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Multi-term fractional differential equations in a nonreflexive Banach space

Ravi P Agarwal[1,2*], Vasile Lupulescu[3], Donal O'Regan[4] and Ghaus ur Rahman[5]

*Correspondence:
agarwal@tamuk.edu
[1]Department of Mathematics, Texas
A&M University-Kingsville, Kingsville,
TX, USA
[2]Department of Mathematics, King
Abdulaziz University, P.O. Box 80203,
Jeddah, 21859, Saudi Arabia
Full list of author information is
available at the end of the article

Abstract

In this paper we establish an existence result for the multi-term fractional differential equation

$$\left(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i} \right) u(t) = f(t, u(t)) \quad \text{for } t \in [0,1], \qquad u(0) = 0, \tag{1}$$

where $D_p^{\alpha_m} y(\cdot)$ and $D_p^{\alpha_i} y(\cdot)$ are fractional pseudo-derivatives of a weakly absolutely continuous and pseudo-differentiable function $u(\cdot) : T \to E$ of order α_m and α_i, $i = 1, 2, \ldots, m-1$, respectively, the function $f(t, \cdot) : T \times E \to E$ is weakly-weakly sequentially continuous for every $t \in T$ and $f(\cdot, y(\cdot))$ is Pettis integrable for every weakly absolutely continuous function $y(\cdot) : T \to E$, T is a bounded interval of real numbers and E is a nonreflexive Banach space, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$ and $a_1, a_2, \ldots, a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$.

Keywords: weak measure of noncompactness; nonreflexive Banach spaces; Pettis integral; multi-term fractional differential equation; fractional pseudo-derivative

1 Introduction

The mathematical field that deals with derivatives of any real order is called fractional calculus. Fractional calculus has been successfully applied in various applied areas like computational biology, computational fluid dynamics and economics *etc.* [1].

In certain situations, we need to solve fractional differential equations containing more than one differential operator, and this type of fractional differential equation is called a multi-term fractional differential equation. Multi-term fractional differential equations have numerous applications in physical sciences and other branches of science [2]. The existence of solutions of multi-term fractional differential equations was studied by many authors [3–8]. The main tool used in [3–5] is the Krasnoselskii's fixed point theorem on a cone, while the main tool used in [7] is the technique associated with the measure of noncompactness and fixed point theorem. In [7], the author established the existence of a monotonic solution for a multi-term fractional differential equation in Banach spaces, using the Riemann-Liouville fractional derivative and in that paper no compactness condition is assumed on the nonlinearity of the function f.

When $\alpha_m = 1$ and $a_1 = a_2 \cdots a_{m-1} = 0$, the existence of weak solutions to multi-term fractional differential equation (1) was discussed in [9–13]. In [6], the author studied the existence of a weak solution of Cauchy problem (1) in reflexive Banach spaces equipped

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with the weak topology, and the author assumed a weak-weak continuity assumption on f. In [8], the author established the existence of a global monotonic solution for Cauchy problem (1), and the author assumed f is Carathéodory with linear growth.

In this present article, we prove the existence of a solution of Cauchy problem (1) in non-reflexive Banach spaces equipped with the weak topology. In comparison to other results in the literature, we use more general assumptions so that the function f is assumed to be weakly-weakly sequentially continuous and $f(\cdot, u(\cdot))$ is Pettis integrable for each weakly absolutely continuous function $u(\cdot): T \to E$.

2 Preliminaries

For convenience here we present some notations and the main properties for Pettis integrable, weakly-weakly continuous functions, and we state some properties of the measure of noncompactness. Also, we present definitions and preliminary facts of fractional calculus in abstract spaces. Let E be a Banach space with the norm $\|\cdot\|$, and let E^* be the topological dual of E. If $x^* \in E^*$, then its value on an element $x \in E$ will be denoted by $\langle x^*, x \rangle$. The space E endowed with the weak topology $\sigma(E, E^*)$ will be denoted by E_w. Consider an interval $T = [0, b]$ of $\mathbb{R}$ endowed with the Lebesgue σ-algebra $\mathcal{L}(T)$ and the Lebesgue measure λ. We will denote by $L^1(T)$ the space of all measurable and Lebesgue integrable real functions defined on T, and by $L^\infty(T)$ the space of all measurable and essentially bounded real functions defined on T.

Definition 2.1 (a) A function $x(\cdot): T \to E$ is said to be *strongly measurable* on T if there exists a sequence of simple functions $x_n(\cdot): T \to E$ such that $\lim_{n\to\infty} x_n(t) = x(t)$ for a.e. $t \in T$.

(b) A function $x(\cdot): T \to E$ is said to be *weakly measurable* (or *scalarly measurable*) on T if for every $x^* \in E^*$, the real-valued function $t \mapsto \langle x^*, x(t) \rangle$ is Lebesgue measurable on T. It is well known that a weakly measurable and almost separable valued function $x(\cdot): T \to E$ is strongly measurable [14, Theorem 1.1].

Definition 2.2 (a) A function $x(\cdot): T \to E$ is said to be *absolutely continuous* on T (*AC*, for short) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| \sum_{k=1}^{m} \left[x(b_k) - x(a_k) \right] \right\| < \varepsilon$$

for every finite disjoint family $\{(a_k, b_k); 1 \le k \le m\}$ of subintervals of T such that $\sum_{k=1}^{m}(b_k - a_k) < \delta$. If the last inequality is replaced by $\sum_{k=1}^{m} \|x(b_k) - x(a_k)\| < \varepsilon$, then we say that $x(\cdot)$ is a *strongly absolutely continuous* (*sAC*) function.

(b) A function $x(\cdot): T \to E$ is said to be *weakly absolutely continuous* (*wAC*) on T if for every $x^* \in E^*$, the real-valued function $t \mapsto \langle x^*, x(t) \rangle$ is *AC* on T.

Remark 2.1 Each *sAC* function is an *AC* function, and each *AC* function is a *wAC* function. If E is a weakly sequentially complete space, then every *wAC* function is an *AC* function [15].

Definition 2.3 (a) A function $x(\cdot) : T \to E$ is said to be *strongly differentiable at $t_0 \in T$* if there exists an element $x'_s(t_0) \in E$ such that

$$\lim_{h \to 0} \left\| \frac{x(t_0 + h) - x(t_0)}{h} - x'_s(t_0) \right\| = 0.$$

The element $x'_s(t_0)$ will also be denoted by $\frac{d_s}{dt} x(t_0)$ and it is called the *strong derivative* of $x(\cdot)$ at $t_0 \in T$.

