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Solutions of the boundary value problem for a $2n$ th-order nonlinear difference equation containing both advance and retardation

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Abstract

In this paper, we consider the boundary value problem for a $2n$ th-order nonlinear difference equation containing both advance and retardation. By using the critical point theory, some sufficient conditions of the existence of solutions of the boundary value problem are obtained. The proof is based on the linking theorem. An example is given to illustrate our results.

Keywords: boundary value problem; difference equation; linking theorem; $2n$ th-order

1 Introduction

Consider the following $2n$ th-order nonlinear difference equation:

$$\Delta^n(r_{t-n}\Delta^n x_{t-n}) + (-1)^{n+1}f(t, x_{t+1}, x_t, x_{t-1}) = 0, \quad t = 1, 2, \dots, T, \quad (1.1)$$

with boundary value conditions

$$x_j = 0, \quad j = 1 - n, 2 - n, \dots, 0, T + 1, \dots, T + n, \quad (1.2)$$

where T and n are given positive integers with $T > n$, $f(t, \cdot) : \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous for $t = 1, 2, \dots, T$, $r_{1-n}, r_{2-n}, \dots, r_T$ are nonzero real numbers and Δ is the forward difference operator defined by $\Delta x_n = x_{n+1} - x_n$, $\Delta^2 x_n = \Delta(\Delta x_n)$.

In the last decade, by using various techniques such as critical point theory, fixed point theory, topological degree theory and coincidence degree theory, a great deal of works have been done on the existence of solutions to boundary value problems of difference equations (see [1–7] and references therein). Among these approaches, the critical point theory seems to be a powerful tool to solving this problem (see [5, 7–9]). However, compared to the boundary value problems of lower order difference equations [6, 8, 10–13], the study of boundary value problems of higher order difference equations is relatively less (see [9, 14, 15]), especially the works done by using the critical point theory. Therefore, there is still spacious room to explore the boundary value problems of higher-order difference equations. For the background on difference equations, we refer to [16].

Recently, in [17], the author considered the $2n$ th-order nonlinear difference equation

$$\Delta^n (r_{t-n} \Delta^n x_{t-n}) + (-1)^{n+1} f(t, x_t) = 0, \quad t = 1, 2, \dots, T, \tag{1.3}$$

with the boundary value conditions (1.2). He distinguished two cases that $f(t, z)$ is super-linear and sublinear in the second variable z , respectively. Equation (1.3) represents a class of no-delay difference equations.

In this paper, we consider the existence of solutions of the boundary value problem of (1.1) with (1.2). First, we construct a functional J such that solutions of boundary value problem (1.1) with (1.2) correspond to critical points of J . Then, by using the linking theorem, we obtain the existence of critical points of J . Finally, an example is given to illustrate our results. We mention that (1.1) is a kind of difference equation containing both advance and retardation. This kind of difference equation has many applications both in theory and practice. For example, as a special case of $n = 1$, (1.1) may be thought to be the discrete analogue of the following forward and backward differential difference equation:

$$c^2 u''(t) = V'(u(t+1) - u(t)) - V'(u(t) - u(t-1)), \quad t \in \mathbb{R}. \tag{1.4}$$

Equation (1.4) has been studied extensively by many scholars. For example, Smets and Willem [18] obtained the existence of solitary waves of (1.4).

In [7], the authors considered (1.1) for the case when $n = 1$ with different boundary value conditions

$$\Delta x_0 = A, \quad x_{T+1} = B. \tag{1.5}$$

In [16], Agarwal took the following difference equation:

$$-\omega^2 M u(k) + f(-u(k-1) + 2u(k) - u(k+1)) = 0 \tag{1.6}$$

with the boundary value conditions

$$u(0) = u(k+1) = 0 \tag{1.7}$$

as an example. It represents the amplitude of the motion of every particle in the string.

Throughout this paper, we assume that there exists a function $F(t, x, y)$ which is differentiable in variable x and y defined for $t = 0, 1, \dots, T$, satisfies

$$\frac{\partial F(t-1, y, z)}{\partial y} + \frac{\partial F(t, x, y)}{\partial y} = f(t, x, y, z), \quad \forall x, y, z \in \mathbb{R}, \tag{1.8}$$

for $t = 1, 2, \dots, T$.

2 Preliminaries and main results

Before we apply the critical point theory, we shall establish the corresponding variational framework for (1.1) with (1.2).

Let \mathbb{N} , \mathbb{Z} and \mathbb{R} denote the sets of all natural numbers, integers and real numbers, respectively. For $a, b \in \mathbb{Z}$, define $\mathbb{Z}(a) = \{a, a+1, \dots\}$, $\mathbb{Z}(a, b) = \{a, a+1, \dots, b\}$ when $a \leq b$.

