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Existence and multiplicity of difference ϕ-Laplacian boundary value problems

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Abstract

Concerned are the difference ϕ-Laplacian boundary value problems. The multiplicity result based on the lower and upper solutions method associated with Brouwer degree is applied to a difference ϕ-Laplacian eigenvalue problem. An existence result of at least three positive solutions is established for the eigenvalue problem with the parameter belonging to an explicit open interval. In addition, an example is given to illustrate the three solutions result.

Keywords: difference ϕ-Laplacian boundary value problem; the lower and upper solution; positive solution; multiplicity; Brouwer degree

1 Introduction

Recently, Kim [1] studied a one-dimensional differential p-Laplacian boundary value problem with a positive parameter and established an existence result of three positive solutions by the lower and upper solutions method associated with Leray-Schauder degree theory. Kim and Shi [2] studied the global continuum and multiple positive solutions of a p-Laplacian boundary value problem. Motivated by the methods in [1, 2], we consider difference ϕ-Laplacian boundary value problems.

For $a, b \in \mathbf{Z}$ with $a < b$, let $[a,b]_{\mathbf{Z}} = \{a, a+1, a+2, \dots, b-1, b\}$. First, by the upper and lower solutions method associated with Brouwer degree theory, we establish the existence and multiplicity results for the following discrete ϕ-Laplacian boundary value problem:

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + f(k, u(k)) = 0, & k \in [1, T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{1}$$

where $T > 1$ is a given positive integer, $\Delta u(k) = u(k+1) - u(k)$, and

(A1) $\phi : \mathbf{R} \to \mathbf{R}$ is an odd and strictly increasing function;

(A2) $f : [1, T]_{\mathbf{Z}} \times \mathbf{R} \to \mathbf{R}$ is continuous.

Then, we apply the multiplicity result of (1) to the following eigenvalue problem:

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + \lambda p(k) g(u(k)) = 0, & k \in [1, T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{2}$$

where λ is a positive parameter. Under some suitable assumptions imposed on g, we establish the existence of three positive solutions of (2) with λ belonging to an explicit open interval.

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The function $\phi(u)$ covers two important cases: $\phi(u) = u$ and $\phi(u) = |u|^{p-2}u$ ($p > 1$). If $\phi(u) = u$, then problem (1) is the classical second order difference Dirichlet boundary value problem. For the case that $\phi(u) = |u|^{p-2}u$, problem (1) is the well-known discrete p-Laplacian problem. The two cases have been widely studied. To name a few, see [3–10] and the references therein.

Problem (1) can be viewed as the discrete analogue of the following differential ϕ-Laplacian problem:

$$\begin{cases} (\phi(u'))' + f(t,u) = 0, & 0 < t < 1, \\ u(0) = u(1) = 0, \end{cases} \tag{3}$$

which rises from the study of radial solutions for p-Laplacian equations ($\phi(u) = |u|^{p-2}u$) on an annular domain (see [11], and references therein). Recently, the differential ϕ-Laplacian problems have been widely studied in many different papers. We refer the readers to [12–19] and the references therein.

For discrete ϕ-Laplacian problems, there are fewer study results than for differential ϕ-Laplacian problems. See Cabda [20], Cabada and Espinar [21] and Bondar [22]. To the best of our knowledge, there are no results on the existence and multiplicity of positive solutions for difference ϕ-Laplacian problems.

The remaining part of this paper is organized as follows. In Section 2, we show the lower and upper solutions method and establish the existence and multiplicity of solutions of (1). In Section 3, we establish the existence of three positive solutions of (2). Finally, we give an example to illustrate our main results.

2 The upper and lower solutions method

In this section, we establish the existence and multiplicity results of solutions for problem (1) by lower and upper solutions method associated with Brouwer degree.

Let $E = \{u : [0, T+1]_{\mathbf{Z}} \to \mathbf{R}^{T+2}\}$ with the norm $\|u\| = \max_{t\in[0,T+1]_{\mathbf{Z}}} |u(t)|$.

Definition 2.1 Given $u, v, w \in E$, we say that

(1) $u \le v$ if for all $k \in [0, T+1]_{\mathbf{Z}}$, $u(k) \le v(k)$.
(2) $u \in [v, w]$ if $v \le u \le w$.
(3) $u \prec v$ if for all $k \in [1, T]_{\mathbf{Z}}$, $u(k) < v(k)$ and $u(0) \le v(0)$, $u(T+1) \le v(T+1)$.

Definition 2.2 $\alpha \in E$ is called a lower solution of problem (1) if

$$\begin{cases} \Delta(\phi(\Delta\alpha(k-1))) + f(k,\alpha(k)) \ge 0, & k \in [1, T]_{\mathbf{Z}}, \\ \alpha(0) \le 0, \qquad \alpha(T+1) \le 0. \end{cases}$$

If the first inequality above is strict, then α is called a strict lower solution of (1).

