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Oscillation of second-order damped differential equations

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Abstract

We study oscillatory behavior of a class of second-order differential equations with damping under the assumptions that allow applications to retarded and advanced differential equations. New theorems extend and improve the results in the literature. Illustrative examples are given.

MSC: 34C10; 34K11

Keywords: oscillation; functional differential equation; damping term

1 Introduction

This paper is concerned with oscillation of solutions to a second-order differential equation with damping

$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)f\big(x\big(\tau(t)\big)\big) = 0, \tag{1.1}$$

where $t \geq t_0 > 0$, $r \in \mathrm{C}^1([t_0,+\infty),(0,+\infty))$, $p,q,\tau \in \mathrm{C}([t_0,+\infty),\mathbb{R})$, $q(t) \geq 0$, q does not vanish eventually, $f \in \mathrm{C}(\mathbb{R},\mathbb{R})$, $f(x)/x \geq \mu$ for some $\mu > 0$ and for all $x \neq 0$. Throughout, we assume that solutions of (1.1) exist for any $t \geq t_0$. A solution x of (1.1) is termed oscillatory if it has arbitrarily large zeros; otherwise, we call it nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

During the past decades, the questions regarding the study of oscillatory properties of differential equations with damping or distributed deviating arguments have become an important area of research due to the fact that such equations arise in many real life problems; see the research papers [1–26] and the references cited therein. In particular, second-order damped differential equations are used in the study of NVH of vehicles. In what follows, we present the background details that motivate the contents of this paper. Yan [25] established an important extension of the celebrated Kamenev oscillation criterion [27] for a second-order damped equation

$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)x(t) = 0.$$

Rogovchenko [19] and Rogovchenko and Tuncay [20] studied a nonlinear damped equation

$$\big(r(t)x'(t)\big)' + p(t)x'(t) + q(t)f\big(x(t)\big) = 0.$$

Rogovchenko and Tuncay [21] extended the results of [20] to a general nonlinear damped equation

$$\big(r(t)\psi\big(x(t)\big)x'(t)\big)' + p(t)x'(t) + q(t)f\big(x(t)\big) = 0.$$

In [8, 15], the authors investigated (1.1) under the assumptions that $r, p, q \in \mathrm{C}([t_0, +\infty), (0, +\infty))$, $\tau(t) \le t$, and $\tau'(t) > 0$. The natural question now is: *Can one extend the results of* [20] *to functional equation* (1.1)*?* The purpose of this paper is to give an affirmative answer to this question.

2 Main results

In the sequel, all functional inequalities are supposed to be satisfied for all sufficiently large t. We use the notation

$$\mathbb{D} := \big\{(t,s) : t_0 \le s \le t < +\infty\big\} \quad \text{and} \quad \mathbb{D}_0 := \big\{(t,s) : t_0 \le s < t < +\infty\big\}.$$

We say that a continuous function $H : \mathbb{D} \to [0, +\infty)$ belongs to the class $\mathcal{W}$ if:

(i) $H(t,t) = 0$ for $t \ge t_0$ and $H(t,s) > 0$ for $(t,s) \in \mathbb{D}_0$;
(ii) H has a nonpositive continuous partial derivative with respect to the second variable satisfying, for some locally integrable continuous function h,

$$\frac{\partial}{\partial s}H(t,s) = -h(t,s)\big(H(t,s)\big)^{\frac{1}{2}}.$$

Using ideas exploited by Rogovchenko and Tuncay [20], we study (1.1) in the cases where

$$\tau(t) \le t \tag{2.1}$$

and

$$\tau(t) \ge t \tag{2.2}$$

for $t \ge t_0$.