(b) A function $x(\cdot) : T \to E$ is said to be *weakly differentiable at $t_0 \in T$* if there exists an element $x'_w(t_0) \in E$ such that

$$\lim_{h \to 0} \left\langle x^*, \frac{x(t_0 + h) - x(t_0)}{h} \right\rangle = \left\langle x^*, x'_w(t_0) \right\rangle$$

for every $x^* \in E^*$. The element $x'_w(t_0)$ will be also denoted by $\frac{d_w}{dt} x(t_0)$ and it is called the *weak derivative* of $x(\cdot)$ at $t_0 \in T$.

Proposition 2.1 [16, Theorem 7.3.3] *If E is a weakly sequentially complete space and $x(\cdot) : T \to E$ is a function such that for every $x^* \in E^*$, the real function $t \mapsto \langle x^*, x(t) \rangle$ is differentiable T, then $x(\cdot)$ is weakly differentiable on T.*

Proposition 2.2 [14, Theorem 1.2] *If $x(\cdot) : T \to E$ is an a.e. weakly differentiable on T, then its weak derivative $x'_w(\cdot)$ is strongly measurable on T.*

Definition 2.4 A function $x(\cdot) : T \to E$ is said to be *pseudo-differentiable* on T to a function $y(\cdot) : T \to E$ if for every $x^* \in E^*$, there exists a null set $N(x^*) \in \mathcal{L}(T)$ such that the real function $t \mapsto \langle x^*, x(t) \rangle$ is differentiable on $T \setminus N(x^*)$ and

$$\frac{d}{dt} \langle x^*, x(t) \rangle = \langle x^*, y(t) \rangle, \quad t \in T \setminus N(x^*). \tag{2}$$

The function $y(\cdot)$ is called a *pseudo-derivative* of $x(\cdot)$ and it will be denoted by $x'_p(\cdot)$ or by $\frac{d_p}{dt} x(\cdot)$.

Definition 2.5 A weakly measurable function $x(\cdot) : T \to E$ is said to be *Pettis integrable* on T if

(a) $x(\cdot)$ is *scalarly integrable*; that is, for every $x^* \in E^*$, the real function $t \mapsto \langle x^*, x(t) \rangle$ is Lebesgue integrable on T;

(b) for every set $A \in \mathcal{L}(T)$, there exists an element $x_A \in E$ such that

$$\langle x^*, x_A \rangle = \int_A \langle x^*, x(s) \rangle \, ds \tag{3}$$

for every $x^* \in E^*$. The element $x_A \in E$ is called the *Pettis integral* on A and it will be denoted by $\int_A x(s) \, ds$.

Remark 2.2 (a) It is known that if $x(\cdot) : T \to E$ is Bochner integrable on T, then the function $y(\cdot) : T \to E$, given by

$$y(t) = (B) \int_0^t x(s) \, ds, \quad t \in T,$$

is *AC* and a.e. differentiable on *T*, and $y'_s(t) = x(t)$ for a.e. $t \in T$. Also, if a function $x(\cdot) : T \to E$ is *AC* and a.e. strongly differentiable on *T*, then $x'_s(\cdot)$ is Bochner integrable on *T* and

$$x(t) = x(0) + (B) \int_0^t x'_s(s)\, ds, \quad t \in T.$$

(b) In the case of the Pettis integral, in [14] it was shown that if $x(\cdot) : T \to E$ is an *AC* and a.e. weakly differentiable on *T*, then $x'_w(\cdot)$ is Pettis integrable on *T* and

$$x(t) = x(0) + \int_0^t x'_w(s)\, ds, \quad t \in T.$$

In 1994 Kadets [17] proved that there exists a strongly measurable and Pettis integrable function $x(\cdot) : T \to E$ such that the indefinite Pettis integral

$$y(t) = \int_0^t x(s)\, ds, \quad t \in T, \tag{4}$$

is not weakly differentiable on a set of positive Lebesgue measures (see also [18, 19]).

Proposition 2.3 [14] *Let* $x(\cdot) : T \to E$ *be a weakly measurable function.*
(a) *If* $x(\cdot)$ *is Pettis integrable on* T, *then the indefinite Pettis integral* (4) *is AC on* T *and* $x(\cdot)$ *is a pseudo-derivative of* $y(\cdot)$.
(b) *If* $y(\cdot) : T \to E$ *is an AC function on* T *and it has a pseudo-derivative* $x(\cdot)$ *on* T, *then* $x(\cdot)$ *is Pettis integrable on* T *and*

$$y(t) = y(0) + \int_0^t x(s)\, ds, \quad t \in T.$$

It is known that the Pettis integrals of two strongly measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ coincide over every Lebesgue measurable set in *T* if and only if $x(\cdot) = y(\cdot)$ a.e. on *T* [14, Theorem 5.2]. Since a pseudo-derivative of the pseudo-differentiable function $x(\cdot) : T \to E$ is not unique and two pseudo-derivatives of $x(\cdot)$ need not be a.e. equal, then the concept of weak equivalence plays an important role in the following.

Definition 2.6 Two weak measurable functions $x(\cdot) : T \to E$ and $y(\cdot) : T \to E$ are said to be *weakly equivalent* on *T* if for every $x^* \in E^*$, we have that $\langle x^*, x(t) \rangle = \langle x^*, y(t) \rangle$ for a.e. $t \in T$.

Proposition 2.4 *A weakly measurable function* $x(\cdot) : T \to E$ *is Pettis integrable on* T *and* $\langle x^*, x(\cdot) \rangle \in L^\infty(T)$ *for every* $x^* \in E^*$ *if and only if the function* $t \mapsto \varphi(t)x(t)$ *is Pettis integrable on* T *for every* $\varphi(\cdot) \in L^1(T)$.