In the same way, we define the upper solution and the strict upper solution of (1) by reversing the inequalities above.

Lemma 2.1 *Let* (A1) *hold. The problem*

$$\begin{cases} -\Delta(\phi(\Delta u(k-1))) + u(k) = 0, & k \in [1, T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0 \end{cases} \tag{4}$$

has the unique solution $u \equiv 0$.

Proof It is clear that 0 is a trivial solution of (4). Suppose that (4) has a nontrivial solution u. Let $|u(m)| = \max_{k\in[1,T]_{\mathbf{Z}}} |u(k)| = \|u\|$. If $u(m) = \max_{k\in[1,T]_{\mathbf{Z}}} u(k)$, then $u(m) > 0$ and $\Delta u(m) \leq 0$, $\Delta u(m-1) \geq 0$, which yields a contradiction:

$$\begin{aligned} u(m) &= \Delta\big(\phi\big(\Delta u(m-1)\big)\big) \\ &= \phi\big(\Delta u(m)\big) - \phi\big(\Delta u(m-1)\big) \leq 0 < u(m). \end{aligned}$$

Similarly, if $u(m) = \min_{k\in[1,T]_{\mathbf{Z}}} u(k)$, then $u(m) < 0$ and $\Delta u(m) \geq 0$, $\Delta u(m-1) \leq 0$, which implies that $u(m) = \Delta(\phi(\Delta u(m-1))) \geq 0 > u(m)$, which is a contradiction. The proof is complete. □

Theorem 2.1 *Let* (A1) *and* (A2) *hold.*

(i) *Assume that there exist α and β, respectively lower and upper solutions of* (1) *such that $\alpha \leq \beta$. Then problem* (1) *has at least one solution u with $\alpha \leq u \leq \beta$.*

(ii) *Assume that problem* (1) *has two pairs of lower and upper solutions (α_1, β_1) and (α_2, β_2) with α_2 and β_1 being strict, satisfying that*

$$\alpha_1 \leq \alpha_2 \leq \beta_2, \qquad \alpha_1 \leq \beta_1 \leq \beta_2,$$

and that there exists $k_0 \in [0, T+1]_{\mathbf{Z}}$ such that $\beta_1(k_0) < \alpha_2(k_0)$. Then problem (1) *has at least three solutions u_1, u_2, u_3 with*

$$\alpha_1 \leq u_1 \prec \beta_1, \qquad \alpha_2 \prec u_2 \leq \beta_2, \qquad u_3 \in [\alpha_1, \beta_2] \backslash \big([\alpha_1, \beta_1] \cup [\alpha_2, \beta_2]\big).$$

Remark We denote that the result (i) has been proved in [20] by Brouwer fixed point theorem. Here, for the convenience of the proof of (ii), it is proven by Brouwer degree theory. The proof of (ii) is motivated by the idea in [1].

Proof of Theorem 2.1. (i) Define $\gamma : [1,T]_{\mathbf{Z}} \times \mathbf{R} \to \mathbf{R}$ by

$$\gamma(k,u) = \begin{cases} \beta(k), & u > \beta(k), \\ u, & \alpha(k) \leq u \leq \beta(k), \\ \alpha(k), & u < \alpha(k). \end{cases}$$

Consider the modified problem

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + f(k, \gamma(k, u(k))) - [u(k) - \gamma(k, u(k))] = 0, & k \in [1,T], \\ u(0) = u(T+1) = 0. \end{cases} \tag{5}$$

Clearly, all solutions u of (5) satisfying $\alpha \leq u \leq \beta$ are solutions of (1). Let u be a solution of (5). By the arguments in [20], we know that $\alpha \leq u \leq \beta$. Now, we prove that problem (5) has at least one solution. Let $E_0 = \{u \in E : u(0) = u(T+1) = 0\}$ and define operator $\widetilde{T} : E_0 \to \mathbf{R}^T$ by

$$(\widetilde{T}u)(k) = \Delta\big(\phi\big(\Delta u(k-1)\big)\big) + f\big(k, \gamma\big(k, u(k)\big)\big) - \big[u(k) - \gamma\big(k, u(k)\big)\big], \quad k \in [1,T]_{\mathbf{Z}}. \tag{6}$$

Obviously, each solution u of $\widetilde{T}u = 0$ solves (5). Define homotopic mapping $\Gamma : [0,1] \times E_0 \to \mathbf{R}^T$ by

$$\begin{aligned}\Gamma(\lambda, u)(k) &= (1-\lambda)\bigl[\Delta\bigl(\phi\bigl(\Delta u(k-1)\bigr)\bigr) - u(k)\bigr] + \lambda \widetilde{T}u(k) \\ &= \Delta\bigl(\phi\bigl(\Delta u(k-1)\bigr)\bigr) - u(k) + \lambda\bigl[f\bigl(k, \gamma\bigl(k, u(k)\bigr)\bigr) + \gamma\bigl(k, u(k)\bigr)\bigr], \quad k \in [1,T]_{\mathbf{Z}}.\end{aligned}$$