Theorem 2.1 *Let* (2.1) *hold and* $\lim_{t\to+\infty} \tau(t) = +\infty$. *Suppose that there exist functions* $H \in \mathcal{W}$ *and* $\rho_1 \in \mathrm{C}^1([t_0, +\infty), \mathbb{R})$ *such that, for some* $\beta \ge 1$,

$$\limsup_{t\to+\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \left[H(t,s)\psi_*(s) - \frac{\beta}{4}\nu_*(s)r(s)h^2(t,s)\right] \mathrm{d}s = +\infty \tag{2.3}$$

for all sufficiently large $t_1 \ge t_0$ *and for* $T_1 > t_1$, *where*

$$\psi_*(t) := \nu_*(t)\left[\mu q(t)\frac{\int_{t_1}^{\tau(t)} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s}{\int_{t_1}^{t} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s} + r(t)\rho_1^2(t) - p(t)\rho_1(t) - \big(r(t)\rho_1(t)\big)'\right] \tag{2.4}$$

and

$$v_*(t) := \exp\left[-2\int^t \left(\rho_1(s) - \frac{p(s)}{2r(s)}\right) \mathrm{d}s\right]. \tag{2.5}$$

Then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $T_0 \geq t_0$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for all $t \geq T_0$. By virtue of (1.1), we have

$$\bigl(r(t)x'(t)\bigr)' + p(t)x'(t) \leq -\mu q(t)x\bigl(\tau(t)\bigr) \leq 0 \quad \text{for } t \geq T_0,$$

which yields

$$\left(r(t)x'(t)\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}\,\mathrm{d}s\right)\right)' \leq 0. \tag{2.6}$$

Hence we have

$$r(t)\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}\,\mathrm{d}s\right)x'(t) > 0 \tag{2.7}$$

or

$$r(t)\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}\,\mathrm{d}s\right)x'(t) < 0 \tag{2.8}$$

for $t \geq t_1 \geq T_0$. Now define the generalized Riccati substitution

$$u(t) := v_*(t)r(t)\left[\frac{x'(t)}{x(t)} + \rho_1(t)\right]. \tag{2.9}$$

We consider each of two cases separately.

Case I. Assume (2.7) holds. Then we have

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^t \frac{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)x'(s)}{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s \\ &\geq x'(t)r(t)\exp\left(\int_{t_0}^t \frac{p(s)}{r(s)}\,\mathrm{d}s\right)\int_{t_1}^t \frac{1}{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s, \end{aligned}$$

which implies that

$$\left(\frac{x(t)}{\int_{t_1}^t \frac{1}{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s}\right)' \leq 0. \tag{2.10}$$

Differentiating (2.9) yields

$$u'(t) = \frac{v_*'(t)}{v_*(t)}u(t) + v_*(t)\frac{(r(t)x'(t))'}{x(t)} - v_*(t)r(t)\left[\frac{u(t)}{v_*(t)r(t)} - \rho_1(t)\right]^2 + v_*(t)\bigl(r(t)\rho_1(t)\bigr)'. \tag{2.11}$$

It follows from (1.1), (2.5), (2.10), and (2.11) that

$$u'(t) \le -\psi_*(t) - \frac{u^2(t)}{v_*(t)r(t)}, \tag{2.12}$$

where ψ_* is defined as in (2.4). Multiplying both sides of (2.12), with t replaced by s, by $H(t,s)$, integrating with respect to s from T_1 to t, we find, for all $\beta \ge 1$ and for all $t \ge T_1 \ge t_1$,

$$\int_{T_1}^t H(t,s)\psi_*(s)\,\mathrm{d}s + \int_{T_1}^t h(t,s)\bigl(H(t,s)\bigr)^{\frac{1}{2}}u(s)\,\mathrm{d}s + \frac{1}{\beta}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,\mathrm{d}s \le H(t,T_1)u(T_1) - \frac{\beta-1}{\beta}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,\mathrm{d}s. \tag{2.13}$$

Define now

$$C := \frac{u(s)}{\sqrt{\beta}}\frac{(H(t,s))^{\frac{1}{2}}}{(v_*(s)r(s))^{\frac{1}{2}}} \quad \text{and} \quad D := -\frac{\sqrt{\beta}}{2}h(t,s)\bigl(v_*(s)r(s)\bigr)^{\frac{1}{2}}.$$

