq \frac{7}{8}$, then $|f(x) - f(y)| \geq \frac{1}{4}|y - x| \geq \frac{1}{8}|y - x|$; if $\frac{7}{8} \geq y \geq \frac{3}{4}$, then

$$\left|f(x) - f(y)\right| \geq \frac{1}{4}|y - x| - \frac{1}{32} \geq \frac{3}{16}|y - x| \geq \frac{1}{8}|y - x|,$$

since $\frac{1}{2} \leq |y - x| \leq \frac{3}{4}$. Similarly, we can show that for any $x, y \in [0, 1]$, $|f(x) - f(y)| \geq \frac{1}{8}|y - x|$. Thus, $m_1 = \frac{1}{8}$. Therefore, we can get a quasi-convex solution $f(x)$ of equation (4.1) by Theorem 3.3, which is continuously dependent on the given function $g(x) \in X(I; M_1)$ with $M = \frac{3-\sqrt{7}}{2}$ by Theorem 3.5. Moreover, equation (4.1) satisfies the Hyers-Ulam stability in $X(I; \frac{131-49\sqrt{7}}{152-48\sqrt{7}}, \frac{3-\sqrt{7}}{2})$ by Theorem 3.6.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All the authors have contributed in all the parts, and they have read and approved the final manuscript.

Acknowledgements
This work was supported by the PhD Start-up Fund of the Natural Science Foundation of Guangdong Province, China (S2011040000464), the Project of Department of Education of Guangdong Province, China (2012KJCX0074), the Natural Science Funds of Zhanjiang Normal University (QL1002, LZL1101), and the Doctoral Project of Zhanjiang Normal University (ZL1109). The authors would like to thank Dr. Shengfu Deng for his very helpful comments and suggestions.

Received: 21 March 2013 Accepted: 23 July 2013 Published: 6 August 2013

References

1. Feigenbaum, MJ: Quantitative universality for a class of non-linear transformations. J. Stat. Phys. **19**, 25-52 (1978)
2. McCarthy, PJ: The general exact bijective continuous solution of Feigenbaum's functional equation. Commun. Math. Phys. **91**(3), 431-443 (1983)
3. Epstein, H: New proofs of the existence of the Feigenbaum function. Commun. Math. Phys. **106**, 395-426 (1986)
4. Eckmann, JP, Wittwer, P: A complete proof of the Feigenbaum conjectures. J. Stat. Phys. **46**, 455-477 (1987)
5. Thompson, CJ, McGuire, JB: Asymptotic and essentially singular solution of the Feigenbaum equation. J. Stat. Phys. **51**(5-6), 991-1007 (1988)
6. Yang, L, Zhang, JZ: The second type of Feigenbaum's functional equation. Sci. China Ser. A **15**, 1061-1069 (1986) (in Chinese)
7. Zhang, WN: Discussion on the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$. Chin. Sci. Bull. **32**, 1441-1451 (1987)
8. Zhang, WN: On continuous solutions of *n*-th order polynomial-like iterative equations. Publ. Math. (Debr.) **76**(1-2), 117-134 (2010)
9. Xu, B, Zhang, WN: Decreasing solutions and convex solutions of the polynomial-like iterative equation. J. Math. Anal. Appl. **329**(1), 483-497 (2007)
10. Si, JG, Zhang, M: Construction of convex solutions of the second type of Feigenbaum's functional equations. Sci. China Ser. A **39**(1), 49-70 (2009) (in Chinese)
11. Xu, B, Zhang, WN, Si, JG: Analytic solutions of an iterative functional differential equation which may violate the Diophantine condition. J. Differ. Equ. Appl. **10**(2), 201-211 (2004)
12. Si, JG: The existence of local analytic solutions of the iterated equations. Acta Math. Sci. **14**(4), 53-63 (1994)
13. Si, JG, Zhang, WN: Analytic solutions of an iterative functional differential equation. Appl. Math. Comput. **150**(3), 647-659 (2004)
14. Zhang, WN, Baker, JA: Continuous solutions of a polynomial-like iterative equation with variable coefficients. Ann. Pol. Math. **73**, 29-36 (2000)
15. Li, XP, Deng, S: Differentiability for the high dimensional polynomial-like iterative equation. Acta Math. Sci. Ser. B **25**(1), 130-136 (2005)
16. Li, XP, Deng, S: An iterative equation on the unit circle. Acta Math. Sci. **26**(3), 145-149 (2006)
17. Deng, S: Remark on invariant curves for planar mappings. Appl. Math. Comput. **217**, 8419-8424 (2011)
18. Zhang, WX, Zhang, WN: Computing iterative roots of polygonal functions. J. Comput. Appl. Math. **205**(1), 497-508 (2007)
19. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
20. Xu, B, Zhang, WN: Construction of continuous solution and stability polynomial-like iterative equation. J. Math. Anal. Appl. **325**, 1160-1170 (2007)
21. Xu, B, Zhang, WN: Hyers-Ulam stability for a nonlinear iterative equation. Colloq. Math. **93**(1), 1-9 (2002)
22. Zhang, WX, Xu, B: Hyers-Ulam-Rassias stability for a multivalued iterative equation. Acta Math. Sci. **28**(1), 54-62 (2008)
23. Roberts, AW, Varberg, DE: Convex Functions. Academic Press, New York (1973)
24. Zhang, WN: Discussion on the differentiable solutions of the iterated equation $\sum_{i=1}^{n} \lambda_i f^i(x) = F(x)$. Nonlinear Anal., Theory Methods Appl. **15**(4), 387-398 (1990)
25. Golubitsky, M, Stewart, IN, Schaeffer, DG: Singularities and Groups in Bifurcation Theory, vol. II. Applied Mathematical Sciences, vol. 69. Springer, New York (1988)
26. Kim, GH: On the stability of generalized gamma functional equation. Int. J. Math. Math. Sci. **23**(8), 513-520 (2000)

doi:10.1186/1687-1847-2013-231
Cite this article as: Liang et al.: The C^0 solutions of the Feigenbaum-like functional equation. *Advances in Difference Equations* 2013 **2013**:231.