By the definition of γ and the continuity of f, there exists an $R > 0$, such that

$$\max_{k\in[1,T]_{\mathbf{Z}}} \max_{u\in\mathbf{R}} \bigl|f\bigl(k, \gamma(k,u)\bigr) + \gamma(k,u)\bigr| < R.$$

Let $B_R(0) = \{u \in E_0 : \|u\| < R\}$. We prove that if $(\lambda, u) \in [0,1] \times E_0$ is a solution of $\Gamma(\lambda, u) = 0$, then $u \in B_R(0)$. Let $|u(m)| = \max_{k\in[1,T]_{\mathbf{Z}}} |u(k)| = \|u\|$. Then there are two cases that $u(m) = \max_{k\in[1,T]_{\mathbf{Z}}} u(k)$ and $u(m) = \min_{k\in[1,T]_{\mathbf{Z}}} u(k)$. For the first case, since $u(m+1) - u(m) \leq 0$, $u(m) - u(m-1) \geq 0$ and ϕ is odd, we have that

$$\begin{aligned}&u(m) - \lambda\bigl[f\bigl(m, \gamma\bigl(m, u(m)\bigr)\bigr) + \gamma\bigl(m, u(m)\bigr)\bigr] \\ &\quad= \Delta\bigl(\phi\bigl(\Delta u(m-1)\bigr)\bigr) \\ &\quad= \phi\bigl(u(m+1) - u(m)\bigr) - \phi\bigl(u(m) - u(m-1)\bigr) \\ &\quad\leq 0,\end{aligned}$$

which implies that

$$u(m) \leq \lambda\bigl[f\bigl(m, \gamma\bigl(m, u(m)\bigr)\bigr) + \gamma\bigl(m, u(m)\bigr)\bigr] < R.$$

Similarly, for the second case, we have that

$$u(m) \geq \lambda\bigl[f\bigl(m, \gamma\bigl(m, u(m)\bigr)\bigr) + \gamma\bigl(m, u(m)\bigr)\bigr] > -R.$$

Therefore, $\|u\| < R$, and $\deg(\Gamma(\lambda, \cdot), B_R(0), 0)$ is well defined. By the homotopy invariance of Brouwer degree, we get that

$$\deg\bigl(\widetilde{T}, B_R(0), 0\bigr) = \deg\bigl(\Gamma(1, \cdot), B_R(0), 0\bigr) = \deg\bigl(\Gamma(0, \cdot), B_R(0), 0\bigr).$$

By Lemma 2.1, the equation $-\Delta(\phi(\Delta u(k-1))) + u(k) = 0$ has the unique solution $u = 0$ in E_0, thus we have

$$\deg\bigl(\Gamma(0, \cdot), B_R(0), 0\bigr) = 1.$$

Therefore, $\deg(\widetilde{T}, B_R(0), 0) = 1$, which implies that problem (5) has at least one solution $u \in E_0$.

(ii) First, we show that if α and β are strict lower and upper solutions, respectively, such that $\alpha \leq \beta$, then $\deg(\widetilde{T}, \Omega_{\alpha\beta}, 0) = 1$, where $\Omega_{\alpha\beta} = \{u \in E_0, \alpha \prec u \prec \beta, \|u\| < R\}$. By the arguments above, each solution u of (5) satisfies that $\alpha \leq u \leq \beta$. We claim that $\alpha \prec u \prec \beta$.

In fact, if it is not true, then there exists an $m \in [1,T]_{\mathbf{Z}}$ such that $\alpha(m) = u(m)$. Since $\Delta u(m-1) \leq \Delta\alpha(m-1)$, $\Delta u(m) \geq \Delta\alpha(m)$, we have by the monotonicity of ϕ that

$$\begin{aligned}
&\Delta\big(\phi\big(\Delta u(m-1)\big)\big) - \Delta\big(\phi\big(\Delta\alpha(m-1)\big)\big) \\
&\quad = \big[\phi\big(\Delta u(m)\big) - \phi\big(\Delta\alpha(m)\big)\big] + \big[\phi\big(\Delta\alpha(m-1)\big) - \phi\big(\Delta u(m-1)\big)\big] \\
&\quad \geq 0.
\end{aligned}$$

It yields a contradiction:

$$\begin{aligned}
\Delta\big(\phi\big(\Delta u(m-1)\big)\big) &= -f\big(m,\gamma\big(m,u(m)\big)\big) + \big[u(m) - \gamma\big(m,u(m)\big)\big] \\
&= -f\big(m,\alpha(m)\big) < \Delta\big(\phi\big(\Delta\alpha(m-1)\big)\big).
\end{aligned}$$

Thus $\alpha \prec u$. Similarly, one can check that $u \prec \beta$. By the excision property of Brouwer degree,

$$\deg(\widetilde{T},\Omega_{\alpha\beta},0) = \deg\big(\widetilde{T},B_R(0),0\big) = 1.$$

