Wang and Han *Advances in Difference Equations* 2013, **2013**:261
http://www.advancesindifferenceequations.com/content/2013/1/261

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

An identity involving certain Hardy sums and Ramanujan's sum

Weiqiong Wang[1,2] and Di Han[2*]

*Correspondence:
handi515@163.com
[2]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China
Full list of author information is available at the end of the article

Abstract
The main purpose of this paper is using the properties of Gauss sums and the mean value theorem of the Dirichlet *L*-functions to study one kind of a hybrid mean value problem involving certain Hardy sums and Ramanujan's sum and to give an exact computational formula for it.
MSC: 11L05; 11L10

Keywords: Gauss sums; Ramanujan's sum; certain Hardy sums; hybrid mean value; computational formula

1 Introduction

Let c be a natural number and d be an integer prime to c. The classical Dedekind sums

$$S(d,c)=\sum_{j=1}^{c}\left(\left(\frac{j}{c}\right)\right)\left(\left(\frac{dj}{c}\right)\right),$$

where

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer,}\end{cases}$$

describe the behavior of the logarithm of the eta-function (see [1] and [2]) under modular transformations. Funakura [3] gave an analogous transformation formula for the logarithm of the classical theta-function

$$\theta(z)=\sum_{n=-\infty}^{+\infty}\exp(\pi i n^2 z),\qquad \operatorname{Im}(z)>0.$$

That is, put $Vz=(az+b)(cz+d)$ with $a,b,c,d\in Z$, $c>0$, and $ad-bc=1$. Then we have

$$\log\theta(Vz)=\log\theta(z)+\frac{1}{2}\log(cz+d)-\frac{1}{4}\pi i+\frac{1}{4}\pi i S_1(d,c), \tag{1.1}$$

where $S_1(d,c)$ is defined as

$$S_1(d,c)=\sum_{j=1}^{c-1}(-1)^{j+1+[\frac{dj}{c}]}.$$

The sums $S_1(d,c)$ (and certain related ones) are sometimes called Hardy sums. They are closely connected with Dedekind sums. Regarding the properties of $S_1(d,c)$ and related sums, some authors studied them and obtained many interesting results; see [4–7] and [8]. For example, Wenpeng Zhang [7] proved the following conclusion: Let p be an odd prime. Then, for any fixed positive integer m, we have the asymptotic formula

$$\sum_{h=1}^{p-1} \left|S_1(h,p)\right|^{2m} = p^{2m} \cdot \frac{\zeta^2(2m)(1-\frac{1}{4^m})}{\zeta(4m)(1+\frac{1}{4^m})} + O\left(p^{2m-1} \cdot \exp\left(\frac{6\ln p}{\ln\ln p}\right)\right),$$

where $\zeta(s)$ is the Riemann zeta-function and $\exp(y) = e^y$.

On the other hand, we define Ramanujan's sum $R_q(c)$ as follows (see Theorem 8.6 of [9]):

$$R_q(c) = \sum_{\substack{k=1 \\ (k,q)=1}}^{q} e^{\frac{2\pi ikc}{q}} = \sum_{d|(c,q)} d\mu(q/d),$$

where $\mu(n)$ is the famous Möbius function. Some related properties of $R_q(c)$ can also be found in [9, 10] and [11].

The main purpose of this paper is using the properties of Gauss sums and the mean square value theorem of the Dirichlet L-functions to study a hybrid mean value problem involving certain Hardy sums and Ramanujan's sum and to give an exact computational formula for it. That is, we shall prove the following.

Theorem *Let $q > 1$ be an odd square-full number. Then we have the identity*

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot S_1(2h,q) = \phi^2(q) \cdot \prod_{p|q}\left(1+\frac{1}{p}\right),$$

where $\phi(q)$ denotes the Euler function, and $\prod_{p|q}$ denotes the product over all distinct prime divisors of q.

It is very interesting that the value is equal to zero in our theorem if we use $R_q(h+1)$ instead of $R_q(2h+1)$.

