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Some properties of solutions of a class of systems of complex q-shift difference equations

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Abstract

In view of Nevanlinna theory, we study the properties of systems of two types of complex difference equations with meromorphic solutions. Some results of this paper improve and extend previous theorems given by Gao, and five examples are given to show the extension of solutions of the system of complex difference equations.

MSC: 39A50; 30D35

Keywords: q-shift; difference equation; transcendental meromorphic function

1 Introduction and main results

In this note, we will investigate the problem of the existence and growth of solutions of complex difference equations. The fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions will be used (see [1–3]). Besides, for the meromorphic function f, $S(r,f)$ denotes any quantity satisfying that $S(r,f) = o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t} < \infty$, and a meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a(z)) = S(r,f)$.

In recent years, difference equations, difference product and q-difference in the complex plane $\mathbb{C}$ have been an active topic of study. Considerable attention has been paid to the growth of solutions of difference equations, value distribution and uniqueness of differences analogues of Nevanlinna's theory [4–8]. Chiang and Feng [9] and Halburd and Korhonen [10] established a difference analogue of the logarithmic derivative lemma independently. After their work, a number of results on meromorphic solutions of complex difference equations were obtained.

The structure of this paper is as follows. In Section 1*, some results on growth of solutions of a complex difference equation are listed, and our theorems are given. In Section* 2*, we introduce some lemmas. Section* 3 *is devoted to proving Theorem* 1.5*. Section* 4 *is devoted to proving Theorem* 1.6*. Finally, Section* 5 *gives some examples to show the accuracy of conclusions of Theorem* 1.5*.*

In 2003, Silvennoinen considered [11] the growth and existence of meromorphic solutions of functional equations of the form $f(p(z)) = R(z,f(z))$, and obtained the following result.

Theorem 1.1 [11] *Let f be a non-constant meromorphic solution of the equation*

$$f\big(g(z)\big) = R\big(z,f(z)\big) := \frac{\sum_{i=0}^{m_1} a_i(z)f^i(z)}{\sum_{j=0}^{n_1} b_j(z)f^j(z)},$$

where g is an entire function, a_i, b_j are small meromorphic functions with respect to f. Then, g is a polynomial.

In 2012, Gao [12, 13] also investigated the growth and existence of meromorphic solutions of two systems of complex difference equations, and obtained some theorems as follows.

Theorem 1.2 [12] *Let (f_1,f_2) be a non-constant meromorphic solution of the system*

$$\begin{cases} f_1(p(z)) = R_1(z,f_2(z)), \\ f_2(p(z)) = R_2(z,f_1(z)). \end{cases} \tag{1}$$

Then $p(z)$ is a polynomial, where

$$R_1\big(z,f_2(z)\big) = \frac{\sum_{i=0}^{s_1} a_i(z)f^i(z)}{\sum_{j=0}^{t_1} b_j(z)f^j(z)}, \qquad R_2\big(z,f_1(z)\big) = \frac{\sum_{i=0}^{s_2} d_i(z)f^i(z)}{\sum_{j=0}^{t_2} e_j(z)f^j(z)}$$

are irreducible rational functions, $a_i(z)$, $b_j(z)$, $d_i(z)$ and $e_j(z)$ are small functions.

Theorem 1.3 [12] *Let $p(z) = az + b$, (f_1,f_2) be a meromorphic solution of system* (1), *and let $\mu(f_1)$, $\mu(f_2)$ be the lower orders of f_1, f_2, respectively. If*

$$\mu(f_1) + \mu(f_2) < \frac{\log d_1d_2}{2\log|a|},$$

then the components f_1 and f_2 in (f_1,f_2) have at least one rational function, where $d_i = \max\{s_i,t_i\}$, $i = 1,2$.

In 2005, Laine *et al.* [14] investigated several higher order difference equations. In particular, they obtained the following result.

