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Symmetric identities for Carlitz's q-Bernoulli numbers and polynomials

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Abstract
In this paper, a further investigation for the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials is performed, and several symmetric identities for these numbers and polynomials are established by applying elementary methods and techniques. It turns out that various known results are derived as special cases.
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1 Introduction

The classical Bernoulli numbers B_n and Bernoulli polynomials $B_n(x)$ are usually defined by the following generating functions

$$\frac{t}{e^t-1}=\sum_{n=0}^{\infty}B_n\frac{t^n}{n!}\quad\bigl(|t|<2\pi\bigr)\tag{1.1}$$

and

$$\frac{te^{xt}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}\quad\bigl(|t|<2\pi\bigr).\tag{1.2}$$

Clearly, $B_n=B_n(0)$. These numbers and polynomials play important roles in many different areas of mathematics such as number theory, combinatorics, special function and analysis, and numerous interesting properties for them have been explored, see, for example, [1, 2].

In the present paper, we will be concerned with the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials. Throughout this paper, it is supposed that $q\in\mathbb{C}$ with $|q|<1$ and $\mathbb{C}$ being a complex number field. For $x\in\mathbb{C}$, the q-number is defined by (see [3])

$$[x]_q=\frac{1-q^x}{1-q}=1+q+\cdots+q^{x-1}.\tag{1.3}$$

Obviously, $\lim_{q\to1}[x]_q=x$.

We now recall the q-Bernoulli numbers $\beta_n = \beta_n(q)$ and q-Bernoulli polynomials $\beta_n(x,q)$, which were introduced by Carlitz [4], as follows

$$\beta_0(q) = 1, \qquad q\bigl(q\beta(q)+1\bigr)^n - \beta_n(q) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1 \end{cases} \tag{1.4}$$

and

$$\beta_n(x,q) = \bigl(q^x\beta(q) + [x]_q\bigr)^n = \sum_{k=0}^{n} \binom{n}{k} \beta_k(q) q^{kx} [x]_q^{n-k} \quad (n \geq 0) \tag{1.5}$$

with the usual convention about replacing β_i by β^i. Since the above Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials appeared, different properties of the q-Bernoulli numbers and q-Bernoulli polynomials have been well studied by many authors, see [5] for a good introduction. In fact, the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials can be defined by the following generating functions (see [6, 7])

$$\sum_{m=0}^{\infty} q^m e^{[m]_q t}\bigl(1 - q - q^m t\bigr) = \sum_{n=0}^{\infty} \beta_n(q) \frac{t^n}{n!} \quad \bigl(|t + \log q| < 2\pi\bigr) \tag{1.6}$$

and

$$\sum_{m=0}^{\infty} q^m e^{[x+m]_q t}\bigl(1 - q - q^{x+m} t\bigr) = \sum_{n=0}^{\infty} \beta_n(x,q) \frac{t^n}{n!} \quad \bigl(|t + \log q| < 2\pi\bigr). \tag{1.7}$$

From (1.6) and (1.7) one can easily obtain

$$\beta_n(q) = \beta_n(0,q), \qquad \lim_{q\to 1} \beta_n(q) = B_n, \qquad \lim_{q\to 1} \beta_n(x,q) = B_n(x). \tag{1.8}$$

If the left-hand side of (1.7) is denoted by $F_q(t,x)$, then by the Mellin transform,

$$\frac{1}{\Gamma(s)} \int_0^{\infty} F_q(-t,x) t^{s-2}\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{q^{x+2n}}{[x+n]_q^s} + \frac{1-q}{s-1} \sum_{n=0}^{\infty} \frac{q^n}{[x+n]_q^{s-1}} \tag{1.9}$$

with $s \in \mathbb{C}$ and $x \neq 0, -1, -2, \ldots$.

Based on the observation on (1.9), Ryoo *et al.* [6] extended the classical Hurwitz zeta function $\zeta(s,x)$

$$\zeta(s,x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad \bigl(s \in \mathbb{C}, \operatorname{Re}(s) > 1; x \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}\bigr), \tag{1.10}$$

to the following q-zeta function

$$\zeta_q(s,x) = \sum_{n=0}^{\infty} \frac{q^n}{[x+n]_q^s} + (1-q)\left(\frac{2-s}{s-1}\right) \sum_{n=0}^{\infty} \frac{q^n}{[x+n]_q^{s-1}}, \tag{1.11}$$

where $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$ and $x \notin \mathbb{Z}_0^- = \{0,-1,-2,\ldots\}$.