Let

$$E = \{x = \{x_t\} : \mathbb{Z}(1-n, T+n) \rightarrow \mathbb{R} | x_{1-i} = x_{T+i} = 0, i \in \mathbb{Z}(1, n)\}. \tag{2.1}$$

Then E is a T -dimensional Hilbert space.

Obviously, E is isomorphic to \mathbb{R}^T . In fact, we can find a map $I : E \rightarrow \mathbb{R}^T$ defined by

$$I : \{x_t\} \rightarrow (x_1, x_2, \dots, x_T)^{\text{tr}},$$

where \cdot^{tr} means the transpose of \cdot .

Define the inner product on E as

$$\langle x, y \rangle = \sum_{i=1}^T x_i y_i, \quad \forall x, y \in E. \tag{2.2}$$

The corresponding norm $\|\cdot\|$ can be induced by

$$\|x\| = \left(\sum_{t=1}^T |x_t|^2 \right)^{\frac{1}{2}}, \quad \forall x \in E. \tag{2.3}$$

For all $x \in E$, define the functional $J(x)$ on E as follows:

$$J(x) = \frac{1}{2} \sum_{t=1-n}^T r_t (\Delta^n x_t)^2 - \sum_{t=0}^T F(t, x_{t+1}, x_t). \tag{2.4}$$

Clearly, $J(x) \in C^1(E, \mathbb{R})$.

By using the fact that

$$\Delta^n x_{t-n} = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i} x_{t-n+i}, \tag{2.5}$$

we can compute the partial derivative as

$$\frac{\partial J(x)}{\partial x_t} = (-1)^n \Delta^n (r_{t-n} \Delta^n x_{t-n}) - f(t, x_{t+1}, x_t, x_{t-1}) \tag{2.6}$$

for $t \in \mathbb{Z}(1, T)$. Therefore, $x \in E$ is a critical point of J if and only if x is a solution of (1.1) with (1.2).

By a simple computation, we see that there exists a $T \times T$ matrix A such that

$$\sum_{t=1-n}^T r_t (\Delta^n x_t)^2 = (AI(x), I(x)) \tag{2.7}$$

for all $x \in E$. Therefore, we can write $J(x)$ as

$$J(x) = \frac{1}{2} (AI(x), I(x)) - F(x), \tag{2.8}$$

where $F(x) = \sum_{t=0}^T F(t, x_{t+1}, x_t)$.

By a straightforward calculation, we obtain that

$$\text{tr}A = \sum_{i=1-n}^0 r_i \sum_{j=0}^{i+n-1} \binom{n}{j}^2 + \sum_{i=1}^{T-n} r_i \sum_{j=0}^{T-n} \binom{n}{j}^2 + \sum_{i=T-n+1}^T r_i \sum_{j=0}^{T-i} \binom{n}{j}^2. \tag{2.9}$$

Let $\lambda(A)$, $\lambda^+(A)$ and $\lambda^-(A)$ be the sets of all the eigenvalues of A , all the positive eigenvalues of A and all the negative eigenvalues of A , respectively. Define $\lambda_{\max} = \max\{\lambda : \lambda \in \lambda(A)\}$, $\lambda_{\min} = \min\{\lambda : \lambda \in \lambda(A)\}$. If $\lambda^+(A) \neq \emptyset$, let $\lambda_{\max}^+ = \max\{\lambda : \lambda \in \lambda^+(A)\}$, $\lambda_{\min}^+ = \min\{\lambda : \lambda \in \lambda^+(A)\}$. Let A^0 , A^+ and A^- denote the eigenspaces associated with the 0 eigenvalue, all positive eigenvalues and all negative eigenvalues of A , respectively. It follows that

$$\frac{1}{2}\lambda_{\min}\|x\|^2 - F(x) \leq J(x) \leq \frac{1}{2}\lambda_{\max}\|x\|^2 - F(x) \tag{2.10}$$

and

$$\mathbb{R}^T = A^0 \oplus A^+ \oplus A^-. \tag{2.11}$$

Definition 2.1 Let E be a real Banach space, the functional $J \in C^1(E, \mathbb{R})$ is said to satisfy the Palais-Smale (P.S. for short) condition if any sequence $\{x_m\}$ in E , such that $\{J(x_m)\}$ is bounded and $J'(x_m) \rightarrow 0$ as $m \rightarrow \infty$, contains a convergent subsequence.