Now, consider the following modified problem:

$$\begin{cases}
\Delta(\phi(\Delta u(k-1))) + f(k,\gamma^*(k,u(k))) - [u(k) - \gamma^*(k,u(k))] = 0, \quad k \in [1,T], \\
u(0) = u(T+1) = 0,
\end{cases} \tag{7}$$

where $\gamma^* : [1,T]_{\mathbf{Z}} \times \mathbf{R} \to \mathbf{R}$ is defined by

$$\gamma^*(k,u) = \begin{cases}
\beta_2(k), & u > \beta_2(k), \\
u, & \alpha_1(k) \leq u \leq \beta_2(k), \\
\alpha_1(k), & u < \alpha_1(k).
\end{cases}$$

It is easy to see that for sufficiently small $\epsilon > 0$, $(\alpha_1 - \epsilon, \beta_1)$ and $(\alpha_2, \beta_2 + \epsilon)$ are two pairs of strict lower and upper solutions of (7). Similarly to (6), let $\widetilde{T}^*$ be the operator corresponding to problem (7). For sufficiently large $R > 0$, define

$$\begin{aligned}
\Omega_{\alpha_1\beta_1} &= \big\{u \in E_0, \alpha_1 - \epsilon \prec u \prec \beta_1, \|u\| < R\big\}, \\
\Omega_{\alpha_1\beta_2} &= \big\{u \in E_0, \alpha_1 - \epsilon \prec u \prec \beta_2 + \epsilon, \|u\| < R\big\},
\end{aligned}$$

and

$$\Omega_{\alpha_2\beta_2} = \big\{u \in E_0, \alpha_2 \prec u \prec \beta_2 + \epsilon, \|u\| < R\big\}.$$

Then $\deg(\widetilde{T}^*,\Omega_{\alpha_1\beta_1},0) = 1$, $\deg(\widetilde{T}^*,\Omega_{\alpha_1\beta_2},0) = 1$ and $\deg(\widetilde{T}^*,\Omega_{\alpha_2\beta_2},0) = 1$. Thus by the additivity property of Brouwer degree, we have $\deg(\widetilde{T}^*,\Omega_{\alpha_1\beta_2}\backslash(\overline{\Omega}_{\alpha_1\beta_1} \cup \overline{\Omega}_{\alpha_2\beta_2},0)) = -1$. Therefore, problem (7) has three solutions u_1, u_2 and u_3 with $u_1 \in \Omega_{\alpha_1\beta_1}$, $u_2 \in \Omega_{\alpha_2\beta_2}$ and $u_3 \in \Omega_{\alpha_1\beta_2}\backslash\overline{\Omega}_{\alpha_1\beta_1} \cup \overline{\Omega}_{\alpha_2\beta_2}$. By the facts that all solutions of (7) satisfy $[\alpha_1,\beta_2]$ and are solution of (1), the proof is complete. □

3 Three positive solutions of eigenvalue problems

Lemma 3.1 *Let* (A1) *hold and u satisfy the following difference inequality*:

$$-\Delta\big(\phi\big(\Delta u(k-1)\big)\big) \geq 0, \quad k \in [1,T]_{\mathbf{Z}} \tag{8}$$

with $u(0) \geq 0$, $u(T+1) \geq 0$. *Then* $u(k) \geq 0$ *for all* $k \in [1,T]_{\mathbf{Z}}$, *and* $\Delta u(k-1) \geq 0$ *for* $k \in [1,k^*]_{\mathbf{Z}}$, $\Delta u(k) \leq 0$ *for* $k \in [k^*,T]_{\mathbf{Z}}$, *where* $k^* \in [0,1+T]_{\mathbf{Z}}$ *satisfies* $u(k^*) = \max_{k\in[0,1+T]_{\mathbf{Z}}} u(k)$.

Proof Since $\Delta[\phi(\Delta u(k-1))] = \phi(\Delta u(k)) - \phi(\Delta u(k-1)) \leq 0$, $k \in [1,T]_{\mathbf{Z}}$, we have by the monotonicity of ϕ that $\Delta u(k) \leq \Delta u(k-1)$, $k \in [1,T]_{\mathbf{Z}}$. If $k^* = 0$ or $T+1$, the result is clear. Now, we assume that $k^* \in [1,T]_{\mathbf{Z}}$. Since

$$\Delta u\big(k^*-1\big) = u\big(k^*\big) - u\big(k^*-1\big) \geq 0,$$
$$\Delta u\big(k^*\big) = u\big(k^*+1\big) - u\big(k^*\big) \leq 0,$$

we have by the monotonicity of $\Delta u(\cdot)$ that $\Delta u(k-1) \geq 0$ for $k \in [1,k^*]_{\mathbf{Z}}$, $\Delta u(k) \leq 0$ for $k \in [k^*,T]_{\mathbf{Z}}$, which implies that $u(k) \geq 0$ holds for all $k \in [1,T]_{\mathbf{Z}}$ by the boundary conditions $u(0) \geq 0$, $u(T+1) \geq 0$. □

Remark If inequality (8) is strict, then $u(k) > 0$ for $k \in [1,T]_{\mathbf{Z}}$, and there exists $k^* \in [0,1+T]_{\mathbf{Z}}$ such that $u(k^*) = \|u\|$, and $\Delta u(k-1) > 0$ for $k \in [1,k^*-1]_{\mathbf{Z}}$, $\Delta u(k^*-1) \geq 0$, and $\Delta u(k) < 0$ for $k \in [k^*,T]_{\mathbf{Z}}$.