The aim of the present paper is to perform a further investigation on the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials. By applying elementary methods and techniques, we establish some symmetric identities for these numbers and polynomials, by virtue of which, various known results are derived as special cases.

2 The restatement of results

In this section, a further investigation for the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials is performed, and several symmetric identities for them are established. We firstly state the following result for the q-zeta function.

Theorem 2.1 *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then for any positive integers a, b,*

$$[b]_q^s \sum_{j=0}^{a-1} q^{bj} \zeta_{q^a}\left(s, bx + \frac{bj}{a}\right) = [a]_q^s \sum_{j=0}^{b-1} q^{aj} \zeta_{q^b}\left(s, ax + \frac{aj}{b}\right). \tag{2.1}$$

Proof By substituting $bx + bj/a$ for x in (1.11), we have

$$\zeta_q\left(s, bx + \frac{bj}{a}\right) = \sum_{n=0}^{\infty} \frac{q^n}{[bx + bj/a + n]_q^s} + (1-q)\left(\frac{2-s}{s-1}\right) \sum_{n=0}^{\infty} \frac{q^n}{[bx + bj/a + n]_q^{s-1}}. \tag{2.2}$$

Note that for any $x \in \mathbb{C}$ and positive integer n, $[nx]_q = [x]_{q^n} \cdot [n]_q$. Hence, by replacing q by q^a in (2.2), we derive

$$\begin{aligned} &\zeta_{q^a}\left(s, bx + \frac{bj}{a}\right) \\ &\quad = \sum_{n=0}^{\infty} \frac{q^{an}[a]_q^s}{[abx + bj + an]_q^s} + \left(1 - q^a\right)\left(\frac{2-s}{s-1}\right) \sum_{n=0}^{\infty} \frac{q^{an}[a]_q^{s-1}}{[abx + bj + an]_q^{s-1}} \\ &\quad = \sum_{n=0}^{\infty} \frac{q^{an}[a]_q^s}{[abx + bj + an]_q^s}\left(1 + (1-q)[abx + bj + an]_q\left(\frac{2-s}{s-1}\right)\right). \end{aligned} \tag{2.3}$$

Since for any non-negative integer n and a positive integer b, there exist unique non-negative integers r and i such that $n = br + i$ with $0 \leq i \leq b-1$. So, the above identity (2.3) can be rewritten as

$$\begin{aligned} \zeta_{q^a}\left(s, bx + \frac{bj}{a}\right) = {}& [a]_q^s \sum_{r=0}^{\infty} \sum_{i=0}^{b-1} \frac{q^{abr+ai}}{[ab(x+r) + bj + ai]_q^s} \\ & \times \left(1 + (1-q)\left[ab(x+r) + bj + ai\right]_q \left(\frac{2-s}{s-1}\right)\right). \end{aligned} \tag{2.4}$$

It follows from (2.4) that

$$\begin{aligned} [b]_q^s \sum_{j=0}^{a-1} q^{bj} \zeta_{q^a}\left(s, bx + \frac{bj}{a}\right) = {}& [a]_q^s [b]_q^s \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} \sum_{n=0}^{\infty} \frac{q^{abn+ai+bj}}{[ab(x+n) + ai + bj]_q^s} \\ & \times \left(1 + (1-q)\left[ab(x+n) + ai + bj\right]_q \left(\frac{2-s}{s-1}\right)\right). \end{aligned} \tag{2.5}$$

In the same way,

$$[a]_q^s \sum_{j=0}^{b-1} q^{aj} \zeta_{q^b}\left(s, ax + \frac{aj}{b}\right) = [a]_q^s [b]_q^s \sum_{j=0}^{a-1} \sum_{i=0}^{b-1} \sum_{n=0}^{\infty} \frac{q^{abn+ai+bj}}{[ab(x+n)+ai+bj]_q^s} \times \left(1 + (1-q)\left[ab(x+n)+ai+bj\right]_q \left(\frac{2-s}{s-1}\right)\right). \tag{2.6}$$