Let B_ρ denote the open ball in E with radius ρ and center 0, and let ∂B_ρ denote its boundary.

In order to obtain the existence of critical points of J on E , we cite the basic lemma, which is crucial in the proof of our main results.

Lemma 2.1 (Linking theorem [19]) *Let E be a real Hilbert space and $E = E_1 \oplus E_2$, where E_1 is a finite-dimensional subspace of E . Assume that $J \in C^1(E, \mathbb{R})$ satisfies the P.S. condition and the following two conditions.*

- (J₁) *There exist constants $a > 0$ and $\rho > 0$ such that $J|_{\partial B_\rho \cap E_2} \geq a$;*
- (J₂) *There exist an $e \in \partial B_1 \cap E_2$ and a constant $R_0 > \rho$ such that $J|_{\partial Q} \leq 0$, where $Q \triangleq (B_{R_0} \cap E_1) \oplus \{re | 0 < r < R_0\}$.*

Then J possesses a critical value $c \geq a$. Moreover, c can be characterized as

$$c = \inf_{h \in \Gamma} \max_{x \in \bar{Q}} J(h(x)),$$

where $\Gamma = \{h \in C(\bar{Q}, E) : h|_{\partial Q} = \text{id}_{\partial Q}\}$ and $\text{id}_{\partial Q}$ is the identity operator on ∂Q .

Now we state our main results.

Theorem 2.1 *Assume that the following conditions are satisfied:*

(A1)

$$\sum_{i=1-n}^0 r_i \sum_{j=0}^{i+n-1} \binom{n}{j}^2 + \sum_{i=1}^{T-n} r_i \sum_{j=0}^{T-n} \binom{n}{j}^2 + \sum_{i=T-n+1}^T r_i \sum_{j=0}^{T-i} \binom{n}{j}^2 > 0. \tag{2.12}$$

(A2)

$$\liminf_{x^2+y^2 \rightarrow \infty} \frac{F(t, x, y)}{x^2 + y^2} > \frac{1}{4} \lambda_{\max}, \quad t \in \mathbb{Z}(0, T). \quad (2.13)$$

(A3)

$$\lim_{(x,y) \rightarrow (0,0)} \frac{F(t, x, y)}{x^2 + y^2} = 0, \quad t \in \mathbb{Z}(0, T). \quad (2.14)$$

Then (1.1) with (1.2) possesses at least two nontrivial solutions.

Corollary 2.1 Assume that (A1) holds, and $F(t, x, y)$ satisfies (A3) and (A4) There exist constants $R > 0$ and $\beta > 2$ such that

$$\begin{aligned} 0 < \beta F(t, x, y) \\ \leq \frac{\partial F(t, x, y)}{\partial x} x + \frac{\partial F(t, x, y)}{\partial y} y, \quad \forall \sqrt{x^2 + y^2} \geq R \text{ and } t \in \mathbb{Z}(0, T). \end{aligned} \quad (2.15)$$

Then (1.1) with (1.2) possesses at least two nontrivial solutions.

Remark 2.1 If $r_i > 0$ for $i \in \mathbb{Z}(1 - n, T)$, then (2.12) clearly holds.

Remark 2.2 Assumption (A4) is a condition similar to the classical Ambrosetti-Rabinowitz superlinear condition [19], which implies that there exist constants $a_1 > 0$ and $a_2 > 0$ such that

$$F(t, x, y) \geq a_1 (\sqrt{x^2 + y^2})^\beta - a_2, \quad \forall (t, x, y) \in \mathbb{Z}(0, T) \times \mathbb{R}^2, \quad (2.16)$$

which leads to

$$\min_{t \in \mathbb{Z}(0, T)} \liminf_{x^2+y^2 \rightarrow \infty} \frac{F(t, x, y)}{x^2 + y^2} = +\infty,$$

and (A2) is satisfied. So Corollary 2.1 holds.

3 Proof of Theorem 2.1

Lemma 3.1 Assume that (A2) holds, then the functional J satisfies the P.S. condition.

Proof According to (A2), there exist positive constants $a_3 > \lambda_{\max}/4$ and a_4 such that

$$F(t, x, y) \geq a_3 (x^2 + y^2) - a_4, \quad \forall (t, x, y) \in \mathbb{Z}(0, T) \times \mathbb{R}^2. \quad (3.1)$$

Let $\{x^{(k)}\}$ be a sequence in E such that $\{J(x^{(k)})\}$ is bounded and $J'(x^{(k)}) \rightarrow 0$ as $k \rightarrow \infty$. Then there exists a positive constant C such that $|J(x^{(k)})| \leq C$.

