Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

# $q$-Fractional calculus for Rubin's $q$-difference operator

Zeinab SI Mansour*

*Correspondence:
zsmansour@ksu.edu.sa;
zeinabs98@hotmail.com
Department of Mathematics,
Faculty of Science, King Saud
University, P.O. Box 2455, Riyadh,
11451, Kingdom of Saudi Arabia

**Abstract**
In this paper we introduce a fractional $q$-integral operator and derivative as a generalization of Rubin's $q$-difference operator. We also reformulate the definition of the $q^2$-Fourier transform and the $q$-analogue of the Fourier multiplier introduced by Rubin in (J. Math. Anal. Appl. 212(2):571-582, 1997; Proc. Am. Math. Soc. 135(3):777-785, 2007). As applications, we give summation formulas for ${}_2\phi_1$ finite series, we also use the $q^2$-Fourier transform and Hahn $q$-Laplace transform to solve a fractional $q$-diffusion equation.

**MSC:** Primary 39A12; secondary 33D15; 42A38; 35R11

**Keywords:** $q^2$-Fourier transform; Rubin's $q$-difference operator; Hahn's $q$-Laplace transform; fractional $q$-integral operator; fractional $q$-diffusion equation

## 1 Introduction and preliminaries

Let $q$ be a positive number, $0 < q < 1$. In the following, we follow the notations and notions of $q$-hypergeometric functions, the $q$-gamma function $\Gamma_q(x)$, Jackson $q$-exponential functions $E_q(x)$, and the $q$-shifted factorial as in [1, 2]. The $q$-difference operator is defined by

$$D_q f(z) := \frac{f(z) - f(qz)}{z - qz} \quad (z \neq 0). \tag{1.1}$$

Jackson [3] introduced an integral denoted by

$$\int_a^b f(x)\, d_q x$$

as a right inverse of the $q$-derivative. It is defined by

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q t \quad (a, b \in \mathbb{C}), \tag{1.2}$$

where

$$\int_0^x f(t)\, d_q t := (1-q) \sum_{n=0}^{\infty} x q^n f\bigl(x q^n\bigr) \quad (x \in \mathbb{C}), \tag{1.3}$$

provided that the series on the right-hand side of (1.3) converges at $x = a$ and $b$.

There is no unique canonical choice for the $q$-integration over $[0,\infty)$. In [4], Hahn defined the $q$-integration for a function $f$ over $[0,\infty)$ by

$$\int_0^\infty f(t)\,d_qt = (1-q)\sum_{n=-\infty}^{\infty} q^n f\bigl(q^n\bigr),$$

while in [5] Matsuo defined $q$-integrations on the interval $[0,\infty)$ and $(-\infty,\infty)$ by

$$\int_0^{\infty/b} f(t)\,d_qt := \frac{1-q}{b}\sum_{n=-\infty}^{\infty} q^n f\bigl(q^n/b\bigr) \quad (b>0), \tag{1.4}$$

$$\int_{-\infty/b}^{\infty/b} f(t)\,d_qt = \frac{1-q}{b}\sum_{-\infty}^{\infty} q^n\bigl(f\bigl(q^n/b\bigr)+f\bigl(-q^n/b\bigr)\bigr), \tag{1.5}$$

respectively, provided that the series converges absolutely. For any $q\in(0,1)$ and $0<b<\infty$, we define the spaces

$$L^p(\mathbb{R}_{b,q}) := \left\{f : \int_{-\infty/b}^{\infty/b} \bigl|f(x)\bigr|^p\,d_qx<\infty, p\geq 1\right\}, \quad \mathbb{R}_{b,q} := \bigl\{\pm q^n/b : n\in\mathbb{Z}\bigr\},$$
$$L^\infty(\mathbb{R}_{b,q}) := \bigl\{f : \|f\|_\infty := \sup\bigl\{f\bigl(\pm q^n/b\bigr) \mid n\in\mathbb{Z}\bigr\}<\infty\bigr\}.$$

We shall use the particular notation $\mathbb{R}_q$, $\widetilde{\mathbb{R}}_q$ and $\widetilde{\mathbb{R}}_{q,+}$ to denote $\mathbb{R}_{1,q}$, $\mathbb{R}_{\sqrt{1-q},q}$ and $\{\frac{q^k}{\sqrt{1-q}}, k\in\mathbb{Z}\}$, respectively. One can verify that $L^2(\mathbb{R}_{b,q})$ associated with the inner product

$$\langle f,g\rangle := \int_{-\infty/b}^{\infty/b} f(t)\overline{g(t)}\,d_qt, \quad f,g\in L^2(\mathbb{R}_{b,q}),$$

is a Hilbert space. The Riemann-Liouville fractional $q$-integral operator is introduced by Al-Salam in [6] and later by Agarwal in [7] and defined by

$$I_q^\alpha f(x) := \frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\int_0^x (qt/x;q)_{\alpha-1} f(t)\,d_qt, \quad \alpha\notin\{-1,-2,\ldots\}. \tag{1.6}$$

Using (1.3), (1.6) reduces to

$$I_q^\alpha f(x) = x^\alpha(1-q)^\alpha\sum_{n=0}^{\infty} q^n\frac{(q^\alpha;q)_n}{(q;q)_n} f\bigl(xq^n\bigr), \tag{1.7}$$

which is valid for all $\alpha$. The Riemann-Liouville fractional $q$-derivative of order $\alpha$, $\alpha>0$, is defined by

$$D_q^\alpha = D_q^k I_q^{k-\alpha} \quad \bigl(k=\lceil\alpha\rceil\bigr).$$

Rubin in [8, 9] introduced the $q$-difference operator

$$\partial_q f(z) = \frac{f(q^{-1}z)+f(-q^{-1}z)-f(qz)+f(-qz)-2f(-z)}{2(1-q)z} \quad (z\neq 0). \tag{1.8}$$

It is straightforward to prove that if a function $f$ is differentiable at a point $z$, then

$$\lim_{q\to 1^-} \partial_q f(z) = f'(z).$$

Also,

$$\delta_q f(z) = \begin{cases} D_q f(z) & \text{if } f \text{ is odd,} \\ \frac{1}{q} D_{q^{-1}} f(z) & \text{if } f \text{ is even.} \end{cases}$$

Let $f$ and $g$ be functions defined on a set $A$, where $A$ satisfies

$$z \in A \to \pm q^{\pm 1} z \in A,$$

and let $f_e$ and $f_o$ be the even and odd parts of $f$, respectively. The following properties of the $\partial_q$ operator are from [9, 10] and hold for all $z \in A \setminus \{0\}$.

(i) $\partial_q f(z) = \frac{1}{q} D_{q^{-1}} f_e(z) + D_q f_o(z)$.

(ii) For two functions $f$ and $g$,

- if $f$ is even and $g$ is odd, then

$$\partial_q(fg)(z) = qg(z)(\partial_q f)(qz) + f(qz)\,\partial_q g(z);$$

- if $f$ and $g$ are even, then

$$\partial_q(fg)(z) = \partial_q f(z)g(z) + f(z/q)\,\partial_q g(z);$$

- if $f$ and $g$ are odd, then

$$\partial_q(fg)(z) = \frac{1}{q}\big(f(z)(\partial_q g)(z/q) + (\partial_q f)(z/q)g(z/q)\big).$$

The $q$-translation $\varepsilon^y$ is introduced by Ismail in [2] and is defined on monomials by

$$\varepsilon^y x^n := x^n(-y/x;q)_n, \tag{1.9}$$

and it is extended to polynomials as a linear operator. Thus

$$\varepsilon^y\left(\sum_{n=0}^{m} f_n x^n\right) := \sum_{n=0}^{m} f_n x^n(-y/x;q)_n. \tag{1.10}$$

The $q$-translation operator is defined for $x^a$, $a > 0$, to be

$$\varepsilon^y x^a := x^a(-y/x;q)_a. \tag{1.11}$$

In [4], Hahn defined the following $q$-analogue of the Laplace transform:

$${}_qL_s f(x) = \phi(s) = \frac{1}{1-q}\int_0^{s^{-1}} E_q(-qsx)f(x)\,d_qx \quad \big(\mathrm{Re}(s) > 0\big). \tag{1.12}$$

Abdi [11] studied certain properties of these $q$-transforms. In [12], he used these analogues to solve linear $q$-difference equations with constant coefficients and certain allied equations. In [4, equation (9.5)], Hahn defined the convolution of two functions $F$, $G$ to be

$$(F * G)(x) = \frac{x}{1-q} \int_0^1 F(tx)G[x - tqx]\, d_q t, \tag{1.13}$$

where $G[x-y]$, for

$$G(x) := \sum_{n=0}^{\infty} a_n x^n,$$

is defined to be

$$G[x-y] := \sum_{n=0}^{\infty} a_n [x-y]_n, \quad \text{with } [x-y]_n := x^n (y/x;q)_n.$$

Using the definition of $q$-integration, $(F * G)$ is nothing but

$$(F * G)(x) = \frac{1}{1-q} \int_0^x F(t)\varepsilon^{-qt} G(x)\, d_q t, \tag{1.14}$$

where $\varepsilon$ is the translation operator (1.10). It is remarked by Hahn [4, p.373] that the convolution theorem

$${}_qL_s(F * G) = {}_qL_s F\, {}_qL_s G \tag{1.15}$$

holds. One can verify that if $\Phi(s) := {}_qL_s F(x)$ and $0 < \alpha < 1$, then

$${}_qL_s D_q^{\alpha} F(x) = \frac{s^{\alpha}}{(1-q)^{\alpha}} \Phi(s) - I_q^{1-\alpha} F\bigl(0^+\bigr) \frac{1}{(1-q)}; \tag{1.16}$$

see [13].