Theorem 1.4 [14] *Suppose that f is a transcendental meromorphic solution of the equation*

$$\sum_{\{J\}} \alpha_J(z) \prod_{j\in J} f(z+c_j) = f\big(p(z)\big), \tag{2}$$

where $\{J\}$ is a collection of all non-empty subsets of $\{1,2,\dots,n\}$, c_j's are distinct complex constants, and $p(z)$ is a polynomial of degree $k \geq 2$. Moreover, we assume that the coefficients $\alpha_J(z)$ are small functions relative to f and that $n \geq k$. Then

$$T(r,f) = O\big((\log r)^{\beta+\varepsilon}\big),$$

where $\beta = \frac{\log n}{\log k}$.

Recently, there were some paper focusing on the properties of solutions of some systems of complex difference equations and q-shift difference equation (see [12, 13, 15–18]). A question is raised naturally, whether the assertion of Theorem 1.4 remains valid, if the equation (2) is replaced by the following

$$\begin{cases} \sum_{\{J_1\}} \alpha_{J_1}(z) \prod_{s\in J_1} f_2(z+c_s) = R_1(z, f_1(p(z))), \\ \sum_{\{J_2\}} \alpha_{J_2}(z) \prod_{t\in J_2} f_1(z+c_t) = R_2(z, f_2(p(z))). \end{cases} \tag{3}$$

In this paper, we study the question above and the problem of the existence of meromorphic solutions for a system of complex difference equations (3), where $p(z)$ is a polynomial, and obtain the following results.

Theorem 1.5 *For systems* (3), $\{J_i\}$ *are two collections of all non-empty subsets of* $\{1, 2, \dots, n_i\}$ *for* $i = 1, 2$, c_j $(j = 1, 2, \dots, n_i)$ *are distinct complex constants, and* $R_i(z, u)$ *are irreducible rational functions in* u *of* $\deg_u \sigma_i = \max\{s_i, t_i\}$ (> 0) $(i = 1, 2)$, *its coefficients of* $R_i(z, u)$ *are all small functions. Let* (f_1, f_2) *be a meromorphic solution of system* (3) *such that* f_1, f_2 *are non-rational meromorphic. All the coefficients of* (3) *are small functions relative to* f_1, f_2, *and* $p(z) = qz + \eta$, $q \neq 0$, η *are complex constants. Thus,*

(i) *if* $0 < |q| < 1$ *and* $\sigma_1\sigma_2 \geq n_1n_2$. *We have*

$$\mu(f_1) + \mu(f_2) \geq \frac{\log \sigma_1\sigma_2 - \log n_1n_2}{-\log |q|}; \tag{4}$$

(ii) *if* $|q| > 1$ *and* $\sigma_1\sigma_2 \leq n_1n_2$, *then*

$$\rho(f_1) + \rho(f_2) \leq \frac{\log n_1n_2 - \log \sigma_1\sigma_2}{\log |q|}; \tag{5}$$

(iii) *if* $|q| = 1$ *and* $\sigma_1\sigma_2 \geq n_1n_2$, *then* $\mu(f_1) + \mu(f_2) \geq \infty$,

where $\mu(f)$ *is the lower order of* f.

Theorem 1.6 *Under the assumptions of Theorem* 1.5, *if* $p(z) = p_kz^k + \cdots + p_1z + p_0$ $(p_0, p_1, \dots, p_k \in \mathbb{C})$ *of degree* $k \geq 2$, (f_1, f_2) *is a meromorphic solution of system* (3) *such that* f_1, f_2 *are non-rational meromorphic, and all the coefficients of* (3) *are small functions relative to* f_1, f_2. *Then*

$$k^2\sigma_1\sigma_2 \leq n_1n_2$$

and

$$T(r, f_1) = O\big((\log r)^{\varsigma+\varepsilon}\big), \qquad T(r, f_2) = O\big((\log r)^{\varsigma+\varepsilon}\big),$$

where $\varepsilon > 0$ *and*

$$\varsigma = \frac{\log n_1n_2 - \log \sigma_1\sigma_2}{2\log k}.$$

2 Some lemmas

Lemma 2.1 (Valiron-Mohon'ko [19]) *Let $f(z)$ be a meromorphic function. Then for all irreducible rational functions in f,*