By (2.8) and (2.10), we have

$$\begin{aligned}
 J(x^{(k)}) &= \frac{1}{2} (Ax^{(k)}, x^{(k)}) - F(x^{(k)}) \\
 &\leq \frac{1}{2} \lambda_{\max} \|x^{(k)}\|^2 - a_3 \sum_{t=0}^T ((x_t^{(k)})^2 + (x_{t+1}^{(k)})^2) + (T+1)a_4 \\
 &\leq \frac{1}{2} (\lambda_{\max} - 4a_3) \|x^{(k)}\|^2 + (T+1)a_4.
 \end{aligned} \tag{3.2}$$

On the other hand, $J(x^{(k)}) \geq -C$. Combining this with (3.2), we have

$$\|x^{(k)}\|^2 \leq \frac{2(T+1)a_4 + 2C}{4a_3 - \lambda_{\max}}. \tag{3.3}$$

Since E is a finite-dimensional space, (3.3) implies $\{x^{(k)}\}$ has a convergent subsequence. Thus P.S. condition is verified. \square

Now we give the proof of Theorem 2.1.

Proof According to (A3), it is easy to see that $F(t, 0, 0) = 0$. Therefore, $\{0\}$ is a solution of (1.1) with (1.2) which is called a trivial solution. From (A1), $\text{tr}(A) > 0$, then we conclude that matrix A has at least one positive eigenvalue and $\lambda_{\max} = \lambda_{\max}^+$. By (3.2), we have

$$J(x) \leq \frac{1}{2} (\lambda_{\max}^+ - 4a_3) \|x\|^2 + (T+1)a_4.$$

This implies that $J(x) \rightarrow -\infty$ as $\|x\| \rightarrow \infty$, then $-J(x)$ is a coercive map. It implies that J is bounded above on E . By the continuity of J on E , there exists $m \in E$ such that

$$J(m) = \max \{J(x) | x \in E\} = \bar{c}.$$

Clearly, m is a critical point of J on E . We prove that m is nontrivial.

In fact, from (A3), there exists $\rho_0 > 0$ such that, for any $(x, y) \in \mathbb{R}^2$ with $x^2 + y^2 \leq \rho_0^2$ and $t \in \mathbb{Z}(0, T)$, we have

$$F(t, x, y) \leq \frac{1}{8} \lambda_{\min}^+ (x^2 + y^2).$$

For any $x \in E$ with $I(x) \in A^+$ and $\|x\| = \rho_0$, by (2.10), it is clear that

$$\begin{aligned}
 J(x) &\geq \frac{1}{2} \lambda_{\min}^+ \|x\|^2 - \frac{1}{8} \lambda_{\min}^+ \sum_{t=0}^T (x_t^2 + x_{t+1}^2) \\
 &= \frac{1}{2} \lambda_{\min}^+ \|x\|^2 - \frac{1}{4} \lambda_{\min}^+ \|x\|^2 \\
 &= \frac{1}{4} \lambda_{\min}^+ \|x\|^2 \\
 &= \frac{1}{4} \lambda_{\min}^+ \rho_0^2.
 \end{aligned}$$

Therefore, $\bar{c} = \max\{J(x)|x \in E\} \geq \frac{1}{4}\lambda_{\min}^+\rho_0^2 > 0$, which implies that $m \neq 0$. That is, m is a nontrivial critical point of J .

Let $E_2 = I^{-1}(A^+)$, $E_1 = I^{-1}(A^- \oplus A^0)$ and $a = \frac{1}{4}\lambda_{\min}\rho_0^2 > 0$, then we have

$$E = E_1 \oplus E_2,$$

and

$$J(x)|_{\partial B_{\rho_0} \cap E_2} \geq a.$$

Let $e \in \partial B_1 \cap E_2$, for any $z \in E_1$ and $r \in \mathbb{R}$, we have

$$\begin{aligned} J(re + z) &= \frac{1}{2}(AI(re + z), I(re + z)) - F(re + z) \\ &\leq \frac{1}{2}r^2(AI(e), I(e)) + \frac{1}{2}(AI(z), I(z)) - 2a_3\|re + z\|^2 + (T + 1)a_4 \\ &\leq \frac{1}{2}\lambda_{\max}^+r^2 - 2a_3(r^2 + \|z\|^2) + (T + 1)a_4 \\ &= \frac{1}{2}(\lambda_{\max}^+ - 4a_3)r^2 - 2a_3\|z\|^2 + (T + 1)a_4. \end{aligned}$$