## 2 Orthogonality relations and completeness criteria

Koornwinder and Swarttouw introduced a $q$-analogue of the cosine and sine Fourier transform in [14] with the functions $\operatorname{Cos}(z;q^2)$ and $\operatorname{Sin}(z;q^2)$ defined by

$$\begin{aligned}
\operatorname{Cos}\bigl(z;q^2\bigr) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} (z(1-q))^{2k}}{(q;q)_{2k}} \\
&= {}_1\phi_1\bigl(0;q;q^2,q^2 z^2 (1-q)^2\bigr), \\
\operatorname{Sin}\bigl(z;q^2\bigr) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k+1)} (z(1-q))^{2k+1}}{(q;q)_{2k+1}} \\
&= z\,{}_1\phi_1\bigl(0;q^3;q^2,q^2 z^2 (1-q)^2\bigr).
\end{aligned} \tag{2.1}$$

A $q$-analogue of the exponential function is introduced in [8, 9] and defined by

$$e\bigl(z;q^2\bigr) := \operatorname{Cos}\bigl(iz;q^2\bigr) - i \operatorname{Sin}\bigl(iz;q^2\bigr).$$

Straightforward calculations give

$$\delta_q \operatorname{Cos}(\lambda x; q^2) = -\lambda \operatorname{Sin}(\lambda x; q^2), \qquad \delta_q \operatorname{Sin}(\lambda x; q^2) = \lambda \operatorname{Cos}(\lambda x; q^2)$$

and

$$\delta_q e(\lambda x; q^2) = \lambda e(\lambda x; q^2),$$

where $x \in \mathbb{C}$ and $\lambda$ is a fixed complex number. Fitouhi *et al.* in [15] proved that

$$\left|\frac{(z;q)_\infty}{(q;q)_\infty} {}_1\phi_1(0;z,q,q^{1+n})\right| \leq \frac{(-|z|,-q;q)_\infty}{(q;q)_\infty} \begin{cases} 1 & \text{if } n \geq 0, \\ |z|^{-n} q^{\frac{n(n+1)}{2}} & \text{if } n < 0. \end{cases}$$

Hence,

$$\left|\operatorname{Sin}\left(\frac{q^n}{1-q}; q^2\right)\right| \leq \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0, \\ q^{n^2} & \text{if } n < 0 \end{cases} \tag{2.2}$$

and

$$\left|\operatorname{Cos}\left(\frac{q^n}{1-q}; q^2\right)\right| \leq \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \begin{cases} 1 & \text{if } n \geq 0, \\ q^{n^2-2n} & \text{if } n < 0. \end{cases} \tag{2.3}$$

Consequently,

$$\left|e\left(\frac{q^n}{1-q}; q^2\right)\right| \leq 2\frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} \begin{cases} 1, & n \geq 0, \\ q^{n^2}, & n < 0. \end{cases} \tag{2.4}$$

The following orthogonality relation is proved in [14].

**Theorem 2.1** *Let* $|z| < 1$ *and* $n$, $m$ *be integers. Then*

$$\delta_{mn} = \sum_{k=-\infty}^{\infty} z^{k+n} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1(0;z^2;q,q^{n+k+1}) z^{k+m} \frac{(z^2;q)_\infty}{(q;q)_\infty} {}_1\phi_1(0;z^2;q,q^{m+k+1}), \tag{2.5}$$

*where the sum converges absolutely and uniformly on compact subsets of the open unit disc.*

The following identity, which follows from (2.5) when we replace $q$ by $q^2$ and $z$ by $q^\alpha$, $\alpha > 0$, is essential in our investigations.

$$\begin{aligned} &q^{-\alpha(n+m)} \frac{(q^2;q^2)_\infty^2}{(q^{2\alpha};q^2)_\infty^2} \delta_{mn} \\ &\quad = \sum_{k=-\infty}^{\infty} q^{2k\alpha} {}_1\phi_1(0;q^{2\alpha};q^2,q^{2n+2k+2}) {}_1\phi_1(0;q^{2\alpha};q^2,q^{2m+2k+2}). \end{aligned} \tag{2.6}$$

**Theorem 2.2** *For* $0 < q < 1$,

$$\int_0^{\infty/\sqrt{1-q}} \operatorname{Sin}\left(\frac{q^n z}{\sqrt{1-q}}; q^2\right) \operatorname{Sin}\left(\frac{q^m z}{\sqrt{1-q}}; q^2\right) d_q z = \frac{\sqrt{1-q}(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} q^{-n} \delta_{n,m}, \tag{2.7}$$

$$\int_0^{\infty/\sqrt{1-q}} \operatorname{Cos}\left(\frac{q^n z}{\sqrt{1-q}}; q^2\right) \operatorname{Cos}\left(\frac{q^m z}{\sqrt{1-q}}; q^2\right) d_q z = \frac{\sqrt{1-q}(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} q^{-n} \delta_{n,m} \tag{2.8}$$

*and*

$$\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e\left(i\frac{q^n z}{\sqrt{1-q}}; q^2\right) e\left(-i\frac{q^m z}{\sqrt{1-q}}; q^2\right) d_q z = 4\frac{\sqrt{1-q}(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} q^{-n} \delta_{n,m}. \tag{2.9}$$

*Proof* We start with proving (2.7). Since

$$\begin{aligned}
&\int_0^{\infty/\sqrt{1-q}} \operatorname{Sin}\left(\frac{q^n}{\sqrt{1-q}}z; q^2\right) \operatorname{Sin}\left(\frac{q^m}{\sqrt{1-q}}z; q^2\right) d_q z \\
&\quad = \sum_{k=-\infty}^{\infty} q^k \sqrt{1-q} \operatorname{Sin}\left(\frac{q^{n+k}}{1-q}; q^2\right) \operatorname{Sin}\left(\frac{q^{m+k}}{1-q}; q^2\right) \\
&\quad = \frac{q^{n+m}}{(1-q)^{3/2}} \sum_{-\infty}^{\infty} q^{3k} {}_1\phi_1\left(0; q^3, q^2; q^{2+2n+2k}\right) {}_1\phi_1\left(0; q^3, q^2; q^{2+2m+2k}\right).
\end{aligned} \tag{2.10}$$

In (2.6), set $\alpha = 3/2$ to obtain

$$\sum_{-\infty}^{\infty} q^{3k} {}_1\phi_1\left(0; q^3; q^2, q^{2+2n+2k}\right) {}_1\phi_1\left(0; q^3, q^2; q^{2+2m+2k}\right) = q^{-3(n+m)/2} \frac{(q^2;q^2)_\infty^2}{(q^3;q^2)_\infty^2} \delta_{n,m}. \tag{2.11}$$

Combining (2.10) and (2.11) yields (2.7). The proof of (2.8) follows similarly and the proof of (2.9) follows by combining (2.7) and (2.8). □

**Theorem 2.3** *For any* $q \in (0,1)$,

(a) *the set* $\{e(\pm\frac{q^n}{\sqrt{1-q}}x; q^2), n \in \mathbb{Z}\}$ *is a complete orthogonal set in* $L_q^2(\widetilde{\mathbb{R}}_q)$,

(b) *both of the sets* $\{\operatorname{Cos}(\frac{xq^n}{\sqrt{1-q}}; q^2), n \in \mathbb{Z}\}$ *and* $\{\operatorname{Sin}(\frac{xq^n}{\sqrt{1-q}}; q^2), n \in \mathbb{Z}\}$ *are complete orthogonal sets in* $L_q^2(\widetilde{\mathbb{R}}_{q,+})$.

*Proof* We only proove (a). The proof of (b) is similar and is omitted. From Theorem 2.2, it remains only to prove that the set $\{e(\pm\frac{q^n}{\sqrt{1-q}}x; q^2), n \in \mathbb{Z}\}$ is complete in $L_q^2(\widetilde{\mathbb{R}}_q)$. This is equivalent to proving that if there exists a function $f \in L^2(\widetilde{\mathbb{R}}_q)$ such that

$$\left\langle f, e\left(\pm\frac{q^n}{\sqrt{1-q}}x; q^2\right)\right\rangle = 0 \quad (n \in \mathbb{Z}), \tag{2.12}$$

then

$$f\left(\pm\frac{q^n}{1-q}\right) = 0 \quad (n \in \mathbb{Z}).$$

From (2.12) we deduce

$$\int_0^{\infty/\sqrt{1-q}} f_e(t)\operatorname{Cos}\left(\frac{q^n}{\sqrt{1-q}}t;q^2\right)d_qt = 0 \quad (n\in\mathbb{Z}),$$
$$\int_0^{\infty/\sqrt{1-q}} f_o(t)\operatorname{Sin}\left(\frac{q^n}{\sqrt{1-q}}t;q^2\right)d_qt = 0 \quad (n\in\mathbb{Z}),$$

where $f_e$ and $f_o$ are the even and odd parts of the function $f$. Then from (3.8)-(3.9) we obtain $f_e(t) = f_o(t) = f(t) = 0$ for all $t \in \widetilde{\mathbb{R}}_q$. Hence $\{e(\pm\frac{q^n}{\sqrt{1-q}}x;q^2), n\in\mathbb{Z}\}$ is a complete orthogonal set in $L^2(\widetilde{\mathbb{R}}_q)$. □

**Theorem 2.4**

(1) *If* $f \in L^2(\widetilde{\mathbb{R}}_q)$, *then*

$$f(x) = \sum_{-\infty}^{\infty} c_n e\left(ix\frac{q^n}{\sqrt{1-q}};q^2\right) + \sum_{-\infty}^{\infty} d_n e\left(-ix\frac{q^n}{\sqrt{1-q}};q^2\right) \quad (x\in\widetilde{\mathbb{R}}_q), \tag{2.13}$$

*where*

$$c_n = \frac{q^n}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)e\left(-it\frac{q^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z}),$$
$$d_n = \frac{q^n}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)e\left(it\frac{q^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z}),$$

*and* $C := \frac{4\sqrt{1-q}(q^2;q^2)^2_\infty}{(q;q^2)^2_\infty}$.

(2) *If* $f \in L^2(\widetilde{\mathbb{R}}_q)$, *then*

$$f(x) = \sum_{n=-\infty}^{\infty} a_n\operatorname{Cos}\left(\frac{xq^n}{\sqrt{1-q}};q^2\right) + \sum_{n=-\infty}^{\infty} b_n\operatorname{Sin}\left(\frac{xq^n}{\sqrt{1-q}};q^2\right) \quad (x\in\widetilde{\mathbb{R}}_q), \tag{2.14}$$

*where*

$$a_n = \frac{4}{C}\int_0^{\infty/\sqrt{1-q}} f_e(t)\operatorname{Cos}\left(\frac{tq^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z})$$

*and*

$$b_n = \frac{4}{C}\int_0^{\infty/\sqrt{1-q}} f_o(t)\operatorname{Sin}\left(\frac{tq^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z}).$$

*Proof* The proof of (1) follows directly from Theorem 2.3 and the orthogonality relations (2.9). In the following we give in detail the proof of (2). Let $f = f_e + f_o$ be any function in $L^2(\widetilde{\mathbb{R}}_q)$. Clearly both $f_e$ and $f_o$ belong to $L^2(\widetilde{\mathbb{R}}_q)$. The restriction of $f_e$ to $\widetilde{\mathbb{R}}_{q,+}$ can be represented in the complete orthogonal set $\{\operatorname{Cos}(\frac{xq^n}{\sqrt{1-q}}), n\in\mathbb{Z}\}$ as

$$f_e(x) = \sum_{n=-\infty}^{\infty} a_n\operatorname{Cos}\left(\frac{xq^n}{\sqrt{1-q}};q^2\right) \quad (x\in\widetilde{\mathbb{R}}_{q,+}), \tag{2.15}$$

where

$$a_n = \frac{4}{C}\int_0^{\infty/\sqrt{1-q}} f_e(t)\,\mathrm{Cos}\left(\frac{tq^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z}).$$