Applying the inequality

$$C^2 - 2CD \ge -D^2, \tag{2.14}$$

we have

$$h(t,s)\bigl(H(t,s)\bigr)^{\frac{1}{2}}u(s) + \frac{1}{\beta}H(t,s)\frac{u^2(s)}{v_*(s)r(s)} \ge -\frac{\beta}{4}v_*(s)r(s)h^2(t,s).$$

Hence, by the latter inequality and (2.13), we obtain

$$\int_{T_1}^t \left[H(t,s)\psi_*(s) - \frac{\beta}{4}v_*(s)r(s)h^2(t,s)\right]\mathrm{d}s \le H(t,T_1)u(T_1) - \frac{\beta-1}{\beta}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,\mathrm{d}s, \tag{2.15}$$

which contradicts (2.3).

Case II. Assume (2.8) holds. Recalling that $x' < 0$ and $\tau(t) \le t$, we have $x(\tau(t)) \ge x(t)$. Using similar proof of the case where (2.7) holds and the fact that

$$\frac{\int_{t_1}^{\tau(t)} \frac{1}{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s}{\int_{t_1}^{t} \frac{1}{r(s)\exp(\int_{t_0}^s \frac{p(v)}{r(v)}\,\mathrm{d}v)}\,\mathrm{d}s} \le 1,$$

one has (2.15), which contradicts (2.3). This completes the proof. □

Theorem 2.2 *Let* (2.1) *hold and* $\lim_{t\to+\infty}\tau(t)=+\infty$. *Suppose that there exist functions* $H\in\mathcal{W}$, $\rho_1\in C^1([t_0,+\infty),\mathbb{R})$, *and* $\phi_*\in C([t_0,+\infty),\mathbb{R})$ *such that, for all sufficiently large* $T>t_1$ *and for some* $\beta>1$,

$$0<\inf_{s\ge t_0}\left[\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right]\le+\infty \tag{2.16}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\psi_*(s)-\frac{\beta}{4}v_*(s)r(s)h^2(t,s)\right]ds\ge\phi_*(T), \tag{2.17}$$

where ψ_* *and* v_* *are as in Theorem* 2.1. *If*

$$\int_{t_0}^{+\infty}\frac{(\phi_{*+}(s))^2}{v_*(s)r(s)}\,ds=+\infty, \tag{2.18}$$

where $\phi_{*+}(t):=\max\{\phi_*(t),0\}$, *then* (1.1) *is oscillatory.*

Proof Without loss of generality, assume again that (1.1) possesses a solution x such that $x(t)>0$ and $x(\tau(t))>0$ on $[T_0,+\infty)$ for some $T_0\ge t_0$. Proceeding as in the proof of Theorem 2.1, we arrive at inequality (2.15), which yields, for all $t>T_1$ and for any $\beta\ge1$,

$$\begin{aligned}\phi_*(T_1)&\le\limsup_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^t\left[H(t,s)\psi_*(s)-\frac{\beta}{4}v_*(s)r(s)h^2(t,s)\right]ds\\&\le u(T_1)-\frac{\beta-1}{\beta}\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,ds.\end{aligned}$$

The latter inequality implies that, for all $t>T_1$ and for all $\beta\ge1$,

$$\phi_*(T_1)+\frac{\beta-1}{\beta}\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,ds\le u(T_1).$$

Consequently,

$$\phi_*(T_1)\le u(T_1) \tag{2.19}$$

and

$$\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^t H(t,s)\frac{u^2(s)}{v_*(s)r(s)}\,ds\le\frac{\beta}{\beta-1}\big(u(T_1)-\phi_*(T_1)\big)<+\infty. \tag{2.20}$$

Assume now that

$$\int_{T_1}^{+\infty}\frac{u^2(s)}{v_*(s)r(s)}\,ds=+\infty. \tag{2.21}$$

Condition (2.16) implies the existence of $\vartheta>0$ such that

$$\inf_{s\ge t_0}\left[\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right]>\vartheta. \tag{2.22}$$