$$R(z,f(z)) = \frac{\sum_{i=0}^{m} a_i(z)f(z)^i}{\sum_{j=0}^{n} b_j(z)f(z)^j},$$

with meromorphic coefficients $a_i(z)$, $b_j(z)$, the characteristic function of $R(z,f(z))$ satisfies that

$$T(r, R(z,f(z))) = dT(r,f) + O(\Psi(r)),$$

where $d = \max\{m,n\}$ and $\Psi(r) = \max_{i,j}\{T(r,a_i), T(r,b_j)\}$.

Lemma 2.2 [14] *Given distinct complex numbers $c_1,\ldots,c_n$, a meromorphic function f, and small functions $\alpha_J(z)$ relative to f, we have*

$$T\left(r, \sum_{\{J\}} \alpha_J(z)\left(\prod_{j\in J} f(z+c_j)\right)\right) \le \sum_{k=1}^{n} T(r, f(z+c_k)) + S(r,f),$$

where $\{J\}$ is a collection of all non-empty subsets of $\{1,2,\ldots,n\}$.

Lemma 2.3 [20] *Suppose that a meromorphic function f is of finite lower order λ. Then, for every constant $c > 1$ and a given ε, there exists a sequence $r_n = r_n(c,\varepsilon) \to \infty$, such that*

$$T(cr_n, f) \le c^{\lambda+\varepsilon} T(r_n, f).$$

Lemma 2.4 [21] *Let $f(z)$ be a transcendental meromorphic function, and let $p(z) = p_k z^k + p_{k-1}z^{k-1} + \cdots + p_1 z + p_0$ be a complex polynomial of degree $k > 0$. For given $0 < \delta < |p_k|$, let $\lambda = |p_k| + \delta$, $\mu = |p_k| - \delta$, then for given $\varepsilon > 0$ and for sufficiently large r,*

$$(1-\varepsilon)T(\mu r^k, f) \le T(r, f\circ p) \le (1+\varepsilon)T(\lambda r^k, f).$$

Lemma 2.5 [5, 19] *Let $g : (0,+\infty) \to R$, $h : (0,+\infty) \to R$ be monotone increasing functions such that $g(r) \le h(r)$ outside of an exceptional set E with the finite linear measure, or $g(r) \le h(r)$, $r \notin H \cup (0,1]$, where $H \subset (1,\infty)$ is a set of the finite logarithmic measure. Then, for any $\alpha > 1$, there exists r_0 such that $g(r) \le h(\alpha r)$ for all $r \ge r_0$.*

Lemma 2.6 [22] *Let $\psi(r)$ be a function of r ($r \ge r_0$), positive and bounded in every finite interval.*

(i) *Suppose that $\psi(\mu r^m) \le A\psi(r) + B$ ($r \ge r_0$), where μ ($\mu > 0$), m ($m > 1$), A ($A \ge 1$), B are constants. Then $\psi(r) = O((\log r)^\alpha)$ with $\alpha = \frac{\log A}{\log m}$, unless $A = 1$ and $B > 0$; and if $A = 1$ and $B > 0$, then for any $\varepsilon > 0$, $\psi(r) = O((\log r)^\varepsilon)$.*

(ii) *Suppose that (with the notation of (i)) $\psi(\mu r^m) \ge A\psi(r)$ ($r \ge r_0$). Then for all sufficiently large values of r, $\psi(r) \ge K(\log r)^\alpha$ with $\alpha = \frac{\log A}{\log m}$, for some positive constant K.*

3 The proof of Theorem 1.5

From the assumptions of Theorem 1.5, we know that f_1 and f_2 are transcendental meromorphic functions.