The orthogonal set $\{\mathrm{Sin}(\frac{xq^n}{\sqrt{1-q}}), n\in\mathbb{Z}\}$ also spans $L^2(\widetilde{\mathbb{R}}_{q,+})$, hence

$$f_o(x) = \sum_{n=-\infty}^{\infty} b_n\,\mathrm{Sin}\left(\frac{xq^n}{\sqrt{1-q}};q^2\right) \quad (x\in\widetilde{\mathbb{R}}_{q,+}), \tag{2.16}$$

where

$$b_n = \frac{4}{C}\int_0^{\infty/\sqrt{1-q}} f_o(t)\,\mathrm{Sin}\left(\frac{tq^n}{\sqrt{1-q}};q^2\right)d_qt \quad (n\in\mathbb{Z}).$$

Because both sides of (2.15) are even functions on $\widetilde{\mathbb{R}}_q$, the equality extends on $\widetilde{\mathbb{R}}_q$; and similarly the two sides of (2.16). Hence we have the representation (2.14) of any $f\in L^2(\widetilde{\mathbb{R}}_q)$. □

## 3 Rubin's $q^2$-Fourier transform

Koornwinder and Swarttouw [14] introduced the pair of $q$-transforms

$$\begin{aligned} g(q^n) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{k=-\infty}^{\infty} q^k \begin{cases}\mathrm{Cos}(\frac{q^{k+n}}{1-q};q^2)\\ \text{or}\\ \mathrm{Sin}(\frac{q^{k+n}}{1-q};q^2)\end{cases} f(q^k),\\ f(q^k) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty} q^n \begin{cases}\mathrm{Cos}(\frac{q^{k+n}}{1-q};q^2)\\ \text{or}\\ \mathrm{Sin}(\frac{q^{k+n}}{1-q};q^2)\end{cases} g(q^n),\end{aligned} \tag{3.1}$$

where $0<q<1$ and $f, g$ are in the space $L^2(\mathbb{R}_q)$. Now assume that $\frac{\log(1-q)}{\log q}\in 2\mathbb{Z}$ or, equivalently,

$$q\in\bigl\{q\in(0,1): 1-q=q^{2m} \text{ for some integer } m\bigr\}. \tag{3.2}$$

Then, by replacing $q^k$ and $q^n$ in (3.1) by $q^k\sqrt{1-q}$ and $q^n\sqrt{1-q}$, and then $f(q^k\sqrt{1-q})$ and $g(q^n\sqrt{1-q})$ by $f(q^k)$ and $g(q^k)$, Koornwinder and Swarttouw obtained the following $q$-analogue of the cosine and sine Fourier transforms:

$$\begin{aligned} g(\lambda) &= \frac{\sqrt{1+q}}{\Gamma_q^2(1/2)}\int_0^\infty f(t)\begin{cases}\mathrm{Cos}(t\lambda;q^2)\\ \text{or}\\ \mathrm{Sin}(t\lambda;q^2)\end{cases} d_qt,\\ f(x) &= \frac{\sqrt{1+q}}{\Gamma_q^2(1/2)}\int_0^\infty f(t)\begin{cases}\mathrm{Cos}(x\lambda;q^2)\\ \text{or}\\ \mathrm{Sin}(x\lambda;q^2)\end{cases} d_q\lambda.\end{aligned} \tag{3.3}$$

Therefore, if we let $q \to 1^-$ for such $q$'s that satisfy (3.2), we obtain the cosine and sine Fourier transforms

$$g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\cos(\lambda t)\,dt, \qquad f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda)\cos(t\lambda)\,d\lambda, \tag{3.4}$$

$$g(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t)\sin(\lambda t)\,dt, \qquad f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(\lambda)\sin(t\lambda)\,d\lambda. \tag{3.5}$$

The pair of functions $\mathrm{Cos}(\lambda x; q^2)$ and $\mathrm{Sin}(\lambda x; q^2)$ satisfy

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) = \begin{cases} \lambda^2 y(x) & \text{if } y(x) = \mathrm{Sin}(\lambda x; q^2), \\ q\lambda^2 y(x) & \text{if } y(x) = \mathrm{Cos}(\lambda x; q^2). \end{cases}$$

Therefore, the eigenfunctions $\{\mathrm{Cos}(\lambda x; q^2), \mathrm{Sin}(\lambda x; q^2)\}$ have two different eigenvalues. Consequently, as remarked by Koornwinder and Swarttouw in [14], no $q$-exponential functions built from $\{\mathrm{Cos}(x; q^2), \mathrm{Sin}(x; q^2)\}$ will satisfy an eigenfunction problem. This motivated Rubin [8] to define the $q$-difference operator (1.8) since for this operator, the functions $\{\mathrm{Cos}(\lambda x; q^2), \mathrm{Sin}(\lambda x; q^2)\}$ are solutions of the eigenvalue problem

$$-\delta_q^2 y(x) = \lambda^2 y(x).$$

Rubin [8] introduced a $q^2$-analogue of the Fourier transform in the form

$$\hat{f}(x; q^2) := \mathcal{F}_q(f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty}^\infty f(t) e(-itx; q^2)\,d_q t, \tag{3.6}$$

where $f \in L^1(\mathbb{R}_q)$ and $q$ satisfies condition (3.2).

**Remark 3.1** Rubin [9] proved that

(1) the $q^2$-Fourier transform defines a bounded linear operator from $L^1(\mathbb{R}_q)$ to $L^\infty(\mathbb{R}_q)$,
(2) the $q^2$-Fourier transform is defined and bounded on $L^1(\mathbb{R}_q) \cap L^2(\mathbb{R}_q)$,
(3) $L^1(\mathbb{R}_q) \cap L^2(\mathbb{R}_q)$ is dense in $L^2(\mathbb{R}_q)$ (consider the functions with finite support).

Consequently, the $q^2$-Fourier transform defines a bounded extension to $L^2(\mathbb{R}_q)$.

Koornwinder and Swarttouw introduced the $q$-Hankel transforms (3.1) which can be written in the form (3.3) only if $q$ satisfies condition (3.2). In fact, we can write the $q$-transforms in (3.1) as $q$-integral on $(-\infty, \infty)$ by using Matsuo definition (1.4) as in the following. Rewrite the transform pair in (3.1) as

$$\begin{aligned} g\left(\frac{q^n}{\sqrt{1-q}}\right) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{k=-\infty}^\infty q^k \begin{cases} \mathrm{Cos}(\frac{q^{k+n}}{1-q}; q^2) \\ \text{or} \\ \mathrm{Sin}(\frac{q^{k+n}}{1-q}; q^2) \end{cases} f\left(\frac{q^k}{\sqrt{1-q}}\right), \\ f\left(\frac{q^k}{\sqrt{1-q}}\right) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^\infty q^n \begin{cases} \mathrm{Cos}(\frac{q^{k+n}}{1-q}; q^2) \\ \text{or} \\ \mathrm{Sin}(\frac{q^{k+n}}{1-q}; q^2) \end{cases} g\left(\frac{q^n}{\sqrt{1-q}}\right), \end{aligned} \tag{3.7}$$

where we assume that the functions $f$ and $g$ are in the space $L^1(\widetilde{\mathbb{R}}_q)\cap L^2(\widetilde{\mathbb{R}}_q)$. Using Matsuo definition of the $q$-integration on $(0,\infty)$, (1.4) with $b=\sqrt{1-q}$, the transformations in (3.7) can be written as

$$\begin{aligned} g(x) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)}\int_0^{\infty/\sqrt{1-q}} f(t)\operatorname{Cos}\bigl(xt;q^2\bigr)\,d_qt, \\ f(t) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)}\int_0^{\infty/\sqrt{1-q}} g(x)\operatorname{Cos}\bigl(xt;q^2\bigr)\,d_qx \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} g(x) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)}\int_0^{\infty/\sqrt{1-q}} f(t)\operatorname{Sin}\bigl(xt;q^2\bigr)\,d_qt, \\ f(t) &= \frac{\sqrt{1+q}}{\Gamma_{q^2}(1/2)}\int_0^{\infty/\sqrt{1-q}} g(x)\operatorname{Sin}\bigl(xt;q^2\bigr)\,d_qx, \end{aligned} \tag{3.9}$$

where $x,t\in\widetilde{\mathbb{R}}_q$ and $f$, $g$ are in $L^1(\widetilde{\mathbb{R}}_q)\cap L^2(\widetilde{\mathbb{R}}_q)$. This is similar to Rubin's work in [9]. Consequently, we set the following reformulation of Rubin's definition of the $q^2$-Fourier transform (3.6).

**Definition 3.2** Let $0<q<1$. We define the $q^2$-Fourier transform for any function $f\in L^1(\widetilde{\mathbb{R}}_q)$ to be

$$\hat{f}\bigl(x;q^2\bigr) := \mathcal{F}_q(f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)e\bigl(-itx;q^2\bigr)\,d_qt. \tag{3.10}$$

It is clear that Rubin's definition of the $q^2$-Fourier transform is a special case of (3.2) because if $1-q=q^{2m}$ for some $m\in\mathbb{Z}$, then

$$\int_0^{\infty/\sqrt{1-q}} f(t)\,d_qt = \int_0^{\infty/q^m} f(t)\,d_qt = \int_0^{\infty} f(t)\,d_qt.$$

However, we get the classical Fourier transform only when $q\to 1^-$ and $q$ satisfies (3.2). Similar to Rubin's results mentioned in Remark 3.1, we can prove that the $q^2$-Fourier transform defines a bounded linear operator from $L^1(\widetilde{\mathbb{R}}_q)$ to $L^\infty(\widetilde{\mathbb{R}}_q)$, and $L^1(\widetilde{\mathbb{R}}_q)\cap L^2(\widetilde{\mathbb{R}}_q)$ is dense in $L^2(\widetilde{\mathbb{R}}_q)$. Therefore, the $q^2$-Fourier transform in (3.2) defines a bounded extension to $L^2(\widetilde{\mathbb{R}}_q)$.

The proofs of the following results, which are valid for any $q\in(0,1)$, are similar to the proofs in [8]. Therefore, we state them without proofs.

(1) If $f\in L^2(\widetilde{\mathbb{R}}_q)$, then

$$f(t) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \mathcal{F}(f)(x)e\bigl(itx(1-q);q^2\bigr)\,d_qx, \quad t\in\widetilde{\mathbb{R}}_q. \tag{3.11}$$

(2) If $f(u)$ and $uf(u)\in L^1_q(\widetilde{\mathbb{R}}_q)$, then

$$\partial_q(\mathcal{F}_qf)(x) = \mathcal{F}_q\bigl(-iuf(u)\bigr)(x).$$

(3) If $f$ and $\partial_q f \in L^1_q(\widetilde{\mathbb{R}}_q)$, then

$$\mathcal{F}_q(\partial_q f)(x) = ix\mathcal{F}_q(f)(x). \tag{3.12}$$

We reformulate the definitions of $q^2$-Fourier multiplier and the $q^2$-Fourier convolution formula introduced by Rubin in [9] with the restriction (3.2) to any $q \in (0,1)$.