Consider the following problem:

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + h(k) = 0, & k \in [1,T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0, \end{cases} \tag{9}$$

where $h : [1,T]_{\mathbf{Z}} \to (0,\infty)$.

In the following arguments, we assume that

(B1) $\phi : \mathbf{R} \to \mathbf{R}$ is an odd and strictly increasing homeomorphism.

Lemma 3.2 *Let* (B1) *hold and u solve* (9). *If h is symmetric on* $[1,T]_{\mathbf{Z}}$, *i.e.*, $h(k) = h(T+1-k)$, $k \in [1,T]_{\mathbf{Z}}$, *then* $u(k)$ *is symmetric on* $[1,T]_{\mathbf{Z}}$. *Moreover*,

(i) *if* $T+1$ $(T \geq 2)$ *is odd, then* $\|u\| = u(\frac{T}{2}) = u(\frac{T}{2}+1)$, *and the solution u of* (9) *can be expressed as*

$$u(k) = \begin{cases} \sum_{s=1}^{k} \phi^{-1}(\sum_{l=s}^{\frac{T}{2}} h(l)), & k \leq \frac{T}{2}, \\ \sum_{s=k}^{T} \phi^{-1}(\sum_{l=\frac{T}{2}+1}^{s} h(l)), & k \geq \frac{T}{2}+1. \end{cases}$$

(ii) *if* $T+1$ $(T \geq 3)$ *is even, then* $\|u\| = u(\frac{T+1}{2})$, *and the solution u of* (9) *can be expressed as*

$$u(k) = \begin{cases} \sum_{s=1}^{k} \phi^{-1}(\sum_{l=s}^{\frac{T+1}{2}-1} h(l) + \frac{1}{2}h(\frac{T+1}{2})), & k \leq \frac{T+1}{2}, \\ \sum_{s=k}^{T} \phi^{-1}(\sum_{l=\frac{T+1}{2}+1}^{s} h(l) + \frac{1}{2}h(\frac{T+1}{2})), & k \geq \frac{T+1}{2}. \end{cases}$$

Proof It is easy to see that

$$u(k) = \sum_{s=1}^{k} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=s}^{T} h(l)\right), \quad k \in [0, T+1]_{\mathbf{Z}}, \tag{10}$$

with

$$\sum_{k=1}^{T+1} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=k}^{T} h(l)\right) = 0. \tag{11}$$

Equivalently,

$$u(k) = \sum_{s=k}^{T} \phi^{-1}\left(-\phi\bigl(u(1)\bigr) + \sum_{l=1}^{s} h(l)\right), \quad k \in [0, T+1]_{\mathbf{Z}}, \tag{12}$$

with

$$\sum_{k=0}^{T} \phi^{-1}\left(-\phi\bigl(u(1)\bigr) + \sum_{l=1}^{k} h(l)\right) = 0. \tag{13}$$

By (10) or (12), one has

$$\phi\bigl(u(1)\bigr) + \phi\bigl(u(T)\bigr) = \sum_{l=1}^{T} h(l). \tag{14}$$

The symmetry of h first implies that $u(1) = u(T)$. In fact, by (11),

$$\begin{aligned}
0 &= \sum_{k=1}^{T+1} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=k}^{T} h(l)\right) \\
&= \sum_{k=1}^{T+1} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=1}^{T+1-k} h(l)\right) \\
&= \sum_{k=0}^{T} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=1}^{k} h(l)\right).
\end{aligned}$$

Since ϕ^{-1} is a homeomorphism from $\mathbf{R}$ onto itself, the solution C of the equation $\sum_{k=1}^{T+1} \phi^{-1}(C + \sum_{l=k}^{T} h(l)) = 0$ is unique. Comparing the equation above with (13), we have $\phi(u(1)) = \phi(u(T))$. Thus for $k \in [1, T]_{\mathbf{Z}}$,

$$\begin{aligned}
u(k) &= \sum_{s=k}^{T} \phi^{-1}\left(-\phi\bigl(u(1)\bigr) + \sum_{l=1}^{s} h(l)\right) = \sum_{s=k}^{T} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=T+1-s}^{T} h(l)\right) \\
&= \sum_{m=1}^{T+1-k} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=m}^{T} h(l)\right) = u(T+1-k),
\end{aligned}$$

the solution u of (9) is symmetric on $[1, T]_{\mathbf{Z}}$.