Let us denote by $P^\infty(T, E)$ the space of all weakly measurable and Pettis integrable functions $x(\cdot) : T \to E$ with the property that $\langle x^*, x(\cdot) \rangle \in L^\infty(T)$ for every $x^* \in E^*$. Since for each $t \in T$ the real-valued function $s \mapsto (t - s)^{\alpha - 1}$ is Lebesgue integrable on $[0, t]$ for every $\alpha > 0$ then, by Proposition 2.4, the *fractional Pettis integral*

$$I^\alpha x(t) := \int_0^t \frac{(t - s)^{\alpha - 1}}{\Gamma(\alpha)} x(s)\, ds, \quad t \in T,$$

exists for every function $x(\cdot) \in P^\infty(T, E)$ as a function from T into E. Moreover, we have that

$$\langle x^*, I^\alpha x(t) \rangle = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \langle x^*, x(s) \rangle \, ds, \quad t \in T$$

for every $x^* \in E^*$, and the real function $t \mapsto \langle x^*, I^\alpha x(t) \rangle$ is continuous (in fact, bounded and uniformly continuous on T if $T = \mathbb{R}$) on T for every $x^* \in E^*$ [20, Proposition 1.3.2].

Proposition 2.5 *If $f(\cdot, \cdot) : T \times E \to E$ is a function such that $f(\cdot, y(\cdot)) \in P^\infty(T, E)$ for every wAC function $y(\cdot) : T \to E$, then the function $f_\alpha(\cdot)$ given by*

$$f_\alpha(t) = \int_0^t \frac{(t-s)^{1-\alpha}}{\Gamma(\alpha)} f(s, y(s)) \, ds, \quad t \in T, \tag{5}$$

has the following properties:
  (a) $f_\alpha(\cdot)$ *is wAC on T;*
  (b) $f(\cdot, y(\cdot))$ *is a pseudo-derivative of $f_\alpha(\cdot)$;*
  (c) $f_\alpha(0) = 0$.
If E is a weakly sequentially complete space, then wAC is replaced by AC.

In the following, consider $\alpha \in (0, 1)$. If $y(\cdot) : T \to E$ is a pseudo-differentiable function with a pseudo-derivative $x(\cdot) \in P^\infty(T, E)$ on T, then the following fractional Pettis integral

$$I^{1-\alpha} x(t) = \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} x(s) \, ds$$

exists on T. The fractional Pettis integral $I^{1-\alpha} x(\cdot)$ is called a *fractional pseudo-derivative* of $y(\cdot)$ on T and it will be denoted by $D_p^\alpha y(\cdot)$; that is,

$$D_p^\alpha x(t) = I^{1-\alpha} x(t), \quad t \in T. \tag{6}$$

If $y(\cdot) : T \to E$ is an a.e. weakly differentiable function with the weak derivative $y'_w(\cdot) \in P^\infty(T, E)$ on T, then

$$D_w^\alpha y(t) := I^{1-\alpha} y'_w(t), \quad t \in T, \tag{7}$$

is called the *fractional weak derivative* of $y(\cdot)$ on T.

The following results will be useful (see [21] and [22]).

Remark 2.3 [21] If $y(\cdot) : T \to E$ is a pseudo-differentiable function with a pseudo-derivative $x(\cdot) \in P^\infty(T, E)$, then (a) $I^\alpha D_p^\alpha y(t) = y(t) - y(0)$ on T; (b) $D_p^\alpha I^\alpha y(t) = y(t)$ on T.

Remark 2.4 [21] The fractional Pettis integral is a linear operator from $P^\infty(T, E)$ into $P^\infty(T, E)$. Moreover, if $x(\cdot) \in P^\infty(T, E)$, then for $\alpha > 0$, $\beta > 0$, we have

$$I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t), \quad t \in T. \tag{8}$$

3 Fractional differential equations

In this section we establish an existence result for the fractional differential equation

$$\left(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i} \right) u(t) = f\big(t, u(t)\big) \quad \text{for } t \in [0,1], \qquad u(0) = 0, \tag{9}$$

where $D_p^{\alpha_m} y(\cdot)$ and $D_p^{\alpha_i} y(\cdot)$ are fractional pseudo-derivatives of the function $u(\cdot) : T \to E$ of order α_m and α_i, $i = 1, 2, \ldots, m-1$, respectively and $f(\cdot, \cdot) : T \times E \to E$ is a given function, $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m < 1$ and $a_1, a_2, \ldots, a_{m-1}$ are real numbers such that $a := \sum_{i=1}^{m-1} \frac{|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} < 1$. Along with Cauchy problem (9), consider the integral equation

$$u(t) = \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-s)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u(s) \, ds + \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f\big(s, u(s)\big) \, ds, \quad t \in T, \tag{10}$$

where the integral is in the sense of Pettis.

Definition 3.1 A *wAC* function (or an *AC* function, if E is a weakly sequentially complete space) $u(\cdot) : T \to E$ is said to be a solution of (9) if

(i) $u(\cdot)$ has pseudo-derivative of order α_i, $i = 1, 2, \ldots, n$,

(ii) the pseudo-derivative of $u(\cdot)$ of order α_i, $i = 1, 2, \ldots, n$, belongs to $P^\infty(T, E)$,

(iii) $(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}) u(t) = f(t, u(t))$ for all $t \in T$,

(iv) $u(0) = 0$.

Lemma 3.1 *Let $f(\cdot, \cdot) : T \times E \to E$ be a function such that $f(\cdot, u(\cdot)) \in P^\infty(T, E)$ for every wAC function $u(\cdot) : T \to E$. Then a wAC function $u(\cdot) : T \to E$ is a solution of (9) if and only if it satisfies the integral equation (10).*

Proof Indeed, if a *wAC* function $u(\cdot) : T \to E$ is a solution of (9), then from Remark 2.3(a) and Remark 2.4 it follows that $u(t) = \sum_{i=1}^{m-1} I^{\alpha_m - \alpha_i} a_i u(t) + I^{\alpha_m} f(t, u(t))$ on T; that is, $u(\cdot)$ satisfies the integral equation (10). Conversely, suppose that a *wAC* function $u(\cdot) : T \to E$ satisfies the integral equation (10). Since $f(\cdot, u(\cdot)) \in P^\infty(T, E)$, then from Proposition 2.5 it follows that the function $t \mapsto I^{\alpha_m} f(t, u(t))$ has a pseudo-derivative belonging to $P^\infty(T, E)$. Thus, using Remark 2.3(b), (10) and (6), we obtain that $(D^{\alpha_m} - \sum_{i=1}^{m-1} a_i D^{\alpha_i}) u(t) = f(t, u(t))$ on T. $\qquad\square$