**Definition 3.3** Let $q \in (0,1)$. We define the $q^2$-Fourier multiplier operator corresponding to translation by $y$ to be

$$(T_y f)(x) = \frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e\big(-ity;q^2\big)(\mathcal{F}_q f)(t) e\big(itx;q^2\big)\, d_q t, \tag{3.13}$$

whenever the $q$-integral makes sense. If $f \in L^2(\widetilde{\mathbb{R}}_q)$ and $g \in L^1(\widetilde{\mathbb{R}}_q)$, we define the multiplier corresponding to Fourier convolution of $f$ with $g$ to be

$$(f * g)(z) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} [T_\lambda f](z) g(\lambda)\, d_q\lambda. \tag{3.14}$$

**Theorem 3.4** *Let $f$ and $g$ be two functions in $(L^1 \cap L^2)(\widetilde{\mathbb{R}}_q)$. Then*

$$\mathcal{F}_q(f * g)(x) = \mathcal{F}_q(f)(x)\mathcal{F}_q(g)(x) \quad (x \in \widetilde{\mathbb{R}}_q). \tag{3.15}$$

*Proof* The proof of (3.15) is completely similar to the proof of [9, Theorem 8] and is omitted. □

## 4 Fractional $q$-operator as a generalization of a $q$-difference operator

Let $f$ be an integrable function of period $2\pi$. Weyl, see Zygmund's book [16], introduced a fractional operator which is more convenient for trigonometric series than the Riemann-Liouville fractional operator. This operator is defined by

$$(I_\alpha f)(x) \sim \sum_{n=-\infty}^{\infty} c_n \frac{e^{inx}}{(in)^\alpha} \quad \text{if } f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}, c_0 = 0, \tag{4.1}$$

where $i^\alpha = e^{i\alpha\pi/2}$. Zygmund [16, p.133] pointed out that

$$I_\alpha f(x) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\Psi_\alpha(x-t)\, dt, \qquad \Psi_\alpha(x) = \sum_{n\neq 0} \frac{e^{inx}}{(in)^\alpha}.$$

He also proved the semigroup identity

$$I_\alpha I_\beta = I_{\alpha+\beta}, \quad \alpha, \beta > 0.$$

In [17], Ismail and Rahman defined a $q$-analogue of the fractional operator $I_\alpha$, so that $\alpha = 1$ represents a right inverse of the Askey-Wilson operator $\mathcal{D}_q$ which is defined by

$$(\mathcal{D}_q f)(x) := \frac{\breve{f}(q^{1/2}e^{i\theta}) - \breve{f}(q^{-1/2}e^{i\theta})}{(q^{1/2} - q^{-1/2})\sin\theta}, \quad x = \cos\theta,$$

where $f(x) = \breve{f}(z)$ with $x = (z + 1/z)/2$.

In this section, we introduce a $q$-analogue of the fractional operator (4.1) as a generalization of the $q$-difference operator defined by Rubin in [8]. From Theorem 2.3, consequently,

$$f(x) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_0(x,t)\,d_qt,$$

where

$$\begin{aligned}\Psi_0(x,t) &= \sum_{-\infty}^{\infty} q^n e\left(ix\frac{q^n}{\sqrt{1-q}}\right)e\left(-it\frac{q^n}{\sqrt{1-q}}\right) + q^n e\left(-ix\frac{q^n}{\sqrt{1-q}}\right)e\left(it\frac{q^n}{\sqrt{1-q}}\right)\\ &= \sum_{-\infty}^{\infty} q^n \operatorname{Cos}\left(x\frac{q^n}{\sqrt{1-q}};q^2\right)\operatorname{Cos}\left(t\frac{q^n}{\sqrt{1-q}}\right)\\ &\quad + q^n \operatorname{Sin}\left(x\frac{q^n}{\sqrt{1-q}};q^2\right)\operatorname{Sin}\left(t\frac{q^n}{\sqrt{1-q}};q^2\right).\end{aligned}$$

**Lemma 4.1** *The series*

$$\sum_{n=-\infty}^{\infty} q^{n(1-\alpha)} e\left(ix\frac{q^n}{\sqrt{1-q}};q^2\right)e\left(-it\frac{q^n}{\sqrt{1-q}};q^2\right) \quad (x,t\in\widetilde{\mathbb{R}}_q) \tag{4.2}$$

*is absolutely convergent only when* $\operatorname{Re}\alpha < 1$.

*Proof* The series in (4.2) can be written as

$$\left(\sum_{n=-\infty}^{-1} + \sum_{n=0}^{\infty}\right) q^{n(1-\alpha)} e\left(ix\frac{q^n}{\sqrt{1-q}};q^2\right)e\left(-it\frac{q^n}{\sqrt{1-q}};q^2\right).$$

From (2.4), the series $\sum_{n=0}^{\infty} q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)$ is absolutely convergent for $\operatorname{Re}\alpha < 1$ and diverges for $\operatorname{Re}\alpha \geq 1$, while the series $\sum_{n=-\infty}^{-1} q^{n(1-\alpha)} e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)$ is absolutely convergent for all $\alpha \in \mathbb{C}$. □

Set

$$\begin{aligned}\Psi_\alpha(x,t) &:= (1-q)^{\alpha/2} \sum_{n=-\infty}^{\infty} q^n \frac{e(ix\frac{q^n}{\sqrt{1-q}};q^2)e(-it\frac{q^n}{\sqrt{1-q}};q^2)}{(iq^n)^\alpha}\\ &\quad + (1-q)^{\alpha/2} \sum_{n=-\infty}^{\infty} q^n \frac{e(-ix\frac{q^n}{\sqrt{1-q}};q^2)e(it\frac{q^n}{\sqrt{1-q}};q^2)}{(-iq^n)^\alpha},\end{aligned}$$

where $x,t \in \widetilde{\mathbb{R}}_q$ and $i^\alpha$ is defined with respect to the principal branch, *i.e.*, $i^\alpha = e^{i\frac{\pi}{2}\alpha}$.

**Lemma 4.2** *For* $k \in \mathbb{N}$ *and* $\operatorname{Re}\alpha < 1$,

$$\delta_{q,x}^k \Psi_\alpha(x,t) = \Psi_{\alpha-k}(x,t).$$

*Proof* The proof follows directly by using that

$$\delta_{q,x}^k e\left(ix\frac{q^n}{\sqrt{1-q}};q^2\right) = \left(\frac{iq^n}{\sqrt{1-q}}\right)^k e\left(ix\frac{q^n}{\sqrt{1-q}};q^2\right). \qquad \square$$

A direct calculation yields the following identity, which holds for $\operatorname{Re}\alpha < 1$,

$$(1-q)^{-\alpha/2}\Psi_\alpha(x,t) = 2\cos\left(\frac{\pi}{2}\alpha\right)A_\alpha(x,t) + 2\sin\left(\frac{\pi}{2}\alpha\right)B_\alpha(x,t), \tag{4.3}$$

where

$$\begin{aligned} A_\alpha(x,t) := \sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right)\operatorname{Cos}\left(\frac{q^{m+k}}{1-q};q^2\right) \\ + q^{k(1-\alpha)} \operatorname{Sin}\left(\frac{q^{n+k}}{1-q};q^2\right)\operatorname{Sin}\left(\frac{q^{m+k}}{1-q};q^2\right) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} B_\alpha(x,t) := \sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Cos}\left(\frac{q^{m+k}}{1-q};q^2\right)\operatorname{Sin}\left(\frac{q^{n+k}}{1-q};q^2\right) \\ - q^{k(1-\alpha)} \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right)\operatorname{Sin}\left(\frac{q^{m+k}}{1-q};q^2\right). \end{aligned} \tag{4.5}$$

**Theorem 4.3** *For* $\operatorname{Re}(\alpha) < 1$,

$$(1-q)^{-\alpha/2}\Psi_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} 2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r} \\ \quad - 2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r} \\ \quad \text{if } n > m + [\frac{1-\operatorname{Re}\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r} \\ \quad - 2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r} \\ \quad \text{if } m > n + [\frac{1-\operatorname{Re}\alpha}{2}], \end{cases} \tag{4.6}$$

*where* $\tau_r = \begin{cases} q^{-\frac{r}{2}}q^{-\frac{1-\alpha}{2}}, & r \text{ is odd}, \\ q^{\frac{r}{2}}, & r \text{ is even}. \end{cases}$

*Moreover*,

$$(1-q)^{-\alpha/2}\Psi_\alpha\left(\frac{q^n}{\sqrt{1-q}},-\frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} 2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty (-1)^r q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r} \\ \quad + 2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}, \\ \quad n > m + [\frac{1-\operatorname{Re}\alpha}{2}], \\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty (-1)^r q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r} \\ \quad + 2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}, \\ \quad m > n + [\frac{1-\operatorname{Re}\alpha}{2}], \end{cases} \tag{4.7}$$

$$(1-q)^{-\alpha/2}\Psi_\alpha\left(-\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)$$

$$=\begin{cases}2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty(-1)^r q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}\\ \quad -2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}\\ \quad \textit{if } n>m+[\frac{1-\operatorname{Re}\alpha}{2}],\\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty(-1)^r q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}\\ \quad -2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}\\ \quad \textit{if } m>n+[\frac{1-\operatorname{Re}\alpha}{2}],\end{cases} \tag{4.8}$$

$$(1-q)^{-\alpha/2}\Psi_\alpha\left(-\frac{q^n}{\sqrt{1-q}},-\frac{q^m}{\sqrt{1-q}}\right)$$

$$=\begin{cases}2\cos(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}\\ \quad +2\sin(\frac{\pi}{2}\alpha)q^{-m(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(n-m)r}\\ \quad \textit{if } n>m+[\frac{1-\operatorname{Re}\alpha}{2}],\\ 2\cos(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]}\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}\\ \quad +2\sin(\frac{\pi}{2}\alpha)q^{-n(1-\alpha)}\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{r=0}^\infty q^{\frac{\alpha r}{2}}\tau_r\frac{(q^{1-\alpha};q)_r}{(q;q)_r}q^{(m-n)r}\\ \quad \textit{if } m>n+[\frac{1-\operatorname{Re}\alpha}{2}].\end{cases} \tag{4.9}$$

*Proof* Using the following formula from [14, p.455]