(i) Assume that $T+1$ ($T \geq 2$) is odd. Since $u(1) = u(T)$, by the symmetry of h and (14), we have

$$\phi\bigl(u(1)\bigr) = \phi\bigl(u(T)\bigr) = \sum_{l=1}^{\frac{T}{2}} h(l) = \sum_{l=\frac{T}{2}+1}^{T} h(l).$$

Then for $k \leq \frac{T}{2}$,

$$\begin{aligned} u(k) &= \sum_{s=1}^{k} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=s}^{T} h(l)\right) = \sum_{s=1}^{k} \phi^{-1}\left(-\sum_{l=\frac{T}{2}+1}^{T} h(l) + \sum_{l=s}^{T} h(l)\right) \\ &= \sum_{s=1}^{k} \phi^{-1}\left(\sum_{l=s}^{\frac{T}{2}} h(l)\right), \end{aligned}$$

and for $k \geq \frac{T}{2} + 1$,

$$\begin{aligned} u(k) &= \sum_{s=k}^{T} \phi^{-1}\left(-\phi\bigl(u(1)\bigr) + \sum_{l=1}^{s} h(l)\right) = \sum_{s=k}^{T} \phi^{-1}\left(-\sum_{l=1}^{\frac{T}{2}} h(l) + \sum_{l=1}^{s} h(l)\right) \\ &= \sum_{s=k}^{T} \phi^{-1}\left(\sum_{l=\frac{T}{2}+1}^{s} h(l)\right). \end{aligned}$$

Clearly, $\|u\| = u(\frac{T}{2}) = u(\frac{T}{2} + 1)$.

(ii) If $T+1$ ($T \geq 3$) is even, then (14) and the symmetry of h imply that

$$\phi\bigl(u(1)\bigr) = \phi\bigl(u(T)\bigr) = \sum_{l=1}^{\frac{T+1}{2}-1} h(l) + \frac{1}{2}h\left(\frac{T+1}{2}\right) = \sum_{l=\frac{T+1}{2}+1}^{T} h(l) + \frac{1}{2}h\left(\frac{T+1}{2}\right).$$

Thus for $k \leq \frac{T+1}{2}$,

$$\begin{aligned} u(k) &= \sum_{s=1}^{k} \phi^{-1}\left(-\phi\bigl(u(T)\bigr) + \sum_{l=s}^{T} h(l)\right) \\ &= \sum_{s=1}^{k} \phi^{-1}\left(-\sum_{l=\frac{T+1}{2}+1}^{T} h(l) - \frac{1}{2}h\left(\frac{T+1}{2}\right) + \sum_{l=s}^{T} h(l)\right) \\ &= \sum_{s=1}^{k} \phi^{-1}\left(\sum_{l=s}^{\frac{T+1}{2}} h(l) - \frac{1}{2}h\left(\frac{T+1}{2}\right)\right) \\ &= \sum_{s=1}^{k} \phi^{-1}\left(\sum_{l=s}^{\frac{T+1}{2}-1} h(l) + \frac{1}{2}h\left(\frac{T+1}{2}\right)\right), \end{aligned}$$

and for $k \geq \frac{T+1}{2}$,

$$\begin{aligned}
u(k) &= \sum_{s=k}^{T} \phi^{-1}\left(-\phi\big(u(1)\big) + \sum_{l=1}^{s} h(l)\right) \\
&= \sum_{s=k}^{T} \phi^{-1}\left(-\sum_{l=1}^{\frac{T+1}{2}-1} h(l) - \frac{1}{2}h\left(\frac{T+1}{2}\right) + \sum_{l=1}^{s} h(l)\right) \\
&= \sum_{s=k}^{T} \phi^{-1}\left(\sum_{l=\frac{T+1}{2}}^{s} h(l) - \frac{1}{2}h\left(\frac{T+1}{2}\right)\right) \\
&= \sum_{s=k}^{T} \phi^{-1}\left(\sum_{l=\frac{T+1}{2}+1}^{s} h(l) + \frac{1}{2}h\left(\frac{T+1}{2}\right)\right).
\end{aligned}$$

Clearly, $\|u\| = u(\frac{T+1}{2})$. The proof is complete. □

Now, we state the existence result of at least three positive solutions of (2). Throughout the following arguments, we suppose that $T \geq 4$. Let v be the unique positive solution of the following boundary value problem:

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + p(k) = 0, & k \in [1,T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0, \end{cases}$$

and $p_0 = \min_{k\in[1,T]_{\mathbf{Z}}} p(k)$.

We make the following assumptions.