Thus, equating (2.5) and (2.6) gives the desired result. □

We next discuss some special cases of Theorem 2.1. Setting $b = 1$ in Theorem 2.1, we have the following distribution formula for the q-zeta function

$$\sum_{j=0}^{a-1} q^j \zeta_{q^a}\left(s, x + \frac{j}{a}\right) = [a]_q^s \zeta_q(s, ax). \tag{2.7}$$

In particular, the case $a = 2$ in (2.7) gives the duplication formula for the q-zeta function

$$\zeta_{q^2}(s, x) + q\zeta_{q^2}\left(s, x + \frac{1}{2}\right) = [2]_q^s \zeta_q(s, 2x). \tag{2.8}$$

Letting $q \to 1$ in (2.7) and (2.8) leads to the familiar distribution formula for the classical Hurwitz zeta function

$$\sum_{j=0}^{a-1} \zeta\left(s, x + \frac{j}{a}\right) = a^s \zeta(s, ax), \tag{2.9}$$

and the duplication formula for the classical Hurwitz zeta function

$$\zeta(s, x) + \zeta\left(s, x + \frac{1}{2}\right) = 2^s \zeta(s, 2x), \tag{2.10}$$

respectively, see, for example, [1].

Now, we are in the position to give some similar symmetric identities for the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials.

Theorem 2.2 *Let a, b be positive integers. Then for any non-negative integer n,*

$$[a]_q^{n-1} \sum_{j=0}^{a-1} q^{bj} \beta_n\left(bx + \frac{bj}{a}, q^a\right) = [b]_q^{n-1} \sum_{j=0}^{b-1} q^{aj} \beta_n\left(ax + \frac{aj}{b}, q^b\right). \tag{2.11}$$

Proof By applying the exponential series $e^{xt} = \sum_{n=0}^{\infty} x^n t^n / n!$ to the generating function of the q-Bernoulli polynomials, we have

$$-\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q^{x+2m} [x+m]_q^n \frac{t^{n+1}}{n!} + (1-q) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q^m [x+m]_q^n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \beta_n(x, q) \frac{t^n}{n!}. \tag{2.12}$$

Comparison of the coefficients of $t^n/n!$ in (2.12) yields

$$\begin{aligned}\beta_n(x,q) &= -n\sum_{m=0}^{\infty} q^{x+2m}[x+m]_q^{n-1} + (1-q)\sum_{m=0}^{\infty} q^m[x+m]_q^n \\ &= -n\sum_{m=0}^{\infty} q^m[x+m]_q^{n-1} + (1-q)(1+n)\sum_{m=0}^{\infty} q^m[x+m]_q^n. \end{aligned} \tag{2.13}$$

So, from (1.11), (2.13) and the analytic continuation of $\zeta_q(s,x)$, one can easily obtain that for any non-negative integer n,

$$\zeta_q(1-n,x) = -\frac{\beta_n(x,q)}{n}. \tag{2.14}$$

In light of the relation (see, *e.g.*, [4, 6])

$$\beta_n(x,q) = \frac{1}{(1-q)^n}\sum_{k=0}^{n}\binom{n}{k}(-1)^k q^{kx}\frac{k+1}{[k+1]_q} \quad (n \geq 0), \tag{2.15}$$

we get the symmetric distribution for the q-Bernoulli polynomials

$$\beta_n\bigl(1-x,q^{-1}\bigr) = (-q)^n\beta_n(x,q) \quad (n \geq 0). \tag{2.16}$$

Thus, the desired result follows by applying (2.14) and (2.16) to Theorem 2.1. □

It follows that we show some special cases of Theorem 2.2. Setting $b = 1$ in Theorem 2.2, we derive the multiplication theorem for the q-Bernoulli polynomials due to Carlitz [4]

$$[a]_q^{n-1}\sum_{j=0}^{a-1} q^j\beta_n\left(x+\frac{j}{a},q^a\right) = \beta_n(ax,q), \tag{2.17}$$

which is a q-analogue of Raabe's multiplication theorem for the classical Bernoulli polynomials (see, *e.g.*, [8]). Letting $q \to 1$ in Theorem 2.2, one can immediately obtain the generalized multiplication theorem for the classical Bernoulli polynomials (see, *e.g.*, [9–11])

$$a^{n-1}\sum_{j=0}^{a-1} B_n\left(bx+\frac{bj}{a}\right) = b^{n-1}\sum_{j=0}^{b-1} B_n\left(ax+\frac{aj}{b}\right). \tag{2.18}$$

Based on the observation of Theorem 2.2, we have the following.