Let us denote by $P_{wk}(E)$ the set of all weakly compact subsets of E. The weak measure of noncompactness is the set function $\beta : P_{wk}(E) \to \mathbb{R}^+$ defined by

$$\beta(A) = \inf\big\{ r > 0; \text{ there exist } K \in P_{wk}(E) \text{ such } A \subset K + r B_1 \big\},$$

where B_1 is the closed unit ball in E. The properties of weak noncompactness measure are analogous to the properties of measure of noncompactness. If $A, B \in P_{wk}(E)$,

(N_1) $A \subseteq B$ implies that $\beta(A) \le \beta(B)$;

(N_2) $\beta(A) = \beta(\mathrm{cl}_w(A))$, where $\mathrm{cl}_w(A)$ denotes the weak closure of A;

(N_3) $\beta(A) = 0$ if and only if $\mathrm{cl}_w(A)$ is weakly compact;

(N_4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$;

(N_5) $\beta(A) = \beta(\text{conv}(A))$;

(N_6) $\beta(A + B) \leq \beta(A) + \beta(B)$;

(N_7) $\beta(x + A) = \beta(A)$, for all $x \in E$;

(N_8) $\beta(\kappa A) = |\kappa|\beta(A)$, for all $\kappa \in \mathbb{R}$;

(N_9) $\beta(\bigcup_{0 \leq r \leq r_0} rA) = r_0\beta(A)$;

(N_{10}) $\beta(A) \leq 2\,\text{diam}(A)$.

Let $T = [0, a]$ and let $C(T, E)$ denote the space of all strong continuous functions $y(\cdot) : T \to E$, endowed with the supremum norm $\|y(\cdot)\|_c = \sup_{t \in T} \|y(t)\|$. Also, $C(T, E_w)$ denotes the space of all weakly continuous functions from T into E_w endowed with the topology of weak uniform convergence. It is known that (see [23, 24])

$$C(T, E)^* = M(T, E^*),$$

where $M(T, E^*)$ is the space of all bounded regular vector measures from $\mathcal{B}(T)$ into E^* which are of bounded variation. Here, $\mathcal{B}(T)$ denotes the σ-algebra of Borel measurable subsets of T.

Lemma 3.2 [25] *Let $Y \subset C(T, E)$ be bounded and equicontinuous. Then*

  (i) *the function $t \to \beta(Y(t))$ is continuous on T,*

  (ii) $\beta_c(Y) = \sup_{t \in T} \beta(Y(t)) = \beta(Y(t))$,

where $\beta_c(\cdot)$ denotes the weak measure of noncompactness in $C(T, E)$ and $Y(t) = \{u(t); u \in Y\}$, $t \in T$.

By a Kamke function we mean a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g(\cdot)$ is continuous, nondecreasing with $g(0) = 0$ and $u \equiv 0$ is the only solution of

$$u(t) \leq \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(u(s))\, ds, \qquad u(0) = 0. \tag{11}$$

We recall that a function $f(\cdot) : E \to E$ is said to be *sequentially continuous* from E_w *into* E_w (or *weakly-weakly sequentially continuous*) if for every weakly convergent sequence $\{x_n\}_{n \geq 1} \subset E$, the sequence $\{f(x_n)\}_{n \geq 1}$ is weakly convergent in E.

Theorem 3.1 *Let $r > 0$. Let $f(\cdot, \cdot) : T \times E \to E$ be a function such that:*

  (h1) $f(t, \cdot)$ *is weakly-weakly sequentially continuous for every $t \in T$;*

  (h2) $f(\cdot, u(\cdot))$ *is Pettis integrable for every wAC function $u(\cdot) : T \to E$;*

  (h3) $\|f(t, u)\| \leq M$ *for all $(t, u) \in T \times B_r$, where $B_r = \{u \in E; \|u - u_0\| \leq r\}$;*

  (h4) *for all $A \subseteq B_r$, we have*

$$\beta(f(T \times A)) \leq g(\beta(A)),$$

  where $g(\cdot)$ is a Kamke function. Then (9) admits a solution $u(\cdot)$ on an interval $[0, a_0]$ with

$$a_0 = \min\left\{1, \left[\frac{r(1 - a)\Gamma(\alpha_m + 1)}{M}\right]^{1/\alpha_m}\right\}.$$

Proof Let $\widetilde{B}_r$ denote the set of all *wAC* functions $u : [0, a_0] \to B_r$, and we consider the nonlinear operator $Q(\cdot)$ defined by

$$(Qu)(t) = \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-s)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u(s)\, ds + \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(s, u(s))\, ds$$

for all $t \in [0, a_0]$. Since the function $f(\cdot, u(t)) \in P^\infty(T, E)$ for every $u(\cdot) \in \widetilde{B}_r$, then by Proposition 2.5 it follows that the real function $t \mapsto (Qu)(t)$ is *wAC* on T for every $u(\cdot) \in \widetilde{B}_r$. Now, since

$$\left\| (Qu)(t) \right\|$$

$$= \left\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u(s)\, ds + \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f(s, u(s))\, ds \right\|$$

$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1}\, ds + \frac{M}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1}\, ds$$

$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M t^{\alpha_m}}{\Gamma(\alpha_m + 1)} \leq ra + (1-a)r,$$

we obtain that

$$\left\| (Qu)(t) \right\| \leq r.$$

Thus Q maps $\widetilde{B}_r$ into $\widetilde{B}_r$. Next, we show that Q is weakly-weakly sequentially continuous. For this let $\{u_n(\cdot)\}_{n \geq 1}$ be a sequence in $C([0, a_0], E)$ such that $u_n(\cdot) \overset{w}{\to} u(\cdot)$ as $n \to \infty$ (that is, $u_n(\cdot)$ converges weakly to $u(\cdot)$ in $C([0, a_0], E)$). Since $(C([0, a_0], E)^*) = M([0, a_0], E^*)$, it follows that

$$\langle m(\cdot), u_n(\cdot) - u(\cdot) \rangle \to 0 \quad \text{as } n \to \infty \tag{12}$$

for all $m(\cdot) \in M([0, a_0], E^*)$. Let $x^* \in E^*$ and $t \in [0, a_0]$. If we take $m(\cdot) = x^* \delta_t(\cdot)$, where δ_t is the Dirac measure concentrated in t, then $m(\cdot) \in M([0, a_0], E^*)$ and from (12) it follows that