For a general odd number $q \geq 3$, whether there exits a computational formula for

$$\sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot S_1(2h,q)$$

is an open problem. Interested readers are welcome to study it with us.

2 Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorem. Hereinafter, we shall use many properties of Gauss sums and character sums, all of which can be found in [9], so they will not be repeated here. First we have the following lemma.

Lemma 1 *Let $q > 1$ be an odd number, and let χ be any non-principal character* mod q. *Then, for any integer m with $(m, q) = 1$, we have the identity*

$$\sum_{c=1}^{q} \chi(c) \cdot R_q(mc+1) = \overline{\chi}(m) \cdot \tau(\chi) \cdot \tau(\overline{\chi}),$$

where $\tau(\chi) = \sum_{a=1}^{q-1} \chi(a) e(\frac{a}{q})$ denotes the classical Gauss sums, and $e(y) = e^{2\pi i y}$.

Proof For any non-principal character χ mod q, from the definition of $R_q(c)$ and the properties of Gauss sums, we have

$$\begin{aligned}
\sum_{c=1}^{q} \chi(c) \cdot R_q(mc+1) &= \sum_{c=1}^{q} \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \chi(c) e\left(\frac{b(mc+1)}{q}\right) \\
&= \sum_{\substack{b=1 \\ (b,q)=1}}^{q} e\left(\frac{b}{q}\right) \sum_{c=1}^{q} \chi(c) e\left(\frac{mbc}{q}\right) \\
&= \overline{\chi}(m) \sum_{\substack{b=1 \\ (b,q)=1}}^{q} \overline{\chi}(b) e\left(\frac{b}{q}\right) \sum_{c=1}^{q} \chi(mbc) e\left(\frac{mbc}{q}\right) \\
&= \overline{\chi}(m) \cdot \tau(\chi) \cdot \sum_{b=1}^{q} \overline{\chi}(b) e\left(\frac{b}{q}\right) = \overline{\chi}(m) \cdot \tau(\chi) \cdot \tau(\overline{\chi}).
\end{aligned}$$

This proves Lemma 1. □

Lemma 2 *Let $q > 2$ be an integer. Then, for any integer a with $(a, q) = 1$, we have the identity*

$$S(a,q) = \frac{1}{\pi^2 q} \sum_{d \mid q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \chi(a) \left| L(1,\chi) \right|^2,$$

where $L(1, \chi)$ denotes the Dirichlet L-function corresponding to character χ mod d.

Proof See Lemma 2 of [12]. □

Lemma 3 *Let $q > 0$ and $(h, q) = 1$. Then we have the identity*

$$S_1(h,q) = -8S(h+q, 2q) + 4S(h,q).$$

Proof This formula is an immediate consequence of (5.9) and (5.10) in [6]. □

Lemma 4 *Let $q > 1$ be an odd number and $0 < h < q$ with $(h, q) = 1$. Then we have the identity*

$$S_1(2h,q) = -20 \cdot S(2h,q) + 8 \cdot S(4h,q) + 8 \cdot S(h,q).$$

Proof From Lemma 2 and Lemma 3, we have

$$\begin{aligned}
S_1(2h,q) &= -8\cdot S(2h+q,2q)+4\cdot S(2h,q)\\
&= -\frac{4}{\pi^2 q}\sum_{d\mid 2q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h+q)\left|L(1,\chi)\right|^2\\
&\quad+\frac{4}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h)\left|L(1,\chi)\right|^2\\
&= -\frac{4}{\pi^2 q}\cdot\sum_{d\mid q}\frac{(2d)^2}{\phi(2d)}\sum_{\substack{\chi \bmod 2d\\ \chi(-1)=-1}}\chi(2h+d)\left|L(1,\chi)\right|^2\\
&= -\frac{16}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h+q)\lambda(2h+q)\left|L(1,\chi\lambda)\right|^2\\
&= -\frac{16}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h)\left|L(1,\chi\lambda)\right|^2, \qquad (2.1)
\end{aligned}$$

where λ is the principal character mod 2.