(B2) There exists an increasing homeomorphism $\psi : (0,\infty) \to (0,\infty)$ such that for all $\mu > 1$,

$$\frac{\phi(\mu x) - \phi(\mu y)}{\phi(x) - \phi(y)} \geq \psi(\mu), \quad \forall x, y \in \mathbf{R} : x < y;$$

(B3) $p : [1,T]_{\mathbf{Z}} \to (0,\infty)$;

(B4) $g \in C([0,\infty),(0,\infty))$ and $\lim_{u\to\infty} \frac{g(u)}{\psi(\frac{u}{\|v\|})} = 0$;

(B5) There exist a, b and M satisfying $\|v\| < a < b < M$ such that g is nondecreasing on $[b,M)$ and

$$\max\left\{\frac{2\phi(b)g^*(a)}{p_0 g(b)\psi(\frac{a}{\|v\|})}, \frac{T\phi(b)}{\phi(\frac{2M}{T+1})}\right\} < 1.$$

Here $g^*(u) = \max_{0\leq s\leq u} g(s)$.

We denote that condition (B4) implies that $\lim_{u\to\infty} \frac{g^*(u)}{\psi(\frac{u}{\|v\|})} = 0$ (see [12], Lemma 2.8). Clearly, g^* is nondecreasing on $[0,\infty)$.

Assumption (B2) is satisfied by two important cases $\phi(x) = x$ and $\phi(x) = |x|^{p-2}x$ ($p > 1$). We also provide the following two functions as examples:

$$\phi(x) = c_1 x^3 + c_2 x, \qquad \phi(x) = c_3 x^{\frac{1}{3}} + c_4 x^{\frac{1}{5}},$$

where $c_i > 0$ ($i = 1,2,3,4$).

Theorem 3.1 *Let* (B1)-(B5) *hold*. *Then for* $\lambda \in (\lambda_1, \lambda_2)$, *problem* (2) *has at least three positive solutions*. *Here*

$$\lambda_1 = \frac{2\phi(b)}{p_0 g(b)}, \qquad \lambda_2 = \min\left\{\frac{\psi(\frac{a}{\|\nu\|})}{g^*(a)}, \frac{2\phi(\frac{2M}{T+1})}{Tp_0 g(b)}\right\}.$$

Proof Let λ be fixed with $\lambda \in (\lambda_1, \lambda_2)$. Clearly, $\alpha_1 \equiv 0$ is a strict lower solution of (2). Let $\beta_1 = \frac{a}{\|\nu\|}\nu$. Note that $a > \|\nu\|$ and $\Delta\nu(k) < \Delta\nu(k-1)$, $k \in [1,T]_{\mathbf{Z}}$. Then by (B2) and the monotonicity of g^*, for $k \in [1,T]_{\mathbf{Z}}$, we have

$$\begin{aligned}
\Delta\big(\phi\big(\Delta\beta_1(k-1)\big)\big) &= \phi\big(\Delta\beta_1(k)\big) - \phi\big(\Delta\beta_1(k-1)\big) \\
&= \phi\left(\frac{a}{\|\nu\|}\Delta\nu(k)\right) - \phi\left(\frac{a}{\|\nu\|}\Delta\nu(k-1)\right) \\
&\le \psi\left(\frac{a}{\|\nu\|}\right)\big(\phi\big(\Delta\nu(k)\big) - \phi\big(\Delta\nu(k-1)\big)\big) \\
&= -\psi\left(\frac{a}{\|\nu\|}\right)p(k) \\
&< -\lambda g^*(a)p(k) \\
&\le -\lambda g^*\big(\beta_1(k)\big)p(k) \\
&\le -\lambda g\big(\beta_1(k)\big)p(k).
\end{aligned}$$

Thus β_1 is a strict upper solution of (2). Now, let α_2 solve the following problem:

$$\begin{cases} \Delta(\phi(\Delta u(k-1))) + \lambda^* p_0 g(b) = 0, & k \in [1,T]_{\mathbf{Z}}, \\ u(0) = u(T+1) = 0, \end{cases}$$

where $\lambda^* \in (\lambda_1, \lambda)$. By the expression (12), we have

$$\begin{aligned}
\alpha_2(1) &= \sum_{s=1}^{T}\phi^{-1}\big(-\phi\big(\alpha_2(1)\big) + s\lambda^* p_0 g(b)\big) \\
&> \phi^{-1}\big(-\phi\big(\alpha_2(1)\big) + \lambda^* p_0 g(b)\big),
\end{aligned}$$

which implies that $\phi(\alpha_2(1)) > \frac{1}{2}\lambda^* p_0 g(b) > \phi(b)$. Thus Lemma 3.2 implies that $\alpha_2(T) = \alpha_2(1) > b$. Consequently, by Lemma 3.1, $\alpha_2(k) > b$ for all $k \in [1,T]_{\mathbf{Z}}$. Again by Lemma 3.2, one can see that if $T+1$ is odd, then

$$\|\alpha_2\| = \sum_{s=1}^{\frac{T}{2}}\phi^{-1}\left(\sum_{l=s}^{\frac{T}{2}}\lambda^* p_0 g(b)\right) < \frac{T+1}{2}\phi^{-1}\left(\frac{T}{2}\lambda^* p_0 g(b)\right) < M,$$

and that if $T+1$ is even, then

$$\|\alpha_2\| = \sum_{s=1}^{\frac{T+1}{2}}\phi^{-1}\left(\sum_{l=s}^{\frac{T+1}{2}-1}\lambda^* p_0 g(b) + \frac{1}{2}\lambda^* p_0 g(b)\right) < \frac{T+1}{2}\phi^{-1}\left(\frac{T}{2}\lambda^* p_0 g(b)\right) < M.$$