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}s^k y^{n+k}\frac{(y^2;q^2)_\infty}{(q^2;q^2)_\infty}{}_1\phi_1\bigl(0;y^2;q^2,q^{2n+2k+2}\bigr)x^{m+k}\frac{(x^2;q^2)_\infty}{(q^2;q^2)_\infty}{}_1\phi_1\bigl(0;x^2;q^2,q^{2m+2k+2}\bigr)\\ &\quad=s^{-m}y^{n-m}\frac{(s^{-1}xy^{-1},y^2;q^2)_\infty}{(sxy,q^2;q^2)_\infty}{}_2\phi_1\bigl(q^2sx^{-1}y,sxy;y^2;q^2,q^{2n-2m}s^{-1}xy^{-1}\bigr)\\ &\quad=s^{-n}x^{m-n}\frac{(s^{-1}yx^{-1},x^2;q^2)_\infty}{(sxy,q^2;q^2)_\infty}{}_2\phi_1\bigl(q^2sxy^{-1},sxy;x^2;q^2,q^{2m-2n}s^{-1}yx^{-1}\bigr),\end{aligned}$$

where $|sxy|<1$, we can prove that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}q^{k(1-\alpha)}\operatorname{Cos}\left(\frac{q^{m+k}}{1-q};q^2\right)\operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right)\\ &\quad=\begin{cases}q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}{}_2\phi_1(q^{2-\alpha},q^{1-\alpha};q;q^2,q^{2n-2m+\alpha}), & n\ge m+[\operatorname{Re}\alpha/2],\\ q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}{}_2\phi_1(q^{2-\alpha},q^{1-\alpha};q;q^2,q^{2m-2n+\alpha}), & m\ge n+[\operatorname{Re}\alpha/2],\end{cases}\end{aligned}$$

where $\operatorname{Re}(1-\alpha)>0$ and

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}q^{k(1-\alpha)}\operatorname{Sin}\left(\frac{q^{m+k}}{1-q};q^2\right)\operatorname{Sin}\left(\frac{q^{n+k}}{1-q};q^2\right)\\ &\quad=\begin{cases}q^{n-m}\frac{q^{-m(1-\alpha)}}{1-q}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{3-\alpha},q;q^2)_\infty}{}_2\phi_1(q^{2-\alpha},q^{3-\alpha};q^3;q^2,q^{2n-2m+\alpha}), & n>m+[\operatorname{Re}\alpha/2],\\ q^{m-n}\frac{q^{-n(1-\alpha)}}{1-q}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{3-\alpha},q;q^2)_\infty}{}_2\phi_1(q^{2-\alpha},q^{3-\alpha};q^3;q^2,q^{2m-2n+\alpha}), & m>n+[\operatorname{Re}\alpha/2].\end{cases}\end{aligned}$$

Also,

$$\sum_{k=-\infty}^{\infty} q^{k(1-\alpha)} \operatorname{Sin}\left(\frac{q^{m+k}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right)$$
$$= \begin{cases} q^{-m(1-\alpha)} \frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} {}_2\phi_1(q^{1-\alpha},q^{2-\alpha};q;q^2,q^{2n-2m+\alpha+1}), \\ q^{m-n} \frac{q^{-n(1-\alpha)}}{1-q} \frac{(q^{\alpha-1},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} {}_2\phi_1(q^{2-\alpha},q^{3-\alpha};q^3;q^2,q^{2m-2n+\alpha-1}). \end{cases}$$

Hence, if $x := \frac{q^n}{\sqrt{1-q}}$ and $t := \frac{q^m}{\sqrt{1-q}}$, then

$$A_\alpha(x,t) = \begin{cases} t^{-(1-\alpha)}(1-q)^{-(1-\alpha)/2} \frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]} \frac{(q^{1-\alpha};q)_r}{(q;q)_r} (\frac{x}{t})^r, \\ \quad n > m + [-\operatorname{Re}\alpha/2], \\ x^{-(1-\alpha)}(1-q)^{-(1-\alpha)/2} \frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty q^{\alpha[\frac{r}{2}]} \frac{(q^{1-\alpha};q)_r}{(q;q)_r} (\frac{t}{x})^r, \\ \quad m > n + [-\operatorname{Re}\alpha/2]. \end{cases} \tag{4.10}$$

Also,

$$A_\alpha(x,-t) = \begin{cases} t^{-(1-\alpha)}(1-q)^{-(1-\alpha)/2} \frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\alpha[\frac{r}{2}]} \frac{(q^{1-\alpha};q)_r}{(q;q)_r} (\frac{x}{t})^r, \\ \quad n > m + [-\operatorname{Re}\alpha/2], \\ x^{-(1-\alpha)}(1-q)^{-(1-\alpha)/2} \frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\alpha[\frac{r}{2}]} \frac{(q^{1-\alpha};q)_r}{(q;q)_r} (\frac{t}{x})^r, \\ \quad m > n + [-\operatorname{Re}\alpha/2], \end{cases} \tag{4.11}$$

$$B_\alpha(x,t) = -\begin{cases} q^{-m(1-\alpha)} \frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}} \tau_r \frac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r}, \\ \quad n \ge m + [\frac{1-\operatorname{Re}\alpha}{2}], \\ q^{-n(1-\alpha)} \frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}} \tau_r \frac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r}, \\ \quad m > n + [\frac{1-\operatorname{Re}\alpha}{2}], \end{cases} \tag{4.12}$$

$$B_\alpha(x,-t) = \begin{cases} q^{-m(1-\alpha)} \frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}} \tau_r \frac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(n-m)r}, \\ \quad n \ge m + [\frac{1-\operatorname{Re}\alpha}{2}], \\ q^{-n(1-\alpha)} \frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty} \sum_{r=0}^\infty (-1)^r q^{\frac{\alpha r}{2}} \tau_r \frac{(q^{1-\alpha};q)_r}{(q;q)_r} q^{(m-n)r}, \\ \quad m > n + [\frac{1-\operatorname{Re}\alpha}{2}]. \end{cases} \tag{4.13}$$

Substituting from (4.10)-(4.13) into (4.3) yields the values $\Psi_\alpha(\pm x, \pm t)$ and the theorem follows. □

**Remark 4.4** In the previous theorem, we calculated the value of $\Psi_\alpha(x,t)$, $x, t \in \widetilde{\mathbb{R}}_q$ and for specific values of $x$, $t$. We can calculate the values of $\Psi_\alpha(x,t)$ for all $x$, $t$ by using the identity

$$\begin{aligned}&\sum_{-\infty}^{\infty} s^{k+m} y^{k+n} \frac{(y^2;q^2)_\infty}{(q^2;q^2)_\infty} {}_1\phi_1\big(0;y^2;q^2,q^{2k+2n+2}\big) x^{k+m} \frac{(x^2;q^2)_\infty}{(q^2;q^2)_\infty} {}_1\phi_1\big(0;x^2;q^2,q^{2k+2m+2}\big)\\ &\quad = y^{n-m} \frac{(s^{-1}xy^{-1},q^{2n-2m+2};q^2)_\infty}{(q^{2n-2m}s^{-1}xy^{-1},q^2;q^2)_\infty} {}_2\phi_1\big(q^{2n-2m}s^{-1}xy^{-1},s^{-1}yx^{-1};q^{2n-2m+2};q^2,sxy\big)\end{aligned}$$

for $|sxy| < 1$. See Proposition 4.1 of [14].

In this case we have

$$\begin{aligned}&\mathrm{CC}_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)\\ &\quad := \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)} \operatorname{Cos}\left(\frac{q^{n+r}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^{m+r}}{1-q};q^2\right)\\ &\quad = q^{-m(1-\alpha)} \frac{(q^\alpha,q^{2n-2m+2},q^2;q^2)_\infty}{(q^{2n-2m+\alpha},q,q;q^2)_\infty} {}_2\phi_1\big(q^{2n-2m+\alpha},q^\alpha;q^{2n-2m+2};q^2,q^{1-\alpha}\big), \qquad (4.14)\end{aligned}$$

$$\begin{aligned}&\mathrm{SS}_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)\\ &\quad := \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)} \operatorname{Sin}\left(\frac{q^{n+r}}{1-q};q^2\right) \operatorname{Sin}\left(\frac{q^{m+r}}{1-q};q^2\right)\\ &\quad = q^{n-m} q^{-m(1-\alpha)} \frac{(q^\alpha,q^{2n-2m+2},q^2;q^2)_\infty}{(q^{2n-2m+\alpha},q,q;q^2)_\infty} {}_2\phi_1\big(q^{2n-2m+\alpha},q^\alpha;q^{2n-2m+2};q^2,q^{3-\alpha}\big), \qquad (4.15)\end{aligned}$$

$$\begin{aligned}&\mathrm{SC}_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)\\ &\quad := \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)} \operatorname{Sin}\left(\frac{q^{n+r}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^{m+r}}{1-q};q^2\right)\\ &\quad = q^{n-m} q^{-m(1-\alpha)} \frac{(q^{\alpha-1},q^{2n-2m+2},q^2;q^2)_\infty}{(q^{2n-2m+\alpha-1},q,q;q^2)_\infty} {}_2\phi_1\big(q^{2n-2m+\alpha-1},q^{\alpha+1};q^{2n-2m+2};q^2,q^{2-\alpha}\big),\end{aligned}$$

$$\begin{aligned}&\mathrm{CS}_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)\\ &\quad := \sum_{r=-\infty}^{\infty} q^{r(1-\alpha)} \operatorname{Sin}\left(\frac{q^{m+r}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^{n+r}}{1-q};q^2\right)\\ &\quad = q^{-m(1-\alpha)} \frac{(q^{\alpha+1},q^{2n-2m+2},q^2;q^2)_\infty}{(q^{2n-2m+\alpha+1},q,q;q^2)_\infty} {}_2\phi_1\big(q^{2n-2m+\alpha+1},q^{\alpha-1};q^{2n-2m+2};q^2,q^{2-\alpha}\big).\end{aligned}$$

**Corollary 4.5** *For each fixed* $x,t \in \widetilde{\mathbb{R}}_q$, *the function* $\Psi_\alpha(x,t)$ *as a function of* $\alpha$ *can be extended to an entire function on* $\mathbb{C}$.

*Proof* If $\alpha$ is a positive integer, then the series on the right-hand sides of (4.6)-(4.9) are finite sums and hence are convergent. Since the zeros of the function $\cos(\frac{\pi}{2}\alpha)$ are the poles of the function $(q^{1-\alpha};q^2)_\infty$ with the same orders. In fact

$$\lim_{\alpha\to(2j+1)} \frac{\cos\frac{\pi}{2}\alpha}{(q^{1-\alpha};q^2)_\infty} = -\frac{\pi}{2\ln q}\frac{q^{j^2+j}}{(q^2;q^2)_j(q^2;q^2)_\infty} \quad (j\in\mathbb{N}_0).$$

Similarly, the zeros of the function $\sin(\frac{\pi}{2}\alpha)$ are the poles of the function $(q^{2-\alpha};q^2)_\infty$ with the same orders and

$$\lim_{\alpha\to(2j)}\frac{\sin\frac{\pi}{2}\alpha}{(q^{2-\alpha};q^2)_\infty}=-\frac{\pi}{2\ln q}\frac{q^{j^2-j}}{(q^2;q^2)_{j-1}(q^2;q^2)_\infty}\quad(j\in\mathbb{N}).$$

Then the left-hand sides of equations (4.6)-(4.9) are entire functions. Hence, as a function of $\alpha$, the functions $\Psi_{-\alpha}(x,t)$ $(x,t\in\widetilde{\mathbb{R}}_q)$ can be analytically extended by defining its values when $\Re\alpha\geq 1$ by the left-hand sides of (4.6)-(4.9). □

It also should be noted that for $\mathrm{Re}(\alpha)\geq 1$, the left-hand sides of (4.6) and (4.7) determine $\Psi_{-\alpha}(x,t)$ for all $x,t\in\widetilde{\mathbb{R}}_q$, which is different from the case of $\mathrm{Re}(\alpha)<1$; see Remark 4.4.