Denote $\Psi_i(r) = \max\{T(r, a_j^i(z))|j = 1, 2, \dots, s_i\}$, $i = 1, 2$, and $C = \max\{|c_1|, |c_2|, \dots, |c_n|\}$. Since $T(r, f(z + c)) \le (1 + o(1))T(r + |c|, f) + M$ (ref. [23]), by applying Lemma 2.1 to (3) and from Lemma 2.2, we have

$$\begin{aligned}
&\sigma_1 T\bigl(r, f_1\bigl(p(z)\bigr)\bigr) + \Psi_1(r) \\
&\quad = T\bigl(r, R_1\bigl(z, f_1\bigl(p(z)\bigr)\bigr)\bigr) \\
&\quad = T\left(r, \sum_{J_1} \alpha_{J_1}(z)\left(\prod_{j\in J_1} f_2(z + c_j)\right)\right) \le \sum_{j=1}^{n_1} T\bigl(r, f_2(z + c_j)\bigr) + S(r, f_2) \\
&\quad \le \left(1 + \frac{\varepsilon_1}{2}\right)\sum_{j=1}^{n_1} T(r + C, f_2) + S(r, f_2) \le n_1\left(1 + \frac{\varepsilon_1}{2}\right)T(\beta_1 r, f_2) + S(r, f_2), \qquad (6) \\
&\sigma_2 T\bigl(r, f_2\bigl(p(z)\bigr)\bigr) + \Psi_2(r) \\
&\quad = T\bigl(r, R_2\bigl(z, f_2\bigl(p(z)\bigr)\bigr)\bigr) \\
&\quad = T\left(r, \sum_{J_2} \alpha_{J_2}(z)\left(\prod_{j\in J_2} f_1(z + c_j)\right)\right) \le \sum_{j=1}^{n_2} T\bigl(r, f_1(z + c_j)\bigr) + S(r, f_1) \\
&\quad \le \left(1 + \frac{\varepsilon_2}{2}\right)\sum_{j=1}^{n_2} T(r + C, f_1) + S(r, f_1) \le n_1\left(1 + \frac{\varepsilon_2}{2}\right)T(\beta_2 r, f_1) + S(r, f_1) \qquad (7)
\end{aligned}$$

for sufficiently large r and any given $\beta_i > 1$, $\varepsilon_i > 0$, $i = 1, 2$. Since $p(z) = qz + \eta$, according to Lemma 2.4 and (6), (7), for $\theta_i = |q| - \delta_i$ $(0 < \delta_i < |q|, 0 < \theta_i < 1)$, $i = 1, 2$ and sufficiently larger r, we get

$$\begin{aligned}
&\sigma_1(1 - \varepsilon_1)T(\theta_1 r, f_1) \le n_1(1 + \varepsilon_1)T(\beta_1 r, f_2), \quad r \notin E_1, \\
&\sigma_2(1 - \varepsilon_2)T(\theta_2 r, f_2) \le n_2(1 + \varepsilon_2)T(\beta_2 r, f_1), \quad r \notin E_2,
\end{aligned}$$

where E_1 and E_2 are the sets of finite linear measure. From Lemma 2.5, for any given $\gamma_i > 1$ $(i = 1, 2)$ and sufficiently large r, we can obtain

$$\begin{aligned}
&\sigma_1(1 - \varepsilon_1)T(\theta_1 r, f_1) \le n_1(1 + \varepsilon_1)T(\beta_1\gamma_1 r, f_2), \\
&\sigma_2(1 - \varepsilon_2)T(\theta_2 r, f_2) \le n_2(1 + \varepsilon_2)T(\beta_2\gamma_2 r, f_1),
\end{aligned}$$

that is,

$$\begin{aligned}
&\frac{\sigma_1(1 - \varepsilon_1)}{n_1(1 + \varepsilon_1)}T(r, f_1) \le T\left(\frac{\beta_1\gamma_1}{\theta_1} r, f_2\right), \\
&\frac{\sigma_2(1 - \varepsilon_2)}{n_2(1 + \varepsilon_2)}T(r, f_2) \le T\left(\frac{\beta_2\gamma_2}{\theta_2} r, f_1\right).
\end{aligned} \qquad (8)$$