Theorem 2.3 *Let a, b be positive integers. Then for any non-negative integer n,*

$$\begin{aligned}&\sum_{i=0}^{n}\binom{n}{i}[a]_q^{n-1-i}[b]_q^i\beta_{n-i}\bigl(bx,q^a\bigr)S_{n,i;q^b}(a) \\ &\quad = \sum_{i=0}^{n}\binom{n}{i}[b]_q^{n-1-i}[a]_q^i\beta_{n-i}\bigl(ax,q^b\bigr)S_{n,i;q^a}(b), \end{aligned} \tag{2.19}$$

where $S_{n,i;q}(a) = \sum_{j=0}^{a-1} q^{(n+1-i)j}[j]_q^i$.

Proof Since $[x+y]_q = [x]_q + q^x[y]_q$ for any $x, y \in \mathbb{C}$, then (1.7) can be rewritten as

$$e^{[x]_q t}\left(\sum_{m=0}^{\infty} q^m e^{[y+m]_q \cdot q^x t}\left(1 - q - q^{y+m} \cdot q^x t\right)\right) = \sum_{n=0}^{\infty} \beta_n(x+y, q)\frac{t^n}{n!}. \tag{2.20}$$

Again, applying the exponential series and (1.7) to the left-hand side of (2.20), with help of the Cauchy product, we get

$$\begin{aligned}\sum_{n=0}^{\infty} \beta_n(x+y, q)\frac{t^n}{n!} &= \left(\sum_{n=0}^{\infty} [x]_q^n \frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty} q^{nx}\beta_n(y, q)\frac{t^n}{n!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i} q^{ix}\beta_i(y, q)[x]_q^{n-i}\frac{t^n}{n!}\right).\end{aligned} \tag{2.21}$$

Hence, comparison of the coefficients of $t^n/n!$ in (2.21) gives the addition theorem for the q-Bernoulli polynomials

$$\beta_n(x+y, q) = \sum_{i=0}^{n}\binom{n}{i} q^{ix}\beta_i(y, q)[x]_q^{n-i} \quad (n \geq 0). \tag{2.22}$$

Since $[x]_{q^a} = [ax]_q/[a]_q$ for any $x \in \mathbb{C}$ and a positive integer a, by (2.22), we obtain

$$\begin{aligned}&\sum_{j=0}^{a-1} q^{bj}\beta_n\left(bx + \frac{bj}{a}, q^a\right) \\ &\quad= \sum_{i=0}^{n}\binom{n}{i}\beta_{n-i}\left(bx, q^a\right)\sum_{j=0}^{a-1} q^{(n+1-i)bj}\left[\frac{bj}{a}\right]_{q^a}^{i} \\ &\quad= \sum_{i=0}^{n}\binom{n}{i}\frac{1}{[a]_q^i}\beta_{n-i}\left(bx, q^a\right)\sum_{j=0}^{a-1} q^{(n+1-i)bj}[bj]_q^i \\ &\quad= \sum_{i=0}^{n}\binom{n}{i}\left(\frac{[b]_q}{[a]_q}\right)^i\beta_{n-i}\left(bx, q^a\right)\sum_{j=0}^{a-1} q^{(n+1-i)bj}[j]_{q^b}^i.\end{aligned} \tag{2.23}$$

Thus, by applying Theorem 2.2 to (2.23), we complete the proof of Theorem 2.3. □

Clearly, Theorem 2.3 above can be regarded as an q-analogue of the corresponding classical formula (see, *e.g.*, [9, 11])

$$\sum_{i=0}^{n}\binom{n}{i} a^{n-1-i} b^i B_{n-i}(bx) S_i(a) = \sum_{i=0}^{n}\binom{n}{i} b^{n-1-i} a^i B_{n-i}(ax) S_i(b), \tag{2.24}$$

where $S_n(a) = 0^n + 1^n + \cdots + (a-1)^n$ is called as the sums of powers. It is worth mentioning that the case $x = 0$, $b = 1$ in (2.24) can be used to give the proofs of the famous von Staudt-Clausen, Frobenius, Ramanujan, *etc.*-type theorems, see [12] for details. For some generalization of (2.24) in other directions, the interested readers may consult [13–16].