From the Euler infinite product formula, we have

$$\begin{aligned}
\left|L(1,\chi\lambda)\right|^2 &= \prod_{p_1}\left|1-\frac{\chi(p_1)\lambda(p_1)}{p_1}\right|^{-2} = \prod_{p_1>2}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2}\\
&= \left|1-\frac{\chi(2)}{2}\right|^2\cdot\prod_{p_1}\left|1-\frac{\chi(p_1)}{p_1}\right|^{-2} = \left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\overline{\chi}(2)}{2}\right)\cdot\left|L(1,\chi)\right|^2, \qquad (2.2)
\end{aligned}$$

where $\prod_{p_1}$ denotes the product over all primes p_1.

Now, combining (2.1), (2.2) and Lemma 2, we have the identity

$$\begin{aligned}
S_1(2h,q) &= -\frac{16}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h)\left|L(1,\chi\lambda)\right|^2\\
&= -\frac{16}{\pi^2 q}\cdot\sum_{d\mid q}\frac{d^2}{\phi(d)}\sum_{\substack{\chi \bmod d\\ \chi(-1)=-1}}\chi(2h)\left(\frac{5}{4}-\frac{\chi(2)}{2}-\frac{\overline{\chi}(2)}{2}\right)\cdot\left|L(1,\chi)\right|^2\\
&= -20\cdot S(2h,q)+8\cdot S(4h,q)+8\cdot S(h,q).
\end{aligned}$$

This proves Lemma 4. □

Lemma 5 *Let $q>1$ be a square-full number (i.e., $q\geq 4$ and a prime $p\mid q$ implies $p^2\mid q$). Then, for any non-primitive character χ* mod q, *we have the identity*

$$\sum_{a=1}^{q}\chi(a)e\left(\frac{a}{q}\right)=0.$$

Proof It is clear that from the multiplicative properties of Gauss sums we know that we only need to prove $q = p^\alpha$, a power of prime, where $\alpha \geq 2$. Suppose that χ is a non-primitive character mod p^α, then χ must be a character mod $p^{\alpha-1}$. So, from the definition of Gauss sums and the properties of a complete residue system mod p^α and trigonometric sums, we have

$$\begin{aligned}\sum_{a=1}^{p^\alpha} \chi(a) e\left(\frac{a}{p^\alpha}\right) &= \sum_{r=0}^{p-1} \sum_{a=1}^{p^{\alpha-1}} \chi\left(rp^{\alpha-1} + a\right) e\left(\frac{rp^{\alpha-1} + a}{p^\alpha}\right) \\ &= \sum_{a=1}^{p^{\alpha-1}} \chi(a) e\left(\frac{a}{p^\alpha}\right) \sum_{r=0}^{p-1} e\left(\frac{r}{p}\right) = 0.\end{aligned}$$

This proves Lemma 5. □

Lemma 6 *Let $q > 3$ be an odd square-full number. Then we have the identities*

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1 + \frac{1}{p}\right);$$

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{24} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q} \left(1 + \frac{1}{p}\right),$$

where $\sum^*_{\substack{\chi \bmod q \\ \chi(-1)=-1}}$ denotes the summation over all odd primitive characters χ mod q.

Proof From the definition of $S(a,q)$, we have the computational formula

$$S(1,q) = \sum_{a=1}^{q-1} \left(\frac{a}{q} - \frac{1}{2}\right)^2 = \frac{(q-1)(q-2)}{12q}. \tag{2.3}$$

From the reciprocity formula of $S(a,q)$, we know that for any positive integer a with $(a,q) = 1$, we have the identity (see [4])

$$S(a,q) + S(q,a) = \frac{q^2 + a^2 + 1}{12aq} - \frac{1}{4}.$$

Applying this formula, we have

$$S(2,q) = \frac{q^2 + 2^2 + 1}{24q} - \frac{1}{4} - S(q,2) = \frac{(q-1)(q-5)}{24q}. \tag{2.4}$$