Case 3.1 $0 < |q| < 1$ and $\sigma_1\sigma_2 \ge n_1n_1$. Since $\beta_i > 1$, $\gamma_i > 1$, $0 < \theta_i < 1$, we have $\frac{\beta_i\gamma_i}{\theta_i} > 1$, $i = 1, 2$. From (8), and by Lemma 2.3, for any given $\varepsilon > 0$, there exists a sequence $r_n \to \infty$ such that

$$\frac{\sigma_1(1-\varepsilon_1)}{n_1(1+\varepsilon_1)} T(r_n, f_1) \le \left(\frac{\beta_1\gamma_1}{\theta_1}\right)^{\mu(f_2)+\varepsilon} T(r_n, f_2),$$
$$\frac{\sigma_2(1-\varepsilon_2)}{n_2(1+\varepsilon_2)} T(r_n, f_2) \le \left(\frac{\beta_2\gamma_2}{\theta_2}\right)^{\mu(f_1)+\varepsilon} T(r_n, f_1)$$

for $r_n > r_0$. From the inequalities above, we have

$$\frac{\sigma_1(1-\varepsilon_1)}{n_1(1+\varepsilon_1)} \frac{\sigma_2(1-\varepsilon_2)}{n_2(1+\varepsilon_2)} \le \left(\frac{\beta_1\gamma_1}{\theta_1}\right)^{\mu(f_2)+\varepsilon} \left(\frac{\beta_2\gamma_2}{\theta_2}\right)^{\mu(f_1)+\varepsilon}. \tag{9}$$

Thus, letting $\varepsilon \to 0$, $\delta_i \to 0$, $\beta_i \to 1$ and $\gamma_i \to 1$ for $i = 1, 2$ and $\varepsilon = \max\{\varepsilon, \varepsilon_1, \varepsilon_2\}$. Since $0 < |q| < 1$ and $\sigma_1\sigma_2 \ge n_1n_2$, from (9), we can get

$$\mu(f_1) + \mu(f_2) \ge \frac{\log \sigma_1\sigma_2 - \log n_1n_2}{-\log |q|}.$$

Hence, (4) holds.

Case 3.2 Suppose that $|q| > 1$. By using the same argument as above, we can get

$$\sigma_1(1-\varepsilon_1)T(\theta_1' r, f_1) \le n_1\left(1 + \frac{\varepsilon_1}{2}\right) T(r + C, f_2) + S(r, f_2),$$
$$\sigma_2(1-\varepsilon_2)T(\theta_2' r, f_2) \le n_2\left(1 + \frac{\varepsilon_2}{2}\right) T(r + C, f_1) + S(r, f_1),$$

where $\theta_i' = |q| - \delta_i$ ($\delta_i > 0$ is chosen to be such that $\theta_i' > 1$), and r is sufficiently large. We can choose sufficiently small $\varepsilon_i > 0$ such that $\frac{1}{\theta_i'} + \varepsilon_i < 1$. Thus, it follows that

$$\begin{aligned}
\sigma_1(1-\varepsilon_1)T(r, f_1) &\le n_1(1+\varepsilon_1)T\left(\frac{r+C}{\theta_1'}, f_2\right) + S(r, f_2) \\
&\le n_1(1+\varepsilon_1)T\left(\left(\frac{1}{\theta_1'} + \varepsilon_1\right) r, f_2\right) + S(r, f_2), \quad r \notin E_3, \\
\sigma_2(1-\varepsilon_2)T(r, f_2) &\le n_2(1+\varepsilon_2)T\left(\frac{r+C}{\theta_2'}, f_1\right) + S(r, f_1) \\
&\le n_2(1+\varepsilon_2)T\left(\left(\frac{1}{\theta_2'} + \varepsilon_2\right) r, f_1\right) + S(r, f_1), \quad r \notin E_4,
\end{aligned}$$

where E_3, E_4 are the sets of the finite logarithmic measure.