It follows from (2.21) that, for any positive constant η, there exists $T_2 > T_1$ such that, for all $t \geq T_2$,

$$\int_{T_1}^{t} \frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s \geq \frac{\eta}{\vartheta}. \tag{2.23}$$

Using integration by parts and (2.23), we have, for all $t \geq T_2$,

$$\begin{aligned}
&\frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s \\
&\quad= \frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\,\mathrm{d}\left[\int_{T_1}^{s}\frac{u^2(\xi)}{\nu_*(\xi)r(\xi)}\,\mathrm{d}\xi\right] \\
&\quad= \frac{1}{H(t,T_1)}\int_{T_1}^{t}\left[\int_{T_1}^{s}\frac{u^2(\xi)}{\nu_*(\xi)r(\xi)}\,\mathrm{d}\xi\right]\left[-\frac{\partial H(t,s)}{\partial s}\right]\mathrm{d}s \\
&\quad\geq \frac{\eta}{\vartheta}\frac{1}{H(t,T_1)}\int_{T_2}^{t}\left[-\frac{\partial H(t,s)}{\partial s}\right]\mathrm{d}s = \frac{\eta}{\vartheta}\frac{H(t,T_2)}{H(t,T_1)} \geq \frac{\eta}{\vartheta}\frac{H(t,T_2)}{H(t,t_0)}.
\end{aligned}$$

By virtue of (2.22), there exists $T_3 \geq T_2$ such that, for all $t \geq T_3$,

$$\frac{H(t,T_2)}{H(t,t_0)} \geq \vartheta,$$

which yields

$$\frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s \geq \eta, \quad t \geq T_3.$$

Since η is an arbitrary positive constant,

$$\liminf_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s = +\infty,$$

and the latter contradicts (2.20). Consequently,

$$\int_{T_1}^{+\infty}\frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s < +\infty,$$

and, by virtue of (2.19),

$$\int_{T_1}^{+\infty}\frac{(\phi_{*+}(s))^2}{\nu_*(s)r(s)}\,\mathrm{d}s \leq \int_{T_1}^{+\infty}\frac{u^2(s)}{\nu_*(s)r(s)}\,\mathrm{d}s < +\infty,$$

which contradicts (2.18). This completes the proof. □

Theorem 2.3 *Let* (2.2) *hold and*

$$\int^{+\infty}\frac{1}{r(s)}\exp\left(-\int_{t_0}^{s}\frac{p(t)}{r(t)}\,\mathrm{d}t\right)\mathrm{d}s < +\infty. \tag{2.24}$$

Suppose that there exist functions $H \in \mathcal{W}$ and $\rho_2 \in C^1([t_0,+\infty),\mathbb{R})$ such that, for some $\beta \geq 1$,

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\varphi_*(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s = +\infty, \tag{2.25}$$

where

$$\varphi_*(t) := \upsilon(t)\left[\mu q(t) \frac{\int_{\tau(t)}^{+\infty} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(z)}{r(z)}\,\mathrm{d}z)}\,\mathrm{d}s}{\int_{t}^{+\infty} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(z)}{r(z)}\,\mathrm{d}z)}\,\mathrm{d}s} + r(t)\rho_2^2(t) - p(t)\rho_2(t) - \big(r(t)\rho_2(t)\big)' \right] \tag{2.26}$$

and

$$\upsilon(t) := \exp\left[-2\int^{t} \left(\rho_2(s) - \frac{p(s)}{2r(s)} \right) \mathrm{d}s \right]. \tag{2.27}$$

Then (1.1) *is oscillatory.*

Proof Let x be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that there exists $T_0 \geq t_0$ such that $x(t) > 0$ for all $t \geq T_0$. From the proof of Theorem 2.1, we have (2.6) and either (2.7) or (2.8) for $t \geq t_1 \geq T_0$. We define the generalized Riccati substitution

$$u(t) := \upsilon(t)r(t)\left[\frac{x'(t)}{x(t)} + \rho_2(t) \right]. \tag{2.28}$$