From (2.3), Lemma 2 with $a = 1$ and the Möbius inversion formula, we have

$$\begin{aligned}\frac{q^2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 &= \pi^2 \cdot \sum_{d|q} \mu(d) \cdot \frac{q}{d} \cdot S\left(1, \frac{q}{d}\right) = \pi^2 \cdot \sum_{d|q} \mu(d) \cdot \frac{(\frac{q}{d}-1)(\frac{q}{d}-2)}{12} \\ &= \frac{\pi^2}{12} \cdot \phi(q) \cdot \left[q \cdot \prod_{p|q} \left(1 + \frac{1}{p}\right) - 3\right]\end{aligned}$$

or

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \frac{\pi^2}{12} \cdot \frac{\phi^2(q)}{q^2} \cdot \left[q \cdot \prod_{p|q}\left(1+\frac{1}{p}\right) - 3\right]. \tag{2.5}$$

Then, using formula (2.5) and the Möbius inversion formula, we have

$$\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 = \sum_{d|q}\mu(d)\left(\sum_{\substack{\chi \bmod q/d \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2\right) = \frac{\pi^2}{12} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right). \tag{2.6}$$

From (2.4), Lemma 2 with $a = 2$ and the Möbius inversion formula, we also have

$$\begin{aligned}\frac{q^2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 &= \pi^2 \cdot \sum_{d|q}\mu(d) \cdot \frac{q}{d} \cdot S\left(2, \frac{q}{d}\right) \\ &= \pi^2 \cdot \sum_{d|q}\mu(d) \cdot \frac{(\frac{q}{d}-1)(\frac{q}{d}-5)}{24} \\ &= \frac{\pi^2}{24} \cdot \phi(q) \cdot \left[q \cdot \prod_{p|q}\left(1+\frac{1}{p}\right) - 6\right]\end{aligned}$$

or

$$\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \frac{\pi^2}{24} \cdot \frac{\phi^2(q)}{q^2} \cdot \left[q \cdot \prod_{p|q}\left(1+\frac{1}{p}\right) - 6\right]. \tag{2.7}$$

Then, using (2.7) and the Möbius inversion formula, we have

$$\begin{aligned}\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 &= \sum_{d|q}\mu(d)\left(\sum_{\substack{\chi \bmod q/d \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2\right) \\ &= \frac{\pi^2}{24} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right). \end{aligned} \tag{2.8}$$

Now Lemma 6 follows from (2.6) and (2.8). □

3 Proof of the theorems

In this section, we shall complete the proof of our theorem. Note that if χ is a primitive character mod q, then $|\tau(\chi)| = \sqrt{q}$, and $\tau(\chi) \cdot \tau(\overline{\chi}) = \overline{\chi}(-1) \cdot \tau(\chi) \cdot \overline{\tau(\chi)} = \overline{\chi}(-1) \cdot q$. From Lemma 1, Lemma 2, Lemma 4, Lemma 5 and Lemma 6, we have

$$\begin{aligned}&\sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot S_1(2h,q) \\ &\quad= \sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot \left(-20 \cdot S(2h,q) + 8 \cdot S(4h,q) + 8 \cdot S(h,q)\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{20}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot \chi(h) \cdot \chi(2) \cdot \left|L(1,\chi)\right|^2 \\
&\quad + \frac{8}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot \chi(h) \cdot \chi(4) \cdot \left|L(1,\chi)\right|^2 \\
&\quad + \frac{8}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \sum_{\substack{h=1 \\ (h,q)=1}}^{q} R_q(2h+1) \cdot \chi(h) \cdot \left|L(1,\chi)\right|^2 \\
&= -\frac{20}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \tau(\chi) \cdot \tau(\overline{\chi}) \cdot \left|L(1,\chi)\right|^2 \\
&\quad + \frac{8}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \tau(\chi) \cdot \tau(\overline{\chi}) \cdot \chi(2) \cdot \left|L(1,\chi)\right|^2 \\
&\quad + \frac{8}{\pi^2 q} \cdot \sum_{d|q} \frac{d^2}{\phi(d)} \sum_{\substack{\chi \bmod d \\ \chi(-1)=-1}} \tau(\chi) \cdot \tau(\overline{\chi}) \cdot \overline{\chi}(2) \cdot \left|L(1,\chi)\right|^2 \\
&= \frac{20}{\pi^2} \cdot \frac{q^2}{\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \left|L(1,\chi)\right|^2 - \frac{8}{\pi^2} \cdot \frac{q^2}{\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 \\
&\quad - \frac{8}{\pi^2} \cdot \frac{q^2}{\phi(q)} \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(2) \cdot \left|L(1,\chi)\right|^2 \\
&= \frac{20}{\pi^2} \cdot \frac{q^2}{\phi(q)} \cdot \frac{\pi^2}{12} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right) - \frac{8}{\pi^2} \cdot \frac{q^2}{\phi(q)} \cdot \frac{\pi^2}{24} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right) \\
&\quad - \frac{8}{\pi^2} \cdot \frac{q^2}{\phi(q)} \cdot \frac{\pi^2}{24} \cdot \frac{\phi^3(q)}{q^2} \prod_{p|q}\left(1+\frac{1}{p}\right) \\
&= \phi^2(q) \cdot \prod_{p|q}\left(1+\frac{1}{p}\right),
\end{aligned}
$$