Since $n_1n_2 \ge \sigma_1\sigma_2$, $\frac{1}{\theta_i'} < 1$, $i = 1, 2$, and f_1, f_2 are transcendental, by applying Lemma 3.1 in [24] and Lemma 2.5 for $\varepsilon_i \to 0$ and $\delta_i \to 0$, we have

$$\rho(f_1) \le \frac{\log n_1n_2 - \log \sigma_1\sigma_2}{2\log |q|}, \qquad \rho(f_2) \le \frac{\log n_1n_2 - \log \sigma_1\sigma_2}{2\log |q|},$$

which implies that (5) is true.

Case 3.3 $|q| = 1$ and $\sigma_1\sigma_2 > n_1n_2$. By using the same argument as in Case 3.1, we can get $\mu(f_1) + \mu(f_2) \geq \infty$.

From Cases 3.1-3.3, the proof of Theorem 1.5 is completed.

4 The proof of Theorem 1.6

By using the same argument as in Theorem 1.5, we can get (6) and (7). Since $p(z) = p_kz^k + \cdots + p_1z + p_0$, by Lemma 2.4, we can get that for $\vartheta_i = |p_k| - \delta_i$ (> 0), $i = 1, 2$ and sufficiently large r,

$$d_1(1-\varepsilon)T(\vartheta_1 r^k, f_1) \leq n_1(1+\varepsilon)T(\beta_1 r, f_2), \quad r \notin E_5,$$
$$d_2(1-\varepsilon)T(\vartheta_2 r^k, f_2) \leq n_2(1+\varepsilon)T(\beta_2 r, f_1), \quad r \notin E_6,$$

where E_5, E_6 are two sets of finite linear measure, and β_1, β_2 are defined as in the proof of Theorem 1.5. In view of Lemma 2.5, we have that for any given γ_1, γ_2 and sufficiently large r,

$$d_1(1-\varepsilon)T(\vartheta_1 r^k, f_1) \leq n_1(1+\varepsilon)T(\beta_1\gamma_1 r, f_2),$$
$$d_2(1-\varepsilon)T(\vartheta_2 r^k, f_2) \leq n_2(1+\varepsilon)T(\beta_2\gamma_2 r, f_1),$$

that is,

$$T\left(\frac{\vartheta_1}{(\beta_1\gamma_1)^k}t_1^k, f_1\right) \leq \frac{n_1(1+\varepsilon)}{\sigma_1(1-\varepsilon)}T(t_1, f_2), \tag{10}$$
$$T\left(\frac{\vartheta_2}{(\beta_2\gamma_2)^k}t_2^k, f_2\right) \leq \frac{n_2(1+\varepsilon)}{\sigma_2(1-\varepsilon)}T(t_2, f_1), \tag{11}$$

where $t_1 = \beta_1\gamma_1 r$ and $t_2 = \beta_2\gamma_2 r$. Combining (10) with (11), we have

$$T\left(\frac{\vartheta_1(\vartheta_2)^k}{(\beta_1\gamma_1)^k(\beta_2\gamma_2)^{2k}}r^{2k}, f_1\right) \leq \frac{n_1(1+\varepsilon)}{\sigma_1(1-\varepsilon)}\frac{n_2(1+\varepsilon)}{\sigma_2(1-\varepsilon)}T(r, f_1), \tag{12}$$
$$T\left(\frac{\vartheta_2(\vartheta_1)^k}{(\beta_2\gamma_2)^k(\beta_1\gamma_1)^{2k}}r^{2k}, f_2\right) \leq \frac{n_2(1+\varepsilon)}{\sigma_2(1-\varepsilon)}\frac{n_1(1+\varepsilon)}{\sigma_1(1-\varepsilon)}T(r, f_2). \tag{13}$$

Since $k \geq 2$, we get $\sigma_1\sigma_2 \leq n_1n_2$. From Lemma 2.6, we obtain

$$T(r, f_1) = O\big((\log r)^{\varsigma_1}\big), \qquad T(r, f_2) = O\big((\log r)^{\varsigma_1}\big),$$

where

$$\varsigma_1 = \frac{\log n_1n_2 - \log\sigma_1\sigma_2 + 2\log(1+\varepsilon) - 2\log(1-\varepsilon)}{2\log k} = \frac{\log n_1n_2 - \log\sigma_1\sigma_2}{2\log k} + \varepsilon_1.$$