Thus $b < \alpha_2(k) < M$ for $k \in [1, T]_{\mathbf{Z}}$. Therefore,

$$\begin{aligned} 0 &= \Delta\big(\phi\big(\Delta\alpha_2(k-1)\big)\big) + \lambda^* p_0 g(b) \\ &\le \Delta\big(\phi\big(\Delta\alpha_2(k-1)\big)\big) + \lambda^* p_0 g\big(\alpha_2(k)\big) \\ &< \Delta\big(\phi\big(\Delta\alpha_2(k-1)\big)\big) + \lambda p(k) g\big(\alpha_2(k)\big), \quad k \in [1, T]_{\mathbf{Z}}, \end{aligned}$$

which implies that α_2 is a strict lower solution of (2). It is easy to see that

$$\alpha_2(k) > b > a \ge \beta_1(k), \quad k \in [1, T]_{\mathbf{Z}}.$$

By $\lim_{u\to\infty} \frac{g^*(u)}{\psi(\frac{u}{\|v\|})} = 0$, one can choose a sufficiently large positive number C_λ, such that

$$\frac{g^*(\lambda C_\lambda)}{\psi(\frac{\lambda C_\lambda}{\|v\|})} < \frac{1}{\lambda},$$

and

$$\frac{\lambda C_\lambda}{\|v\|} > 1, \qquad \alpha_2 \le \beta_2,$$

where $\beta_2 = \lambda C_\lambda \frac{v}{\|v\|}$. Then by (B2) and the monotonicity of g^*, β_2 is a strict upper solution of (2). In fact, for $k \in [1, T]_{\mathbf{Z}}$,

$$\begin{aligned} \Delta\big(\phi\big(\Delta\beta_2(k-1)\big)\big) &= \phi\big(\Delta\beta_2(k)\big) - \phi\big(\Delta\beta_2(k-1)\big) \\ &= \phi\left(\frac{\lambda C_\lambda}{\|v\|}\Delta v(k)\right) - \phi\left(\frac{\lambda C_\lambda}{\|v\|}\Delta v(k-1)\right) \\ &\le \psi\left(\frac{\lambda C_\lambda}{\|v\|}\right)\big(\phi\big(\Delta v(k)\big) - \phi\big(\Delta v(k-1)\big)\big) \\ &= -\psi\left(\frac{\lambda C_\lambda}{\|v\|}\right)p(k) \\ &< -\lambda g^*(\lambda C_\lambda)p(k) \\ &\le -\lambda g^*\big(\beta_2(k)\big)p(k) \\ &\le -\lambda g\big(\beta_2(k)\big)p(k). \end{aligned}$$

Thus by Theorem 2.1, problem (2) has three positive solutions for $\lambda \in (\lambda_1, \lambda_2)$. □

Remark If g is nondecreasing on $[0, \infty)$, then we take $M = \infty$ and $\lambda_2 = \frac{\psi(\frac{a}{\|v\|})}{g(a)}$.

4 An example

Taking $\phi(u) = u^{\frac{1}{3}} + u^{\frac{1}{5}}$, $g(u) = (u+1)^{\frac{1}{2}}$, $p(k) \equiv 1$, $T = 4$, consider

$$\begin{cases} \Delta((\Delta u(k-1))^{\frac{1}{3}} + (\Delta u(k-1))^{\frac{1}{5}}) + \lambda(u(k)+1)^{\frac{1}{2}} = 0, & k \in [1, T]_{\mathbf{Z}}, \\ u(0) = u(5) = 0. \end{cases} \tag{15}$$

Let $\psi(u) = u$. It is easy to see that (B1)-(B4) hold. Choose $a = 14$, $b = 15$, then (B5) is satisfied. In fact, after some simple calculations, we get that $\|v\| \approx 1.089$ and that

$$\lambda_1 = \frac{2\phi(b)}{p_0 g(b)} = \frac{1}{2}\left(15^{\frac{1}{3}} + 15^{\frac{1}{5}}\right) \approx 2.091,$$

$$\lambda_2 = \frac{\psi\left(\frac{a}{\|v\|}\right)}{g(a)} = \frac{14}{\sqrt{15}\|v\|} \approx 3.319.$$

Thus by Theorem 3.1, problem (15) has at least three positive solutions for λ: $2.091 < \lambda < 3.319$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors jointly worked on the results, and they read and approved the final manuscript.

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