$$\langle x^*, u_n(t) - u(t) \rangle \to 0 \quad \text{as } n \to \infty.$$

Therefore, for each $t \in [0, a_0]$, $u_n(t)$ converges weakly to $u(t)$ in E. Further, by (h1) it follows that $f(s, u_n(s))$ converges weakly to $f(s, u(s))$ in E for each $t \in [0, a_0]$. Hence, using Lemma 2.2 from [22], it follows that $I^\alpha u_n(\cdot)$ converges weakly to $I^\alpha u(\cdot)$ in E for all $t \in [0, a_0]$, and also we have that $\{u_n\}$ converges to u weakly. Then, by the Lebesgue dominated convergence theorem for Pettis integral (see [23]), we obtain

$$\lim_{n \to \infty} \langle m(\cdot), Qu_n(\cdot) - Qu(\cdot) \rangle = \lim_{n \to \infty} \int_0^{a_0} \left[\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u_n(s)\, ds \right.$$

$$\left. + \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f(s, u_n(s))\, ds \right.$$

$$- \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u(s)\, ds$$

$$- \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f\big(s, u(s)\big)\, ds \Bigg] dm(s),$$

$$\lim_{n \to \infty} \langle m(\cdot), Qu_n(\cdot) - Qu(\cdot) \rangle = \lim_{n \to \infty} \int_0^{a_0} \Bigg\{ \sum_{i=1}^{m-1} \int_0^t \frac{(t-s)^{\alpha_m - \alpha_i - 1} a_i [u_n(s) - u(s)]}{\Gamma(\alpha_m - \alpha_i)}\, ds$$

$$+ \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} \big[f\big(s, u_n(s)\big) - f\big(s, u(s)\big) \big]\, ds \Bigg\} dm(s)$$

$$= \lim_{n \to \infty} \int_0^{a_0} \Bigg[\frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f\big(s, u_n(s)\big)\, ds$$

$$- \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f\big(s, u(s)\big)\, ds \Bigg] dm(s)$$

$$= \lim_{n \to \infty} \int_0^{a_0} \big[I^\alpha u_n(t) - I^\alpha u(t) \big]\, dm(s) = 0$$

for all $m(\cdot) \in M([0, a_0], E^*)$. Therefore, Q is weakly-weakly continuous. Next, for any $n \geq 1$, we define the sequence $\{u_n(\cdot)\}_{n \geq 1}$ as follows:

$$u_n(t) = 0 \quad \text{if } 0 \leq t \leq \frac{a_0}{n},$$

$$u_n(t) = \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-s)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u_n(t)\, ds + \int_0^t \frac{(t-s)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f\big(s, u_n(t)\big)\, ds \quad \text{if } \frac{a_0}{n} \leq t \leq a_0.$$

Obviously, $u_n \in \widetilde{B}_r$ for all $n \geq 1$. Further, for all $n \geq 1$, if $t \in [0, a_o/n]$, then

$$\big\| u_n(t) - \big(Qu_n(t)\big) \big\| = \big\| (Qu_n)(t) \big\|$$

$$= \Bigg\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u_n(s)\, ds$$

$$+ \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f\big(s, u_n(s)\big)\, ds \Bigg\|$$

$$\leq \sum_{i=1}^{m-1} \frac{r|a_i| t^{\alpha_m - \alpha_i}}{\Gamma(\alpha_m - \alpha_i)(\alpha_m - \alpha_i)} + \frac{M t^{\alpha_m}}{\Gamma(\alpha_m)}$$

$$= \sum_{i=1}^{m-1} \frac{r|a_i| (a_0/n)^{\alpha_m - \alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M(a_0/n)^{\alpha_m}}{\Gamma(\alpha_m + 1)}.$$

If $t \in [a_0/n, a_0]$, then

$$\big\| u_n(t) - (Qu_n)(t) \big\| = \Bigg\| \Bigg(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t - a_0/n} (t-s)^{\alpha_m - \alpha_i - 1} u_n(s)\, ds$$

$$+ \frac{1}{\Gamma(\alpha_m)} \int_0^{t - a_0/n} (t-s)^{\alpha_m - 1} f\big(s, u_n(s)\big)\, ds$$

$$- \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u_n(s) \, ds$$

$$- \frac{1}{\Gamma(\alpha_m)} \int_0^t (t-s)^{\alpha_m - 1} f(s, u_n(s)) \, ds \Bigg) \Bigg\|$$

$$= \Bigg\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \Bigg[\int_0^t (t-s)^{\alpha_m - \alpha_i - 1} u_n(s) \, ds$$

$$- \int_0^{t - a_0/n} (t-s)^{\alpha_m - \alpha_i - 1} u_n(s) \, ds \Bigg]$$

$$+ \frac{1}{\Gamma(\alpha_m)} \Bigg[\int_0^t (t-s)^{\alpha_m - 1} f(s, u_n(s)) \, ds$$

$$- \int_0^{t - a_0/n} (t-s)^{\alpha_m - 1} f(s, u_n(s)) \, ds \Bigg] \Bigg\|$$

$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|(a_0/n)^{\alpha_m - \alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M(a_0/n)^{\alpha_m}}{\Gamma(\alpha_m + 1)}.$$

Thus we obtain that for all $t \in [0, a_0]$,

$$\lim_{n \to \infty} \| u_n - Q u_n \|_c = 0. \tag{13}$$

Let $V = \{u_n(\cdot); n \geq 1\}$ and $W = Q(V) = \{Q u_n(\cdot); n \geq 1\}$. If $s, t \in [0, a_0/n]$ and $s < t$, then

$$\| Q u_n(s) - Q u_n(t) \| = \Bigg\| \int_0^t \frac{(t - \tau)^{\alpha_m - 1} f(\tau, 0)}{\Gamma(\alpha_m)} \, d\tau - \int_0^s \frac{(s - \tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(\tau, 0) \, d\tau \Bigg\|$$

$$= \Bigg\| \int_0^s \frac{(t - \tau)^{\alpha_m - 1} f(\tau, 0)}{\Gamma(\alpha_m)} \, d\tau + \int_s^t \frac{(t - \tau)^{\alpha_m - 1} f(\tau, 0)}{\Gamma(\alpha_m)} \, d\tau$$