Case I. Assume (2.7) holds. Differentiating (2.28), we have

$$u'(t) = \frac{\upsilon'(t)}{\upsilon(t)}u(t) + \upsilon(t)\frac{(r(t)x'(t))'}{x(t)} - \upsilon(t)r(t)\left[\frac{u(t)}{\upsilon(t)r(t)} - \rho_2(t) \right]^2 + \upsilon(t)\big(r(t)\rho_2(t)\big)'. \tag{2.29}$$

It follows from (1.1), (2.27), and (2.29) that

$$u'(t) \leq -\varphi(t) - \frac{u^2(t)}{\upsilon(t)r(t)}, \tag{2.30}$$

where

$$\varphi(t) := \upsilon(t)\big[\mu q(t) + r(t)\rho_2^2(t) - p(t)\rho_2(t) - \big(r(t)\rho_2(t)\big)' \big].$$

Multiplying both sides of (2.30), with t replaced by s, by $H(t,s)$, integrating with respect to s from T_1 to t, we find, for all $\beta \geq 1$ and for all $t \geq T_1 \geq t_1$,

$$\begin{aligned} &\int_{T_1}^{t} H(t,s)\varphi(s)\,\mathrm{d}s + \int_{T_1}^{t} h(t,s)\big(H(t,s)\big)^{\frac{1}{2}}u(s)\,\mathrm{d}s + \frac{1}{\beta}\int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\upsilon(s)r(s)}\,\mathrm{d}s \\ &\quad \leq H(t,T_1)u(T_1) - \frac{\beta-1}{\beta}\int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\upsilon(s)r(s)}\,\mathrm{d}s. \end{aligned} \tag{2.31}$$

Now define

$$C_* := \frac{u(s)}{\sqrt{\beta}} \frac{(H(t,s))^{\frac{1}{2}}}{(\upsilon(s)r(s))^{\frac{1}{2}}} \quad \text{and} \quad D_* := -\frac{\sqrt{\beta}}{2} h(t,s)\bigl(\upsilon(s)r(s)\bigr)^{\frac{1}{2}}.$$

Applying inequality (2.14) (replace C and D with C_* and D_*), we have

$$h(t,s)\bigl(H(t,s)\bigr)^{\frac{1}{2}} u(s) + \frac{1}{\beta} H(t,s)\frac{u^2(s)}{\upsilon(s)r(s)} \ge -\frac{\beta}{4}\upsilon(s)r(s)h^2(t,s).$$

Hence, by the latter inequality and (2.31), we have

$$\begin{aligned}
&\int_{T_1}^{t} \left[H(t,s)\varphi(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s \\
&\quad \le H(t,T_1)u(T_1) - \frac{\beta - 1}{\beta} \int_{T_1}^{t} H(t,s)\frac{u^2(s)}{\upsilon(s)r(s)} \,\mathrm{d}s.
\end{aligned} \tag{2.32}$$

Using monotonicity of H, we conclude that, for all $t \ge T_1$,

$$\int_{T_1}^{t} \left[H(t,s)\varphi(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s \le H(t,T_1)\bigl|u(T_1)\bigr| \le H(t,t_0)\bigl|u(T_1)\bigr|.$$

Thus

$$\int_{t_0}^{t} \left[H(t,s)\varphi(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s \le H(t,t_0)\left[\bigl|u(T_1)\bigr| + \int_{t_0}^{T_1} \bigl|\varphi(s)\bigr| \,\mathrm{d}s \right].$$

Hence we have

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\varphi(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s \le \bigl|u(T_1)\bigr| + \int_{t_0}^{T_1} \bigl|\varphi(s)\bigr| \,\mathrm{d}s < +\infty,$$

which contradicts (2.25) due to the fact that $\varphi_*(t) \le \varphi(t)$, where φ_* is defined as in (2.26).