Set $\varsigma = \frac{\log n_1n_2 - \log\sigma_1\sigma_2}{2\log k}$. Then we have

$$T(r, f_1) = O\big((\log r)^{\varsigma+\varepsilon_1}\big), \qquad T(r, f_2) = O\big((\log r)^{\varsigma+\varepsilon_1}\big).$$

Next, we will prove that $k^2\sigma_1\sigma_2 \le n_1n_2$. Suppose that $k^2\sigma_1\sigma_2 > n_1n_2$, then we can get $\varsigma = \frac{\log n_1n_2 - \log\sigma_1\sigma_2}{2\log k} < 1$. For sufficiently small $\varepsilon_1 > 0$, we have $\varsigma_1 = \varsigma + \varepsilon_1 < 1$. This contradicts the condition on the transcendency of f_1, f_2.

Thus, the proof of Theorem 1.6 is completed.

5 Some examples for Theorem 1.5

The following examples show that the conclusions (4) and (5) in Theorem 1.5 are sharp.

Example 5.1 The solution $(f_1(z), f_2(z)) = (e^z, e^{-z})$ satisfies the system, where $p(z) = -\frac{1}{2}z + \eta$, c, η are any nonzero complex constants,

$$a_1(z) = e^{4\eta}, \qquad b_1(z) = e^{2\eta}\left(e^c + e^{-c}\right), \qquad a_2(z) = e^{-4\eta}, \qquad b_2(z) = e^{-2\eta}\left(e^c + e^{-c}\right).$$

Thus, we have

$$\mu(f_1) + \mu(f_2) = 1 + 1 = \frac{\log\sigma_1\sigma_2 - \log n_1n_2}{-\log|q|},$$

where $\sigma_1 = \sigma_2 = 4$, $n_1 = n_2 = 2$ and $|q| = \frac{1}{2} < 1$. This example shows that the equality in (4) can be achieved.

Example 5.2 The solution $(f_1(z), f_2(z)) = (e^{z^2}, e^{(z+1)^2})$ satisfies

$$\begin{cases} f_1(z+c) + f_1(z-c) = a_1(z)(f_2(qz+\eta))^4, \\ f_2(z+c) + f_2(z-c) = a_2(z)(f_1(qz+\eta))^4, \end{cases}$$

where c is any nonzero complex constant, $q = \frac{1}{2}$, $\eta = -1$, and

$$a_1(z) = e^{c^2}\left(e^{2zc} + e^{-2zc}\right), \qquad a_2(z) = e^{(c+1)^2}\left(e^{2z(c+1)} + e^{-2z(c+1)-4c}\right)e^{4z-4}.$$

We note that $a_1(z)$, $a_2(z)$ are small functions relative to e^{z^2}, $e^{(z+1)^2}$. Thus, we have

$$\mu(f_1) + \mu(f_2) = 4 > 2 = \frac{\log\sigma_1\sigma_2 - \log n_1n_2}{-\log|q|},$$

where $\sigma_1 = \sigma_2 = 4$, $n_1 = n_2 = 2$ and $|q| = \frac{1}{2} < 1$. This example shows that the inequality (4) is true.