$$- \int_0^s \frac{(s - \tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(\tau, 0) \, d\tau \Bigg\|$$

$$\leq \frac{M}{\Gamma(\alpha_m)} \int_0^s \Big[(s - \tau)^{\alpha_m - 1} - (t - \tau)^{\alpha_m - 1} \Big] \, d\tau$$

$$+ \frac{M}{\Gamma(\alpha_m)} \int_s^t (t - \tau)^{\alpha_m - 1} \, d\tau$$

$$\leq \frac{2M}{\Gamma(\alpha_m + 1)} \Big[(t - s)^{\alpha_m} \Big].$$

If $s, t \in [a_0/n, a_0]$ and $s < t$, then

$$\| Q u_n(t) - Q u_n(s) \| = \Bigg\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \Bigg[\int_0^t (t - \tau)^{\alpha_m - \alpha_i - 1} u_n(\tau) \, d\tau$$

$$- \int_0^s (s - \tau)^{\alpha_m - \alpha_i - 1} u_n(\tau) \, d\tau \Bigg]$$

$$+ \frac{1}{\Gamma(\alpha_m)} \Bigg[\int_0^t (t - \tau)^{\alpha_m - 1} f(\tau, u_n(\tau)) \, d\tau$$

$$- \int_0^s (s - \tau)^{\alpha_m - 1} f(\tau, u_n(\tau)) \, d\tau \Bigg] \Bigg\|$$

$$\leq \left\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \left[\int_0^s \left[(s-\tau)^{\alpha_m - \alpha_i - 1} - (t-\tau)^{\alpha_m - \alpha_i - 1} \right] u_n(\tau) \, d\tau \right. \right.$$

$$\left. \left. - \int_s^t (t-\tau)^{\alpha_m - \alpha_i - 1} u_n(\tau) \, d\tau \right] \right\|$$

$$+ \left\| \frac{1}{\Gamma(\alpha_m)} \left[\int_0^s \left[(s-\tau)^{\alpha_m - 1} - (t-\tau)^{\alpha_m - 1} \right] f(\tau, u_n(\tau)) \, d\tau \right. \right.$$

$$\left. \left. - \int_s^t (t-\tau)^{\alpha_m - 1} f(\tau, u_n(\tau)) \, d\tau \right] \right\|$$

$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} \left[\int_0^s \left[(s-\tau)^{\alpha_m - \alpha_i - 1} - (t-\tau)^{\alpha_m - \alpha_i - 1} \right] d\tau \right]$$

$$+ \int_s^t (t-\tau)^{\alpha_m - \alpha_i - 1} \, d\tau$$

$$+ \frac{M}{\Gamma(\alpha_m)} \left[\int_0^s \left[(s-\tau)^{\alpha_m - 1} - (t-\tau)^{\alpha_m - 1} \right] d\tau \right.$$

$$\left. + \int_s^t (t-\tau)^{\alpha_m - 1} \, d\tau \right]$$

$$\leq 2 \left[\sum_{i=1}^{m-1} \frac{r|a_i|}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M}{\Gamma(\alpha_m + 1)} \right] (t-s)^{\alpha_m}.$$

If $s \in [0, a_0/n]$ and $t \in [a_0/n, a_0]$, then

$$\| Q u_n(s) - Q u_n(t) \| = \left\| \int_0^s \frac{(s-\tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(\tau, 0) \, ds - \sum_{i=1}^{m-1} \int_0^t \frac{a_i (t-\tau)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u_n(\tau) \, d\tau \right.$$

$$\left. - \int_0^t \frac{(t-\tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(\tau, u_n(\tau)) \, ds \right\|$$

$$= \left\| \frac{1}{\Gamma(\alpha_m)} \int_0^s \left[(t-\tau)^{\alpha_m - 1} - (s-\tau)^{\alpha_m - 1} \right] f(\tau, 0) \, d\tau \right.$$

$$+ \int_s^t \frac{(t-\tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} f(\tau, u_n(\tau)) \, ds$$

$$\left. + \sum_{i=1}^{m-1} \int_s^t \frac{a_i (t-\tau)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} u_n(\tau) \, d\tau \right\|$$

$$\leq \frac{M}{\Gamma(\alpha_m)} \int_0^s \left[(t-\tau)^{\alpha_m - 1} - (s-\tau)^{\alpha_m - 1} \right] d\tau$$

$$+ \int_s^t \frac{M(t-\tau)^{\alpha_m - 1}}{\Gamma(\alpha_m)} \, d\tau + \sum_{i=1}^{m-1} \int_s^t \frac{|a_i| r (t-\tau)^{(\alpha_m - \alpha_i - 1)}}{\Gamma(\alpha_m - \alpha_i)} \, d\tau$$

$$\leq \frac{M}{\Gamma(\alpha_m + 1)} \left[s^{\alpha_m} + (t-s)^{\alpha_m} - t^{\alpha_m} + (t-s)^{\alpha_m} \right]$$

$$+ \frac{r \sum_{i=1}^{m-1}}{\Gamma(\alpha_m - \alpha_i + 1)} (t-s)^{\alpha_m - \alpha_i}$$

$$\leq \left[\frac{2M}{\Gamma(\alpha_m + 1)} + \frac{r \sum_{i=1}^{m-1}}{\Gamma(\alpha_m - \alpha_i + 1)} \right] (t-s)^{\alpha_m}.$$

It follows from the above three cases that W is equicontinuous, and by (13) we obtain that V is also equicontinuous. By virtue of Lemma 3.2 and (13), we have that

$$\beta_c(V - W) = \beta\big((I - Q)(V)\big)$$
$$= \sup_{t \in [0, a_0]} \beta(I - Q)\big(V(t)\big) = 0.$$

Since for all $t \in [0, a_0]$ we have that $V(t) \subseteq V(t) - W(t) + W(t)$, it follows that

$$\beta\big(V(t)\big) \leq \beta\big(W(t)\big) \quad \text{for } t \in [0, a_0].\tag{14}$$

On the other hand, for all $t \in [0, a_0]$, we have $W(t) \subseteq W(t) - V(t) + V(t) = V(t) - (I - Q)(V(t))$, and so

$$\beta\big(W(t)\big) \leq \beta\big(V(t)\big) \quad \text{for } t \in [0, a_0].\tag{15}$$