We finally present another type symmetric identities for the Carlitz's q-Bernoulli numbers and q-Bernoulli polynomials.

Theorem 2.4 *Let m, n be any non-negative integers. Then*

$$\begin{aligned}&\sum_{k=0}^{m}\binom{m}{k}q^{(n+k)x}\beta_{n+k}(y,q)[x]_q^{m-k}\\&\quad=\sum_{k=0}^{n}\binom{n}{k}q^{(n-k)x}\beta_{m+k}(x+y,q)[-x]_q^{n-k}.\end{aligned}\tag{2.25}$$

Proof Observe that

$$[x]_q u+q^x[y+m]_q(u+v)=[x+y+m]_q(u+v)-[x]_q v.\tag{2.26}$$

It follows from (2.26) that

$$\begin{aligned}&e^{[x]_q u}\sum_{m=0}^{\infty}q^m e^{[y+m]_q\cdot q^x(u+v)}\bigl(1-q-q^{y+m}\cdot q^x(u+v)\bigr)\\&\quad=e^{-[x]_q v}\sum_{m=0}^{\infty}q^m e^{[x+y+m]_q\cdot(u+v)}\bigl(1-q-q^{x+y+m}\cdot(u+v)\bigr).\end{aligned}\tag{2.27}$$

We next consider the left-hand side of (2.27). In light of (1.7), we have

$$\begin{aligned}&e^{[x]_q u}\sum_{m=0}^{\infty}q^m e^{[y+m]_q\cdot q^x(u+v)}\bigl(1-q-q^{y+m}\cdot q^x(u+v)\bigr)\\&\quad=e^{[x]_q u}\sum_{n=0}^{\infty}q^{nx}\beta^n(y,q)\frac{(u+v)^n}{n!}.\end{aligned}\tag{2.28}$$

Applying $(u+v)^n=\sum_{m=0}^{n}\binom{n}{m}u^m v^{n-m}$ to the summation on the right-hand side of (2.28), and then changing the order of summation, we get

$$\begin{aligned}\sum_{n=0}^{\infty}q^{nx}\beta^n(y,q)\frac{(u+v)^n}{n!}&=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}q^{nx}\beta_n(y,q)\frac{u^m}{m!}\frac{v^{n-m}}{(n-m)!}\\&=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{(m+n)x}\beta_{m+n}(y,q)\frac{u^m}{m!}\frac{v^n}{n!}.\end{aligned}\tag{2.29}$$

Hence, putting (2.29) into (2.28), and then using the exponential series, with the help of the Cauchy product, we obtain

$$\begin{aligned}&e^{[x]_q u}\sum_{m=0}^{\infty}q^m e^{[y+m]_q\cdot q^x(u+v)}\bigl(1-q-q^{y+m}\cdot q^x(u+v)\bigr)\\&\quad=\left(\sum_{m=0}^{\infty}[x]_q^m\frac{u^m}{m!}\right)\left(\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{(m+n)x}\beta_{m+n}(y,q)\frac{u^m}{m!}\frac{v^n}{n!}\right)\\&\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{m}\binom{m}{k}q^{(n+k)x}\beta_{n+k}(y,q)[x]_q^{m-k}\right)\frac{u^m}{m!}\frac{v^n}{n!}.\end{aligned}\tag{2.30}$$

Similarly, we have

$$\begin{aligned} &e^{-[x]_q v}\sum_{m=0}^{\infty} q^m e^{[x+y+m]_q\cdot(u+v)}\bigl(1-q-q^{x+y+m}\cdot(u+v)\bigr)\\ &\quad=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{n}{k}\bigl(-[x]_q\bigr)^{(n-k)}\beta_{n+k}(x+y,q)\right)\frac{u^m}{m!}\frac{v^n}{n!}. \end{aligned} \tag{2.31}$$