**Definition 4.6** For $\mathrm{Re}\,\alpha>0$, we define a fractional $q$-integral operator $J_q^\alpha$ on $L^2(\widetilde{\mathbb{R}}_q)\cap L^1(\widetilde{\mathbb{R}}_q)$ by

$$J_q^\alpha f(x):=\frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}f(t)\Psi_\alpha(x,t)\,d_qt.$$

The following properties follow at once from (4.3) and their analytic continuation on $\mathbb{C}$.

- If $f\in L^2(\widetilde{\mathbb{R}}_q)\cap L(\widetilde{\mathbb{R}}_q)$ and $f$ is even, then

$$J_q^\alpha f(x)=2\int_0^{\infty/\sqrt{1-q}}f(t)\varphi_\alpha(x,t)\,d_qt,$$

where

$$(1-q)^{-\alpha/2}\varphi_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)=\begin{cases}2q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2n-2m+\alpha)}\\ \quad-2q^{n-m-(m+1)(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha-1)},\\ 2q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha)}\\ \quad+2q^{-n(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha+1)}\end{cases}$$

and

$$(1-q)^{-\alpha/2}\varphi_\alpha\left(-\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right)=\begin{cases}2q^{-m(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{\alpha+1};q)_{2j}}{(q;q)_{2j}}q^{i(2n-2m+\alpha)}\\ \quad+2q^{n-m-m(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha-1)},\\ 2q^{-n(1-\alpha)}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha)}\\ \quad-2q^{-n(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2m-2n+\alpha+1)}.\end{cases}$$

Hence, if $f$ is an even function, then $\delta_q^{2k}f(x)$ is an even function and $\delta_q^{2k+1}f(x)$ is an odd function for all $k\in\mathbb{N}_0$.

- If $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L(\widetilde{\mathbb{R}}_q)$ and $f$ is odd, then

$$J_q^\alpha f(x) = 2\int_0^{\infty/\sqrt{1-q}} f(t)\psi_\alpha(x,t)\,d_qt,$$

where

$$(1-q)^{-\alpha/2}\psi_\alpha\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} 2q^{-m(1-\alpha)}q^{n-m}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha)} \\ \quad -2q^{-m(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2n-2m+\alpha+1)}, \\ 2q^{-n(1-\alpha)}q^{m-n}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha)} \\ \quad +2q^{-(n+1)(1-\alpha)}q^{m-n}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha-1)} \end{cases}$$

and

$$(1-q)^{-\alpha/2}\psi_\alpha\left(-\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} -2q^{-m(1-\alpha)}q^{n-m}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2n-2m+\alpha)} \\ \quad -2q^{-m(1-\alpha)}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j}}{(q;q)_{2j}}q^{j(2n-2m+\alpha+1)}, \\ 2q^{-n(1-\alpha)}q^{m-n}\frac{(q^\alpha,q^2;q^2)_\infty}{(q^{1-\alpha},q;q^2)_\infty}\cos\frac{\pi}{2}\alpha\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha)} \\ \quad -2q^{-(n+1)(1+\alpha)}q^{m-n}\sin\frac{\pi}{2}\alpha\frac{(q^{1+\alpha},q^2;q^2)_\infty}{(q^{2-\alpha},q;q^2)_\infty}\sum_{j=0}^\infty\frac{(q^{1-\alpha};q)_{2j+1}}{(q;q)_{2j+1}}q^{j(2m-2n+\alpha-1)}. \end{cases}$$

Hence, if $f$ is an odd function, then $\delta_q^{2k}f(x)$ is an odd function and $\delta_q^{2k+1}f(x)$ is an even function for all $k \in \mathbb{N}_0$.

**Theorem 4.7** *Let $f \in L^2(\widetilde{\mathbb{R}}_q)$. Then*

$$\delta_q(J_qf)(x) = f(x) \quad \textit{for all } x \in \widetilde{\mathbb{R}}_q. \tag{4.16}$$

*Moreover, if $f$ is $q$-regular at zero, then*

$$J_q(\delta_qf)(x) = f(x) \quad \textit{for all } x \in \widetilde{\mathbb{R}}_q. \tag{4.17}$$

*Proof* If $f \in L^2(\widetilde{\mathbb{R}}_q)$, then

$$J_qf(x) = \frac{2}{C}\int_0^{\infty/\sqrt{1-q}} f_e(t)\varphi_1(x,t)\,d_qt + \frac{2}{C}\int_0^{\infty/\sqrt{1-q}} f_e(t)\psi_1(x,t)\,d_qt,$$

where

$$\varphi_1\left(\frac{q^n}{\sqrt{1-q}},\frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} \frac{\pi\sqrt{1-q}}{\ln q}, & n > m, \\ \frac{\pi\sqrt{1-q}}{\ln q} + \frac{C}{2}, & m \ge n \end{cases}$$

and

$$\psi_1\left(\frac{q^n}{\sqrt{1-q}}, \frac{q^m}{\sqrt{1-q}}\right) = \begin{cases} -\frac{C}{2}, & n > m-1, \\ 0, & m > n. \end{cases}$$

Hence,

$$J_q f(x) = \left(1 + \frac{2\pi\sqrt{1-q}}{C\ln q}\right)\int_0^x f_e(t)\,d_qt + \frac{2\pi\sqrt{1-q}}{C\ln q}\int_x^\infty f_e(t)\,d_qt - \int_{qx}^\infty f_0(t)\,d_qt. \tag{4.18}$$

One can verify that the first and second $q$-integrals of (4.18) are odd functions, while the last $q$-integral is an even function. Consequently,

$$\delta_q J_q f(x) = \left(1 + \frac{2\pi\sqrt{1-q}}{C\ln q}\right)D_{q,x}\int_0^x f_e(t)\,d_qt + \frac{2\pi\sqrt{1-q}}{C\ln q}D_{q,x}\int_x^\infty f_e(t)\,d_qt - D_{q,x}\int_x^\infty f_0(t)\,d_qt. \tag{4.19}$$

Using the fundamental theorem of $q$-calculus, see [13], we obtain (4.16). To prove (4.17), we assume that $f$ is $q$-regular at zero,

$$\begin{aligned} J_q\delta_q f(x) &= J_q(\delta_q f_e)(x) + J_q(\delta_q f_0)(x) \\ &= \frac{2}{C}\int_0^{\infty/\sqrt{1-q}} \delta_q f_e(t)\psi_1(x,t)\,d_qt + \frac{2}{C}\int_0^{\infty/\sqrt{1-q}} \delta_q f_0(t)\varphi_1(x,t)\,d_qt. \end{aligned}$$

But

$$\begin{aligned} \int_0^{\infty/\sqrt{1-q}} \delta_q f_e(t)\psi_1(x,t)\,d_qt &= \int_0^{\infty/\sqrt{1-q}} D_{q,t} f_e(t/q)\psi_1(x,t)\,d_qt \\ &= -\frac{C}{2}\int_{qx}^{\infty/\sqrt{1-q}} D_{q,t} f_e(t/q)\,d_qt = \frac{C}{2} f_e(x) \end{aligned}$$

and

$$\begin{aligned} &\int_0^{\infty/\sqrt{1-q}} \delta_q f_0(t)\varphi_1(x,t)\,d_qt \\ &\quad= \int_0^{\infty/\sqrt{1-q}} D_{q,t} f_0(t)\varphi_1(x,t)\,d_qt \\ &\quad= \left(\frac{\pi\sqrt{1-q}}{\ln q} + \frac{C}{2}\right)\int_0^x D_{q,t} f_0(t)\,d_qt + \frac{\pi\sqrt{1-q}}{\ln q}\int_x^\infty D_{q,t} f_0(t)\,d_qt \\ &\quad= \frac{C}{2} f_o(x) - \left(\frac{\pi\sqrt{1-q}}{\ln q} + \frac{C}{2}\right) f_0(0) = \frac{C}{2} f_o(x) \end{aligned}$$

since $f_0(0) = 0$. This proves (4.17) and completes the proof of the theorem. □

We can also prove that

$$\psi_2(x,t) := \begin{cases} -\frac{c}{2}(x-qt) + (x-t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \le q^{-1}t, \\ -\frac{c}{2}(t-qx) + (x-t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \ge qt, \end{cases}$$

$$\psi_2(x,-t) := \begin{cases} \frac{c}{2}(x+qt) + (x+t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \le q^{-1}t, \\ \frac{c}{2}(t+qx) + (x+t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \ge qt \end{cases}$$

and

$$\psi_2(-x,t) := \begin{cases} \frac{c}{2}(x+qt) - (x+t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \le q^{-1}t, \\ \frac{c}{2}(t+qx) - (x+t)\frac{\pi\sqrt{1-q}}{\ln q}, & x \ge qt. \end{cases}$$

Hence,

$$\begin{aligned} J_q^2 f(x) &= \frac{\pi\sqrt{1-q}}{c\ln q}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}(x-t)f(t)\,d_qt \\ &\quad + \int_0^{q^{-1}x}\bigl(qxf_e(t) - tf_0(t)\bigr)\,d_qt + \int_{q^{-1}x}^{\infty/\sqrt{1-q}}\bigl(xf_e(t) - qtf_0(t)\bigr)\,d_qt. \end{aligned}$$

**Definition 4.8** For $\alpha > 0$, we define a fractional $q$-difference operator $\delta_q^\alpha$ on $L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$ by

$$\delta_q^\alpha f(x) := J_q^{-\alpha} f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{-\alpha}(x,t)\,d_qt. \tag{4.20}$$

**Lemma 4.9** *The operator $\delta_q^\alpha$ coincides with Rubin's q-difference operator when $\alpha$ is a positive integer.*

*Proof* Let $\alpha = k$ for some $k \in \mathbb{N}$, and let $f \in L^2(\widetilde{\mathbb{R}}_q) \cap L^1(\widetilde{\mathbb{R}}_q)$. Using (2.7), we conclude

$$f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_0(x,t)\,d_qt.$$

Then from Lemma 4.2 we obtain

$$\delta_q^k f(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{-k}(x,t)\,d_qt,$$

and the lemma follows. □

**Lemma 4.10** *If $\alpha > 0$ and $f \in L^2(\widetilde{\mathbb{R}}_q)$, then*

$$\delta_q^\alpha f(x) = \delta_q^k J_q^{k-\alpha} f(x) \quad \bigl(\alpha > 0; k = \lceil\alpha\rceil; x \in \mathbb{R}_q\bigr). \tag{4.21}$$