Since $J(z) \leq 0$ for $z \in E_1$, we see that there exists a positive constant $R_0 > \rho_0$ such that $J(x) \leq 0$ for any $x \in \partial Q$, where $Q \triangleq (B_{R_0} \cap E_1) \oplus \{re|0 < r < R_0\}$. By the linking theorem, J possesses a critical value $\tilde{c} \geq a$, where

$$\tilde{c} = \inf_{h \in \Gamma} \max_{x \in Q} J(h(x)) \quad \text{and} \quad \Gamma = \{h \in C(\bar{Q}, \mathbb{R}^T) : h|_{\partial Q} = \text{id}_{\partial Q}\}.$$

For convenience, we note $J(m^*) = \tilde{c}$, $m^* \in E$. Obviously, $m^* \neq \{0\}$.

If $m \neq m^*$, then the proof is finished. Otherwise, $\bar{c} = J(m) = J(m^*) = \tilde{c}$. Since e is arbitrary, we can choose $-e \in \partial B_1 \cap E_2$. Similarly, there exists $R_* > \rho_0$ such that $J(x) \leq 0$ for any $x \in \partial Q_1$, where $Q_1 \triangleq (B_{R_*} \cap E_1) \oplus \{-re|0 < r < R_*\}$. Again, by the linking theorem, J possesses a critical value $\hat{c} \geq a$, where

$$\hat{c} = \inf_{h \in \Gamma_1} \max_{x \in Q_1} J(h(x)) \quad \text{and} \quad \Gamma_1 = \{h \in C(\bar{Q}_1, \mathbb{R}^T) : h|_{\partial Q_1} = \text{id}_{\partial Q_1}\}.$$

Note $J(\hat{m}) = \hat{c}$. We will assert that $m^* \neq \hat{m}$.

By way of contradiction, assume that $m^* = \hat{m}$. Then $\hat{c} = \tilde{c} = \bar{c} = \max\{J(x)|x \in E\}$. Since $J(x)|_{\partial Q} \leq 0$ and $J(x)|_{\partial Q_1} \leq 0$, \hat{m} must be an interior point of both sets Q and Q_1 . However, $Q \cap Q_1 \subset E_1$ and $J(x) \leq 0$ for any $x \in E_1$. This implies that $\bar{c} < 0$. It is a contradiction, so the conclusion of Theorem 2.1 holds.

Now the proof is complete. □

4 Example

As an application of Theorem 2.1, we give an example to illustrate our result.

Example 4.1 Consider the following $2n$ th-order difference equation:

$$\Delta^n(r_{t-n}\Delta^n x_{t-n}) + (-1)^{n+1}f(t, x_{t+1}, x_t, x_{t-1}) = 0, \quad t \in \mathbb{Z}(1, T) \tag{4.1}$$

with boundary value conditions

$$x_j = 0, \quad j \in \mathbb{Z}(1-n, 0) \cup \mathbb{Z}(T+1, T+n), \quad (4.2)$$

where T and n are given positive integers with $T > n$, $r_t = -1$ for $t \in \mathbb{Z}(1-n, 0)$, $r_t = 2$ for $t \in \mathbb{Z}(1, T)$, and

$$\begin{aligned} f(t, x, y, z) = & 2y \left(2 + \sin \frac{t\pi}{T} \right) \left(\frac{x^2 + y^2}{2 + y^2} \ln(2 + x^2) + \ln(2 + x^2) \ln(2 + y^2) - \ln^2 2 \right) \\ & + 2y \left(2 + \sin \frac{(t-1)\pi}{T} \right) \left(\frac{y^2 + z^2}{2 + y^2} \ln(2 + z^2) + \ln(2 + y^2) \ln(2 + z^2) - \ln^2 2 \right) \end{aligned}$$

for $t \in \mathbb{Z}(1, T)$. Clearly, (2.12) holds. If we let

$$F(t, x, y) = \left(2 + \sin \frac{t\pi}{T} \right) (\ln(2 + x^2) \ln(2 + y^2) - \ln^2 2) (x^2 + y^2), \quad t \in \mathbb{Z}(0, T),$$

then (1.8) is satisfied. Since

$$\lim_{x^2 + y^2 \rightarrow \infty} \frac{F(t, x, y)}{x^2 + y^2} = +\infty, \quad \lim_{x^2 + y^2 \rightarrow 0} \frac{F(t, x, y)}{x^2 + y^2} = 0$$

for $t \in \mathbb{Z}(0, T)$, we see that (2.13) and (2.14) hold. By Theorem 2.1, we know that (4.1) with (4.2) has at least two nontrivial solutions.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors jointly worked on the results, and they read and approved the final manuscript.

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