From (14) and (15), we infer that

$$\beta\big(W(t)\big) = \beta\big(V(t)\big) \quad \text{for } t \in [0, a_0].\tag{16}$$

Further, fix $t \in T_0$, $\varepsilon > 0$ and choose $\eta > 0$ such that $\eta < \big(\frac{\varepsilon \Gamma(\alpha_m+1)}{M+r\Gamma(\alpha_m+1)}\big)^{1/\alpha_m}$. Then

$$\left\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} u(s)\, ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f\big(s, u(s)\big)\, ds \right\|$$
$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m - \alpha_i}}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m + 1)}$$
$$\leq \sum_{i=1}^{m-1} \frac{r|a_i|\eta^{\alpha_m}}{\Gamma(\alpha_m - \alpha_i + 1)} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m + 1)}$$
$$\leq r\eta^{\alpha_m} + \frac{M\eta^{\alpha_m}}{\Gamma(\alpha_m + 1)}$$
$$\leq \frac{M + r\Gamma(\alpha_m + 1)}{\Gamma(\alpha_m + 1)}\eta^{\alpha_m} < \varepsilon.$$

Hence we conclude that

$$\left\| \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s)\, ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f\big(s, u(s)\big)\, ds \right\| < \varepsilon.$$

Using the property (N_{10}) of noncompactness measure, we infer

$$\beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s)\, ds + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f\big(s, V(s)\big)\, ds \right)$$
$$\leq 2\varepsilon.\tag{17}$$

Since by Lemma 3.2 the function $t \to \nu(t) := \beta(V(t))$ is continuous on $[0, t - \eta]$, it follows that $s \to (t-s)^{\alpha_m-1}g(V(s))$ is continuous on $[0, t - \eta]$. Hence, there exists $\delta > 0$ such that

$$\left\| (t - \tau)^{\alpha_m-1}g\big(V(\tau)\big) - (t-s)^{\alpha_m-1}g\big(V(s)\big) \right\| < \frac{\varepsilon}{2}$$

and

$$\left\| g\big(V(\xi)\big) - g\big(V(\tau)\big) \right\| < \frac{\varepsilon}{2\eta^{\alpha_m-1}}.$$

If $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$ with $\tau, s, \xi \in [0, t - \eta]$, then it follows that

$$
\begin{aligned}
\left| (t-\tau)^{\alpha_m-1}g\big(V(\xi)\big) - (t-s)^{\alpha_m-1}g\big(V(s)\big) \right| &\le \left| (t-\tau)^{\alpha_m-1}g\big(V(\tau)\big) - (t-s)^{\alpha_m-1}g\big(V(s)\big) \right| \\
&\quad + (t-\tau)^{\alpha_m-1}\left| g\big(V(\xi)\big) - g\big(V(\tau)\big) \right| \\
&< \varepsilon,
\end{aligned}
$$

that is,

$$\left| (t-\tau)^{\alpha_m-1}g\big(V(\xi)\big) - (t-s)^{\alpha_m-1}g\big(V(s)\big) \right| < \varepsilon \tag{18}$$

for all $\tau, s, \xi \in [0, t - \xi]$ with $|\tau - s| < \delta$ and $|\tau - \xi| < \delta$. Consider the following partition of the interval $[0, t - \eta]$ into n parts $0 = t_0 < t_1 < \cdots < t_n = t - \eta$ such that $t_{i-1} - t_i < \delta$ ($i = 1, 2, \ldots, n$). By Lemma 3.2, for each i, there exists $s_i \in [t_{i-1}, t_i]$ such that $\beta(V([t_{i-1}, t_i])) = \nu(s_i)$, $i = 1, 2, \ldots, n$. Then we have (see [26], Theorem 2.2)

$$
\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1}V(s)\,ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m-1}f\big(s, V(s)\big)\,ds
$$

$$
\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1}V(s)\,ds + \sum_{j=1}^{n} \frac{1}{\Gamma(\alpha_m)} \int_{t_{i-1}}^{t_i} (t-s)^{\alpha_m-1}f\big(s, V(s)\big)\,ds
$$

$$
\subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m-\alpha_i-1}V(s)\,ds
$$

$$
+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1})\overline{\mathrm{conv}}\big\{ (t-s)^{\alpha_m-1}f\big(s, u(s)\big); s \in [t_{i-1}, t_i], u \in V \big\},
$$

and so

$$
\beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1}V(s)\,ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m-1}f\big(s, V(s)\big)\,ds \right)
$$

$$
\le \beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m-\alpha_i-1}V(s)\,ds \right)
$$

$$
+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1})\beta\big(\overline{\mathrm{conv}}\big\{ (t-s)^{\alpha_m-1}f\big(s, u(s)\big); s \in [t_{i-1}, t_i], u \in V \big\}\big)
$$

$$= \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds \right)$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1}) \beta \left(\left\{ (t-s)^{\alpha_m - 1} f(s, u(s)); s \in [t_{i-1}, t_i], u \in V \right\} \right)$$

$$\leq \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds \right)$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha_m - 1} \beta \left(f \left([0, a_0] \times V[t_{i-1}, t_i] \right) \right)$$

$$\leq \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds \right)$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha_m - 1} g \left(V[t_{i-1}, t_i] \right)$$

$$\leq \beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds \right)$$

$$+ \frac{1}{\Gamma(\alpha_m)} \sum_{i=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha_m - 1} g \left(V(s_i) \right).$$

Using (18) we have that

$$\left| (t - t_i)^{\alpha_m - 1} g \left(V(s_i) \right) - (t - s)^{\alpha_m - 1} g \left(V(s) \right) \right| < \varepsilon.$$

This implies that

$$\frac{1}{\Gamma(\alpha_m)} \sum_{j=1}^{n} (t_i - t_{i-1})(t - t_i)^{\alpha_m - 1} g \left(V(s_i) \right)$$

$$\leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} g \left(V(s) \right) \, ds + \varepsilon (t - \eta) / \Gamma(\alpha_m). \tag{19}$$

By using (17) we claim that

$$\beta \left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds \right) \leq 2\varepsilon. \tag{20}$$