Now, if $\alpha$ is a positive integer, then $\lceil\alpha\rceil = \alpha$ and from (4.21)

$$\partial_q^\alpha = \partial_q^k.$$

*Proof* Since

$$\delta_q^k J_q^{k-\alpha} f(x) = \frac{1}{C}\delta_q^k \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(t)\Psi_{k-\alpha}(x,t)\,d_qt$$

and $\delta_{q,x}^k \Psi_{k-\alpha}(x,t) = \Psi_{-\alpha}(x,t)$, the proof follows. □

**Theorem 4.11** *If $f \in L^1(\widetilde{\mathbb{R}}_q) \cap L^2(\widetilde{\mathbb{R}}_q)$ and $\alpha$, $\beta$ are complex numbers such that* $\operatorname{Re}(\alpha) < 1$ *and* $\operatorname{Re}(\beta) < 1$ *such that* $\operatorname{Re}(\alpha+\beta) < 1$, *then*

$$J_q^\alpha J_q^\beta f = J_q^\beta J_q^\alpha f = J_q^{\alpha+\beta} f.$$

*Proof* Let $x \in \widetilde{\mathbb{R}}_q$. Since

$$\begin{aligned} J_q^\alpha\big(J_q^\beta f\big)(x) &= \frac{1}{C^2}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u)\Psi_\alpha(x,t)\Psi_\beta(t,u)\,d_qu\,d_qt \\ &= \frac{1}{C^2}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u)\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \Psi_\alpha(x,t)\Psi_\beta(t,u)\,d_qt\,d_qu, \end{aligned}$$

using the orthogonality relation (2.9), we obtain

$$\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \Psi_\alpha(x,t)\Psi_\beta(t,u)\,d_qt = C\Psi_{\alpha+\beta}(x;u).$$

Consequently,

$$J_q^\alpha\big(J_q^\beta f\big)(x) = \frac{1}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f(u)\Psi_{\alpha+\beta}(x;u)\,d_qu = J_q^{\alpha+\beta} f(x).$$

□

**Example 4.12** Let $n \in \mathbb{Z}$ and let $f_n(x)$ be the even function defined on $\widetilde{\mathbb{R}}_q$ by

$$f_n(x) := \begin{cases} 1, & x = \pm x_n, x_n := \frac{q^n}{\sqrt{1-q}}, \\ 0, & \text{otherwise}. \end{cases}$$

Hence,

$$f_n(x) = \sum_{k=-\infty}^{\infty} a_k \operatorname{Cos}\left(x\frac{q^k}{\sqrt{1-q}};q^2\right) + \sum_{k=-\infty}^{\infty} b_k \operatorname{Sin}\left(x\frac{q^k}{\sqrt{1-q}};q^2\right).$$

Since $f_n$ is an even function, then $b_k = 0$ for all $k \in \mathbb{Z}$ and

$$a_k = \frac{q^k}{C}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} f_n(t)\operatorname{Cos}\left(t\frac{q^k}{\sqrt{1-q}};q^2\right)d_qt = \frac{2q^{n+k}\sqrt{1-q}}{C}\operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right).$$

Consequently,

$$f_n(x) = \frac{2q^n\sqrt{1-q}}{C}\sum_{k=-\infty}^{\infty} q^k \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right)\operatorname{Cos}\left(\frac{q^k x}{\sqrt{1-q}};q^2\right).$$

Hence,

$$\sum_{k=-\infty}^{\infty} q^k \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^k x}{\sqrt{1-q}};q^2\right) = \begin{cases} \frac{Cq^{-n}}{2\sqrt{1-q}}, & x = x_n, \\ \text{zero}, & \text{otherwise.} \end{cases}$$

Since

$$\delta_q f_n(x) = \frac{1}{q} D_{q^{-1}} f_n(x) := \begin{cases} \frac{\pm 1}{x_n(1-q)}, & x = \pm x_n, \\ \frac{\mp 1}{q x_n(1-q)}, & x = \pm q x_n, \\ \text{zero}, & \text{otherwise} \end{cases}$$

and

$$\delta_q f_n(x) = -\frac{2q^n}{C} \sum_{k=-\infty}^{\infty} q^{2k} \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right) \operatorname{Sin}\left(\frac{q^k x}{\sqrt{1-q}};q^2\right) \quad (x \in \widetilde{\mathbb{R}}_q).$$

We obtain

$$\sum_{k=-\infty}^{\infty} q^{2k} \operatorname{Sin}\left(\frac{x q^k}{\sqrt{1-q}};q^2\right) \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right) = \begin{cases} \text{zero}, & x \notin \{\pm x_n, \pm q x_n\}, \\ \pm\frac{Cq^{-n}}{2x_n(1-q)}, & x = \pm x_n, \\ \pm\frac{Cq^{-n}}{2q x_n(1-q)}, & x = \pm q x_n. \end{cases} \tag{4.22}$$

Similarly,

$$\delta_q^2 f_n(x) = \begin{cases} \frac{-1}{q x_n^2(1-q)}, & x = \pm x_n, \\ \frac{-1}{q^2 x_n^2(1-q)^2}, & x = \pm q x_n, \\ \frac{-q}{x_n^2(1-q)^2}, & x = \pm q^{-1} x_n, \\ \text{zero}, & \text{otherwise.} \end{cases}$$

But

$$\delta_q^2 f_n(x) = -2\frac{q^n}{C\sqrt{1-q}} \sum_{k=-\infty}^{\infty} q^{3k} \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^k x}{\sqrt{1-q}}\right) \quad (x \in \widetilde{\mathbb{R}}_q).$$

Hence,

$$\sum_{k=-\infty}^{\infty} q^{3k} \operatorname{Cos}\left(\frac{q^{n+k}}{1-q};q^2\right) \operatorname{Cos}\left(\frac{q^k x}{\sqrt{1-q}};q^2\right) = \begin{cases} \frac{Cq^{-n-1}}{2x_n^2\sqrt{1-q}}, & x = \pm x_n, \\ \frac{Cq^{-n-2}}{2x_n^2(\sqrt{1-q})^3}, & x = \pm q x_n, \\ \frac{Cq^{-n+1}}{2x_n^2(\sqrt{1-q})^3}, & x = \pm q^{-1} x_n, \\ 0, & \text{otherwise.} \end{cases} \tag{4.23}$$

In the following two examples, we show how we can use (4.20) when $\alpha$ is a positive integer to obtain new summation formulae.

**Example 4.13** Let $f$ be an even function. Then, for each $k \in \mathbb{N}_0$, we have

$$\delta_q^{2k} f(x) = q^{k(k-1)} D_{q,x}^{2k} f\bigl(x/q^k\bigr), \qquad \delta_q^{2k+1} f(x) = q^{k(k+1)} D_{q,x}^{2k+1} f\bigl(x/q^{k+1}\bigr).$$

Hence,

$$\delta_q^{2k} f(x) = q^{k(k-1)}(1-q)^{-2k} x^{-2k} \sum_{r=0}^{2k} q^r \frac{(q^{-2k};q)_r}{(q;q)_r} f\bigl(xq^{r-k}\bigr).$$

On the other hand,

$$\delta_q^{2k} f(x) = 4\frac{(-1)^k}{C} \int_0^{\infty/\sqrt{1-q}} f(t)\mathrm{CC}_{-2k}(x,t)\, d_q t.$$

Let $m \in \mathbb{Z}$ and let $f_m(x)$ be the even function defined on $\widetilde{\mathbb{R}}_q$ by

$$f_m(x) := \begin{cases} 1, & x = \pm x_m, x_m := \frac{q^n}{\sqrt{1-q}}, \\ 0, & \text{otherwise.} \end{cases}$$

Set $x = \frac{q^n}{\sqrt{1-q}}$. Hence,

$$\begin{aligned} \delta_q^{2k} f(x) &= 4\frac{(-1)^k\sqrt{1-q}}{C} q^{-2mk}(1-q)^{-k} \frac{(q^{-2k}, q^{2n-2m+2}, q^2; q^2)_\infty}{(q^{2n-2m-2k}, q, q; q^2)_\infty} \\ &\quad \times {}_2\phi_1\bigl(q^{2n-2m-2k}, q^{-2k}; q^{2n-2m+2}; q^2, q^{2k+1}\bigr) \\ &= q^{k^2+m-n-2nk}(1-q)^{-k} \frac{(q^{-2k};q)_{m-n+k}}{(q;q)_{m-n+k}}. \end{aligned}$$

Since

$$\frac{(q^{-2k}, q^{2n-2m+2}; q^2)_\infty}{(q^{2n-2m-2k}; q^2)_\infty} = 0 \quad \text{if } n \geq m+k \text{ or } n < m.$$

Hence, if $m \leq n \leq m+k-1$, we obtain

$$\begin{aligned} &{}_2\phi_1\bigl(q^{2n-2m-2k}, q^{-2k}; q^{2n-2m+2}; q^2, q^{2k+1}\bigr) \\ &\quad = (-1)^k q^{k^2+(m-n)(2k+1)} \frac{(q^{-2k};q)_{m-n+k}}{(q;q)_{m-n+k}} \frac{(q^2, q^{2n-2m-2k}; q^2)_\infty}{(q^{-2k}, q^{2n-2m+2}; q^2)_\infty}. \end{aligned}$$

**Example 4.14** Let $f$ be an odd function. Then, for each $k \in \mathbb{N}_0$, we have

$$\delta_q^{2k} f(x) = q^{k^2} D_{q,x}^{2k} f\bigl(x/q^k\bigr), \qquad \delta_q^{2k+1} f(x) = q^{k^2} D_{q,x}^{2k+1} f\bigl(x/q^k\bigr).$$

Hence,

$$\delta_q^{2k} f(x) = q^{k^2}(1-q)^{-2k} x^{-2k} \sum_{r=0}^{2k} q^r \frac{(q^{-2k};q)_r}{(q;q)_r} f\bigl(xq^{r-k}\bigr).$$

On the other hand,

$$\delta_q^{2k} f(x) = 4\frac{(-1)^k}{C}\int_0^{\infty/\sqrt{1-q}} f(t)\mathrm{CS}_{-2k}(x,t)\,d_qt.$$

Let $m \in \mathbb{Z}$ and let $f_m(x)$ be the odd function defined on $\widetilde{\mathbb{R}}_q$ by

$$f_m(x) := \begin{cases} \pm 1, & x = \pm x_m, x_m := \frac{q^m}{\sqrt{1-q}}, \\ 0, & \text{otherwise.} \end{cases}$$