Thus, by equating (2.30) and (2.31) and comparing the coefficients of $u^m v^n/m!n!$, we get

$$\sum_{k=0}^{m}\binom{m}{k}q^{(n+k)x}\beta_{n+k}(y,q)[x]_q^{m-k}=\sum_{k=0}^{n}\binom{n}{k}\bigl(-[x]_q\bigr)^{n-k}\beta_{m+k}(x+y,q), \tag{2.32}$$

which together with $-[x]_q = q^x[-x]_q$ implies the desired result. We are done. □

It becomes obvious that (2.22) is a special case of Theorem 2.4 by setting $n = 0$ and replacing m by n. As another special case, in view of (2.16), we discover that for any non-negative integers m, n,

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}q^{n+k}\beta_{n+k}(x,q)=(-1)^n\sum_{k=0}^{n}\binom{n}{k}q^{-(m+k)}\beta_{m+k}\bigl(-x,q^{-1}\bigr). \tag{2.33}$$

In particular, the case of $x = 0$ in (2.33) is an q-analogue of the formula due to Gessel [17]

$$(-1)^m\sum_{k=0}^{m}\binom{m}{k}B_{n+k}=(-1)^n\sum_{k=0}^{n}\binom{n}{k}B_{m+k}\quad(m,n\geq 0). \tag{2.34}$$

In fact, there exists a similar symmetric identity to (2.33), which is as follows.

Theorem 2.5 *Let m, n be any non-negative integers. Then*

$$\begin{aligned} &(-1)^m\sum_{k=0}^{m+1}\binom{m+1}{k}q^{n+k}(n+k+1)\beta_{n+k}(x,q)\\ &\quad+(-1)^n\sum_{k=0}^{n+1}\binom{n+1}{k}q^{-(m+k)}(m+k+1)\beta_{m+k}\bigl(-x,q^{-1}\bigr)=0. \end{aligned} \tag{2.35}$$

Proof Since $k\binom{m+1}{k}=(m+1)\binom{m}{k-1}$ for any non-negative integers k and m, then

$$\begin{aligned} &(-1)^m\sum_{k=0}^{m+1}\binom{m+1}{k}q^{n+k}(n+k+1)\beta_{n+k}(x,q)\\ &\quad=(-1)^m(n+1)\sum_{k=0}^{m+1}\binom{m+1}{k}q^{n+k}\beta_{n+k}(x,q)\\ &\qquad+(-1)^m(m+1)\sum_{k=0}^{m+1}\binom{m}{k-1}q^{n+k}\beta_{n+k}(x,q). \end{aligned} \tag{2.36}$$

It follows from (2.33) that

$$\begin{aligned}
&(-1)^m(n+1)\sum_{k=0}^{m+1}\binom{m+1}{k}q^{n+k}\beta_{n+k}(x,q) \\
&= (-1)^{n+1}(n+1)\sum_{k=0}^{n}\binom{n}{k}q^{-(m+k+1)}\beta_{m+k+1}\left(-x,q^{-1}\right) \\
&= (-1)^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}q^{-(m+k)}k\beta_{m+k}\left(-x,q^{-1}\right)
\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}
&(-1)^m(m+1)\sum_{k=0}^{m+1}\binom{m}{k-1}q^{n+k}\beta_{n+k}(x,q) \\
&= (-1)^m(m+1)\sum_{k=0}^{m}\binom{m}{k}q^{n+k+1}\beta_{n+k+1}(x,q) \\
&= (-1)^{n+1}\sum_{k=0}^{n+1}\binom{n+1}{k}q^{-(m+k)}(m+1)\beta_{m+k}\left(-x,q^{-1}\right).
\end{aligned} \tag{2.38}$$

Thus, putting (2.37) and (2.38) to the right-hand side of (2.36), we are given the desired result, and this completes the proof. □

Noticing that the case $x = 0$ in Theorem 2.5 above can be regarded as a q-analogue of the following formula

$$\begin{aligned}
&(-1)^m\sum_{k=0}^{m}\binom{m+1}{k}(n+k+1)B_{n+k} \\
&+ (-1)^n\sum_{k=0}^{n}\binom{n+1}{k}(m+k+1)B_{m+k} = 0 \quad (m+n \geq 1),
\end{aligned} \tag{2.39}$$

which was discovered and used to give a brief proof of the famous Kummer congruence by Momiyama [18]. For the generalization of (2.34) and (2.39) in the other directions, one is referred to [19, 20].

Competing interests
The author declares that they have no competing interests.

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