If we let

$$A(t) = \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f(s, V(s)) \, ds,$$

$$B(t) = \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s) \, ds,$$

then $a + B(t) \subset A(t) + B(t)$ for $a \in A(t)$, which implies that $\beta(B(t)) \leq \beta(A(t) + B(t)) < 2\varepsilon$. From relations (19) and (20), we obtain

$$\beta\left(\sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s)\,ds + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} f\big(s, V(s)\big)\,ds\right)$$

$$\leq 2\varepsilon + \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} g\big(V(s)\big)\,ds + \varepsilon(t-\eta)/\Gamma(\alpha_m). \qquad (21)$$

Since

$$(QV)(t) \subset \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_0^{t-\eta} (t-s)^{\alpha_m - \alpha_i - 1} V(s)\,ds$$

$$+ \sum_{i=1}^{m-1} \frac{a_i}{\Gamma(\alpha_m - \alpha_i)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - \alpha_i - 1} V(s)\,ds$$

$$+ \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} f\big(s, V(s)\big)\,ds$$

$$+ \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f\big(s, V(s)\big)\,ds,$$

then by virtue of (17) and (21), we have

$$\beta\big((QV)(t)\big) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t-\eta} (t-s)^{\alpha_m - 1} g\big(V(s)\big)\,ds + \varepsilon(t-\eta)/\Gamma(\alpha_m) + 4\varepsilon$$

$$\leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g\big(V(s)\big)\,ds + \varepsilon\big((t+4)/\Gamma(\alpha_m)\big).$$

As the last inequality is true for every $\varepsilon > 0$, we infer

$$\beta\big((QV)(t)\big) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g\big(V(s)\big)\,ds, \quad t \in [0, a_0],$$

and thus by using (16) it follows that

$$V(t) \leq \frac{1}{\Gamma(\alpha_m)} \int_0^{t} (t-s)^{\alpha_m - 1} g\big(V(s)\big)\,ds \quad \text{for } t \in [0, a_0].$$

Since $g(\cdot)$ is a Kamke function, then $V(t) = 0$ for $t \in T_0$. Using Lemma 3.2, we infer

$$\beta_c\big(V(T_0)\big) = \sup_{t \in T_0} \beta\big(V(t)\big) = 0.$$

Thus V is relatively compact in $C(T_0, E_w)$. Therefore, taking a subsequence if necessary, we can assume that $\{u_n(\cdot)\}_{n \geq 1}$ converges weakly in $\widetilde{B}_r$ to a function $u(\cdot)$. Since Q is weakly-weakly sequentially continuous, then $\{u_n - Qu_n\}_{n \geq 1}$ converges weakly to $u - Qu$ in $C(T_0, E_w)$. Recalling that the norm is weakly lower semicontinuous [27], we obtain that

$$\lim_{n \to \infty} \sup \|u_n - Qu_n\|_c \geq \|u - Qu\|_c.$$

Then from (13) it follows that $\|u - Qu\|_c = 0$, and so

$$u(t) = u_0 + \frac{1}{\Gamma(\alpha_m)} \int_{t-\eta}^{t} (t-s)^{\alpha_m - 1} f\big(s, u(s)\big)\, ds, \quad t \in T_0.$$

Using Lemma 3.1, we conclude that $u(\cdot)$ is a solution of (9). $\qquad\square$

Remark 3.1 If $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m = \alpha \in (0,1)$, then we obtain Theorem 5.1 from [21].

Let E be a weakly sequentially complete space. It is known that if $f(\cdot, \cdot)$ is a continuous function from $T \times E_w$ into E_w, then the function $t \mapsto f(t, y(t))$ is Pettis integrable for every AC function $y(\cdot) : E \to E$ (see [28, Lemma 15]). Therefore, in the case of weakly sequentially complete spaces, we obtain the following result (see also [29]).

Corollary 3.1 *Let E be a weakly sequentially complete space. If $f(\cdot, \cdot) : T \times E \to E$ is a continuous function from $T \times E_w$ into E_w and conditions* (h3)-(h4) *in Theorem* 3.1 *hold, then* (9) *admits a solution $y(\cdot)$ on an interval $[0, a]$ with*

$$a_0 = \min\left\{ 1, \left[\frac{r(1-a)\Gamma(\alpha_m + 1)}{M} \right]^{1/\alpha_m} \right\}.$$

Remark 3.2 If $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m = 1$, then we obtain some known results. In this case, Corollary 3.1 is a generalization of a result from [30] and [25]. Also, for any Banach space, the following result is a generalization of Theorem 2.1 in [31] (see also [13, 32–34]) for $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m = 1$.

Corollary 3.2 *If $f : T \times E \to E$ is a continuous function from $T \times E_w$ into E_w such that* $\mathrm{cl}_w f(T \times B) \in \mathcal{P}_{wk}(E)$, *then* (9) *admits a solution on $[0, a]$ with*

$$a_0 = \min\left\{ 1, \left[\frac{r(1-a)\Gamma(\alpha_m + 1)}{M} \right]^{1/\alpha_m} \right\}.$$

If E is a reflexive Banach space, it is not necessary to assume any compactness conditions since in this case a subset of E is weakly compact if and only if it is weakly closed and norm bounded. Thus, arguing similarly as in the proof of Theorem 3.1, we obtain the following result.

Corollary 3.3 [35] *Let E be a reflexive Banach space. If $f : T \times E \to E$ is a continuous function from $T \times E_w$ into E_w such that $\|f(t, y)\| \leq M$ for all $(t, y) \in T \times B_r$, then* (9) *admits a solution on $[0, a]$ with*

$$a_0 = \min\left\{ 1, \left[\frac{r(1-a)\Gamma(\alpha_m + 1)}{M} \right]^{1/\alpha_m} \right\}.$$

Remark 3.3 If $\alpha_1 = \alpha_2 = \cdots = \alpha_{m-1} = 0$ and $\alpha_m = 1$, then we obtain Theorem 8 from [30], Theorem 3.1 from [25], and using the conditions of Corollary 3.3, we obtain some known results from [36].

Competing interests

All authors declare that they have no competing interests.

Authors' contributions

All authors carried out the proof. All authors conceived of the study, and participated in its design and coordination. All authors read and approved the final manuscript.

Author details

[1]Department of Mathematics, Texas A&M University-Kingsville, Kingsville, TX, USA. [2]Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah, 21859, Saudi Arabia. [3]Constantin Brancusi University, Republicii 1, Targu-Jiu, 210152, Romania. [4]School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland. [5]Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan.

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