Set $x = \frac{q^n}{\sqrt{1-q}}$. Hence,

$$\begin{aligned} \delta_q^{2k} f(x) &= 4\frac{(-1)^k\sqrt{1-q}}{C} q^{-2mk}(1-q)^{-k-\frac{1}{2}} \frac{(q^{-2k}, q^{2n-2m+2}, q^2; q^2)_\infty}{(q^{2n-2m-2k}, q, q; q^2)_\infty} \\ &\quad \times {}_2\phi_1\bigl(q^{2n-2m-2k}, q^{-2k-2}; q^{2n-2m+2}; q^2, q^{2k+3}\bigr) \\ &= q^{k^2+k+m-n}(1-q)^{-k-\frac{1}{2}} \frac{(q^{-2k-1}; q)_{m-n+k}}{(q;q)_{m-n+k}}. \end{aligned}$$

Since

$$\frac{(q^{-2k}, q^{2n-2m+2}; q^2)_\infty}{(q^{2n-2m-2k}; q^2)_\infty} = 0 \quad \text{if } n \geq m+k \text{ or } n < m.$$

Hence, if $m \leq n \leq m+k-1$, we obtain

$$\begin{aligned} &{}_2\phi_1\bigl(q^{2n-2m-2k}, q^{-2k-2}; q^{2n-2m+2}; q^2, q^{2k+3}\bigr) \\ &\quad = (-1)^k q^{k^2+k+(m-n)(2k+1)} \frac{(q^{-2k-1}; q)_{m-n+k}}{(q;q)_{m-n+k}} \frac{(q^2, q^{2n-2m-2k}; q^2)_\infty}{(q^{-2k}, q^{2n-2m+2}; q^2)_\infty}. \end{aligned}$$

## 5 Application of the $q^2$-analogue of the Fourier transform to solve $q$-fractional difference equations

In [18], Ho explored the possibility of using the classical Fourier and Mellin integral transforms to solve the class of $q$-difference differential equations

$$D_{q,t}^n u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad x \in \mathbb{R}, t > 0, n \geq 1, \tag{5.1}$$

with the initial conditions

$$y(x,0) = f(x), \qquad D_{q,t}^k y(x,t)|_{t=0^+} = g_k(x) \quad (k = 1, \dots, n-1),$$

where the functions $f(x)$ and $g_k(x)$ are assumed to vanish as $x \to \pm\infty$. In [19] Brahim and Quanes used the $q^2$-Fourier transform and the $q$-Mellin transform to solve equation (5.1) in case of $n = 1, 2$, and only for $q$ satisfying condition (3.2). In this section, we use the $q^2$-Fourier transform with the ${}_qL_s$ transform to solve the $q$-fractional diffusion equation

$$\begin{aligned} &D_{q,t}^\alpha u(x,t) = \lambda\, \partial_{q,x}^2 u(x,t), \\ &\quad x \in \widetilde{\mathbb{R}}_q, t \in \mathbb{R}, 0 \leq \alpha < 1, 0 < q < 1, \end{aligned} \tag{5.2}$$

with the initial conditions

$$I_{q,t}^{1-\alpha}u(x,t)|_{t=0^+} = \phi(x), \qquad \partial_{q,x}^k u(x,t) \in L_q^1(\widetilde{\mathbb{R}}_q) \quad (k = 0,1), \tag{5.3}$$

$$\phi \in \left(L_q^1 \cap L_q^2\right)(\widetilde{\mathbb{R}}_q). \tag{5.4}$$

**Theorem 5.1** *The solution of q-fractional diffusion equation* (5.2) *subject to the initial conditions* (5.3)-(5.4) *is given by*

$$u(x,t) = \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} \left[T_y G(x,t)\right](x)\phi(y)\, d_q y,$$

*where*

$$\left[T_y G(x,t)\right](x) = \frac{(1+q)^{1/2}}{2\Gamma_{q^2}(\frac{1}{2})} \int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}} e\left(-iy\xi; q^2\right) g(\xi,t) e\left(ix\xi; q^2\right) d_q\xi$$

*and*

$$g(\xi,t) := \begin{cases} t^{\alpha-1} e_{\alpha,\alpha}(-\lambda\xi^2 t^\alpha; q), & |\lambda\xi^2 t^\alpha| < \frac{1}{(1-q)^\alpha}, \\ 0, & \textit{otherwise}, \end{cases}$$

*where, in general,* $e_{\alpha,\beta}(x;q)$ *is the q-analogue of the q-Mittag-Leffler function defined for* $\mathrm{Re}(\alpha > 0)$ *and* $\beta \in \mathbb{C}$ *by*

$$e_{\alpha,\beta}(x;q) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma_q(\alpha k + \beta)}, \quad \left|x(1-q)^\alpha\right| < 1.$$

*Proof* First we calculate the $q^2$-Fourier transform of (5.2) with respect to the variable $x$. Hence, applying (3.12) yields

$$D_{q,t}^\alpha U(\xi,t) = -\lambda\xi^2 U(\xi,t), \tag{5.5}$$

where

$$U(\xi,t) := \mathcal{F}_{q,x}\left(u(x,t)\right)(\xi).$$

Now we calculate the ${}_qL_s$ transform of (5.5) with respect to the variable $t$. Using (1.16) we obtain

$$\left(P^\alpha + \lambda\xi^2\right) V(\xi,s) = \frac{\mathcal{F}_q(\phi)(\xi)}{1-q}, \tag{5.6}$$

where

$$V(\xi,s) = {}_{q,t}L_s\left(U(\xi,t)\right)(s), \qquad p = \frac{s}{1-q}.$$

One can verify that

$$ {}_{q,t}L_s\left(t^{\alpha-1}e_{\alpha,\alpha}\left(-\lambda\xi^2t^\alpha;q\right)\right)=\frac{1}{1-q}\frac{1}{p^\alpha+\lambda\xi^2}. $$

Consequently,

$$ U(\xi,t)=\mathcal{F}_q(\phi)(\xi)t^{\alpha-1}e_{\alpha,\alpha}\left(-\lambda\xi^2t^\alpha\right),\quad \left|\lambda\xi^2t^\alpha\right|<\frac{1}{(1-q)^\alpha}. $$

It follows from the inversion formula of the $q^2$-Fourier transform that

$$ t^{\alpha-1}e_{\alpha,\alpha}\left(-\lambda\xi^2t^\alpha;q\right)=\mathcal{F}_{q,x}\big(G(x,t)\big)(\xi), $$

$$ G(x,t)=\frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}t^{\alpha-1}e_{\alpha,\alpha}\left(-\lambda\xi^2t^\alpha;q\right)e\left(i\xi x;q^2\right)d_q\xi, $$

where the variable of the $q$-integration $\xi$ runs only over all $\xi\in\widetilde{\mathbb{R}}_q$ such that

$$ \left|\lambda\xi^2t^\alpha\right|<\frac{1}{(1-q)^\alpha}. $$

Consequently,

$$ u(x,t)=\phi(x)*G(x,t). $$

Applying the $q^2$-Fourier convolution formula gives

$$ u(x,t)=\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}\big[T_yG(x,t)\big](x)\phi(y)\,d_qy, $$

where

$$ \big[T_yG(x,t)\big](x)=\frac{\sqrt{1+q}}{2\Gamma_{q^2}(1/2)}\int_{-\infty/\sqrt{1-q}}^{\infty/\sqrt{1-q}}e\left(-iy\xi;q^2\right)t^{\alpha-1}e_{\alpha,\alpha}\left(-\lambda\xi^2t^\alpha;q\right)e\left(ix\xi;q^2\right)d_q\xi. \qquad \square $$

**Competing interests**
The author declares that they have no competing interests.

**Acknowledgements**
This research is supported by NPST Program of King Saud University; project number 10-MAT1293-02.

Received: 10 June 2013 Accepted: 15 August 2013 Published: 23 September 2013

**References**

1. Gasper, G, Rahman, M: Basic Hypergeometric Series, 2nd edn. Cambridge University Press, Cambridge (2004)
2. Ismail, MEH: Classical and Quantum Orthogonal Polynomials in One Variable. Cambridge University Press, Cambridge (2005)
3. Jackson, FH: On *q*-definite integrals. Q. J. Pure Appl. Math. **41**, 193-203 (1910)
4. Hahn, W: Beiträge zur Theorie der Heineschen Reihen. Math. Nachr. **2**, 340-379 (1949) (in German)
5. Matsuo, A: Jackson integrals of Jordan-Pochhammer type and quantum Knizhnik-Zamolodchikov equations. Commun. Math. Phys. **151**(2), 263-273 (1993)
6. Al-Salam, WA: Some fractional *q*-integrals and *q*-derivatives. Proc. Edinb. Math. Soc. **2**(15), 135-140 (1966/1967)
7. Agarwal, RP: Certain fractional *q*-integrals and *q*-derivatives. Proc. Camb. Philos. Soc. **66**, 365-370 (1969)
8. Rubin, RL: A $q^2$-analogue operator for $q^2$-analogue Fourier analysis. J. Math. Anal. Appl. **212**(2), 571-582 (1997)

9. Rubin, RL: Duhamel solutions of non-homogeneous $q^2$-analogue wave equations. Proc. Am. Math. Soc. **135**(3), 777-785 (2007)
10. Fitouhi, A, Safraoui, A: Paley-Wiener theorem for the $q^2$-Fourier-Rubin transform. Tamsui Oxford Univ. J. Math. Sci. **26**(3), 287-304 (2010)
11. Abdi, WH: On $q$-Laplace transforms. Proc. Natl. Acad. Sci. India, Sect. A **29**, 389-408 (1960)
12. Abdi, WH: On certain $q$-difference equations and $q$-Laplace transforms. Proc. Natl. Acad. Sci. India, Part A **28**, 1-15 (1962)
13. Annaby, MH, Mansour, ZS: $q$-Fractional Calculus and Equations. Springer, Heidelberg (2012)
14. Koornwinder, TH, Swarttouw, RF: On $q$-analogues of Fourier and Hankel transforms. Trans. Am. Math. Soc. **333**(1), 445-461 (1992)
15. Fitouhi, A, Moncef, HM, Bouzeffour, F: The $q$-$j_\alpha$ Bessel function. J. Approx. Theory **115**(1), 144-166 (2002)
16. Zygmund, A: Trigonometric Series, vol. II. Cambridge University Press, Cambridge (1959)
17. Ismail, MEH, Rahman, M: Inverse operators, $q$-fractional integrals and $q$-Bernoulli polynomials. J. Approx. Theory **114**(2), 269-307 (2002)
18. Ho, CL: On the use of Mellin transform to a class of $q$-difference-differential equations. Phys. Lett. A **268**(4-6), 217-223 (2000)
19. Brahim, K, Quanes, R: Some applications of the $q$-Mellin transform. Tamsui Oxford Univ. J. Math. Sci. **26**(3), 335-343 (2010)

doi:10.1186/1687-1847-2013-276
**Cite this article as:** Mansour: $q$-**Fractional calculus for Rubin's** $q$-**difference operator.** *Advances in Difference Equations* 2013 **2013**:276.