where we have used the identity

$$
\sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \chi(2) \cdot \left|L(1,\chi)\right|^2 = \sideset{}{^*}\sum_{\substack{\chi \bmod q \\ \chi(-1)=-1}} \overline{\chi}(2) \cdot \left|L(1,\chi)\right|^2.
$$

This completes the proof of our theorem.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
WW carried out the hybrid mean value problem involving certain Hardy sums and Ramanujan's sum and gave an exact computational formula. DH participated in the research and summary of the study. All authors read and approved the final manuscript.

Author details
[1]School of Science, Chang'an University, Xi'an, Shaanxi, P.R. China. [2]Department of Mathematics, Northwest University, Xi'an, Shaanxi, P.R. China.

Acknowledgements
The authors would like to thank the referees for their very helpful and detailed comments, which have significantly improved the presentation of this paper. This work is supported by the N.S.F. (11071194, 61202437) of P.R. China, and partly by the Fundamental Research Funds for the Central Universities of P.R. China (CHD2010JC101)

Received: 24 July 2013 Accepted: 9 August 2013 Published: 26 August 2013

References
1. Apostol, TM: Introduction to Analytic Number Theory. Springer, New York (1976)
2. Chowla, S: On Kloosterman's sums. Norske Vid. Selsk. Forh. **40**, 70-72 (1967)
3. Funakura, T: On Kronecker's limit formula for Dirichlet series with periodic coefficients. Acta Arith. **55**, 59-73 (1990)
4. Guy, RK: Unsolved Problems in Number Theory, 2nd edn. Springer, New York (1994)
5. Malyshev, AV: A generalization of Kloosterman sums and their estimates. Vestn. Leningr. Univ. **15**, 59-75 (1960) (in Russian)
6. Xi, P, Yi, Y: On character sums over flat numbers. J. Number Theory **130**, 1234-1240 (2010)
7. Zhang, W: On a problem of D. H. Lehmer and its generalization. Compos. Math. **86**, 307-316 (1993)
8. Zhang, W: A problem of D. H. Lehmer and its mean square value formula. Jpn. J. Math. **29**, 109-116 (2003)
9. Zhang, W: A problem of D. H. Lehmer and its generalization (II). Compos. Math. **91**, 47-56 (1994)
10. Zhang, W: A mean value related to D. H. Lehmer's problem and the Ramanujan's sum. Glasg. Math. J. **54**, 155-162 (2012)
11. Zhang, W: On the mean values of Dedekind sums. J. Théor. Nr. Bordx. **8**, 429-442 (1996)
12. Zhang, W: On the difference between an integer and its inverse modulo *n* (II). Sci. China Ser. A **46**, 229-238 (2003)

doi:10.1186/1687-1847-2013-261
Cite this article as: Wang and Han: An identity involving certain Hardy sums and Ramanujan's sum. *Advances in Difference Equations* 2013 **2013**:261.