Example 5.3 The solution $(f_1(z), f_2(z)) = (e^{2z}, e^{-2z})$ satisfies

$$\begin{cases} f_1(z+c)f_1(z-c)f_1(z+2c)f_1(z-2c) + f_1(z+c)f_1(z-c) \\ \quad = a_1(z)(f_2(qz+\eta))^2 + b_1(z)f_2(qz+\eta), \\ f_2(z+c)f_2(z-c)f_2(z+2c)f_2(z-2c) + f_2(z+c)f_2(z-c) \\ \quad = a_2(z)(f_1(qz+\eta))^2 + b_2(z)f_1(qz+\eta), \end{cases}$$

where c, η are any nonzero complex constants, $q = -2$, and

$$a_1(z) = e^{4\eta}, \qquad b_1(z) = e^{2\eta}, \qquad a_2(z) = e^{-4\eta}, \qquad b_2(z) = e^{-2\eta}.$$

Thus, we have

$$\rho(f_1) + \rho(f_2) = 2 = \frac{\log n_1 n_2 - \log \sigma_1 \sigma_2}{\log |q|},$$

where $n_1 = n_2 = 4$, $\sigma_1 = \sigma_2 = 2$ and $|q| = 2 > 1$. This example shows that the equality in (5) can be achieved.

Example 5.4 The solution $(f_1(z), f_2(z)) = (e^{z^2}, e^{(z+1)^2})$ satisfies

$$\begin{cases} d_1(z)f_1(z+c)f_1(z-c)f_1(z+2c)f_1(z-2c)f_1(z+3c)f_1(z-3c) \\ \qquad \times f_1(z+4c)f_1(z-4c) + d_2(z)f_1(z+c)f_1(z-c)f_1(z+5c)f_1(z-5c) \\ \quad = a_1(z)(f_2(qz+\eta))^2 + b_1(z)f_2(qz+\eta), \\ e_1(z)f_2(z+c)f_2(z-c)f_2(z+2c)f_2(z-2c)f_2(z+3c)f_2(z-3c) \\ \qquad \times f_2(z+4c)f_2(z-4c) + e_2(z)f_2(z+c)f_2(z-c)f_2(z+5c)f_2(z-5c) \\ \quad = a_2(z)(f_1(qz+\eta))^2 + b_2(z)f_1(qz+\eta), \end{cases}$$

where c is a nonzero constant, $q = 2$, $\eta = -1$,

$$d_1(z) = e^{-60c^2}, \qquad d_2(z) = e^{-52c^2}, \qquad e_1(z) = e^{-60c^2-8}e^{-16z}, \qquad e_2(z) = e^{-52c^2-4}e^{-8z}$$

and

$$a_1(z) = 1, \qquad b_1(z) = 1, \qquad a_2(z) = e^{8z-2}, \qquad b_2(z) = e^{4z-1}.$$

We note that $a_i(z)$, $b_i(z)$, $d_i(z)$, $e_i(z)$, $i = 1, 2$ are small functions relative to e^{z^2}, $e^{(z+1)^2}$. Thus, we have

$$\rho(f_1) + \rho(f_2) = 4 \le \log_2 25 = \frac{\log n_1 n_2 - \log \sigma_1 \sigma_2}{\log |q|},$$

where $n_1 = n_2 = 10$, $\sigma_1 \sigma_2 = 2$ and $|q| = 2 > 1$. This example shows that the inequality in (5) is true.

Example 5.5 The solution $(f_1(z), f_2(z)) = (e^{e^z}, e^{e^{-z}})$ satisfies the following system

$$\begin{cases} f_1(z+3\log 2) + f_1(z+4\log 2) + f_1(z+3\log 2)f_1(z+4\log 2) \\ \quad = f_2(-z-3\log 2) + (f_2(-z-3\log 2))^2 + (f_2(-z-3\log 2))^3, \\ f_2(z+3\log 2) + f_2(z+4\log 2) + f_2(z+3\log 2)f_2(z+4\log 2) \\ \quad = f_2(-z-3\log 2) + (f_2(-z-3\log 2))^2 + (f_2(-z-3\log 2))^3. \end{cases}$$

We have $\mu(f_1) + \mu(f_2) = \infty$. Thus, it shows that (iii) in Theorem 1.5 is true when $\sigma_1 = \sigma_2 = n_1 = n_2 = 3$, $c_1 = 3\log 2$, $c_2 = 4\log 2$, $q = -1$ and $\eta = -3\log 2$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
HYX completed the main part of this article, HYX and ZXX corrected the main theorems. All authors read and approved the final manuscript.

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