Case II. Assume (2.8) holds. From (2.6), we have

$$x'(s) \le \frac{r(t)\exp(\int_{t_0}^{t} \frac{p(z)}{r(z)} \,\mathrm{d}z)}{r(s)\exp(\int_{t_0}^{s} \frac{p(z)}{r(z)} \,\mathrm{d}z)} x'(t), \quad s \ge t.$$

Hence we get

$$x(l) - x(t) \le x'(t)r(t)\exp\left(\int_{t_0}^{t} \frac{p(z)}{r(z)} \,\mathrm{d}z \right) \int_{t}^{l} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(z)}{r(z)} \,\mathrm{d}z)} \,\mathrm{d}s.$$

Letting $l \to +\infty$, we obtain

$$x(t) \ge -x'(t)r(t)\exp\left(\int_{t_0}^{t} \frac{p(z)}{r(z)} \,\mathrm{d}z \right) \int_{t}^{+\infty} \frac{1}{r(s)\exp(\int_{t_0}^{s} \frac{p(z)}{r(z)} \,\mathrm{d}z)} \,\mathrm{d}s.$$

This inequality yields

$$\left(\frac{x(t)}{\int_t^{+\infty}\frac{1}{r(s)\exp(\int_{t_0}^s\frac{p(z)}{r(z)}\,\mathrm{d}z)}\,\mathrm{d}s}\right)'\geq 0,$$

and so

$$\frac{x(\tau(t))}{x(t)}\geq\frac{\int_{\tau(t)}^{+\infty}\frac{1}{r(s)\exp(\int_{t_0}^s\frac{p(z)}{r(z)}\,\mathrm{d}z)}\,\mathrm{d}s}{\int_t^{+\infty}\frac{1}{r(s)\exp(\int_{t_0}^s\frac{p(z)}{r(z)}\,\mathrm{d}z)}\,\mathrm{d}s}.$$

The rest of the proof is similar to that of the case where (2.7) holds. Then one can get a contradiction to (2.25). This completes the proof. □

On the basis of Theorem 2.3, similar as in the proof of Theorem 2.2, we have the following result immediately.

Theorem 2.4 *Let* (2.2) *and* (2.24) *hold. Suppose that there exist functions* $H\in\mathcal{W}$, $\rho_2\in\mathrm{C}^1([t_0,+\infty),\mathbb{R})$, *and* $\phi\in\mathrm{C}([t_0,+\infty),\mathbb{R})$ *such that, for all* $T\geq t_0$ *and for some* $\beta>1$, *one has* (2.16) *and*

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\varphi_*(s)-\frac{\beta}{4}\upsilon(s)r(s)h^2(t,s)\right]\mathrm{d}s\geq\phi(T),\tag{2.33}$$

where φ_* *and* υ *are as in Theorem* 2.3. *If*

$$\int_{t_0}^{+\infty}\frac{(\phi_+(s))^2}{\upsilon(s)r(s)}\,\mathrm{d}s=+\infty,\tag{2.34}$$

where $\phi_+(t):=\max\{\phi(t),0\}$, *then* (1.1) *is oscillatory.*

Remark 2.1 Efficient oscillation tests can be derived from Theorems 2.1-2.4 with different choices of the functions H, ρ_1, and ρ_2. For example, for $(t,s)\in\mathbb{D}$, Kamenev's weight function H defined by $H(t,s)=(t-s)^m$, where $m\geq 1$, belongs to the class $\mathcal{W}$. The details are left to the reader.

3 Applications and discussion

The following three examples illustrate applications of theoretical results in the previous section.

Example 3.1 For $t\geq 1$, consider a second-order ordinary damped differential equation

$$x''(t)+\frac{1}{t}x'(t)+\frac{1}{t^2}x(t)=0,\tag{3.1}$$

where $r(t)=1$, $p(t)=1/t$, $q(t)=1/t^2$, $f(x)=x$, and $\tau(t)=t$. Letting $\mu=1$, $\rho_1(t)=0$, and $H(t,s)=(t-s)^2$, then $\nu_*(t)=t$, $h^2(t,s)=4$, and so $\psi_*(t)=1/t$ and

$$\begin{aligned}&\limsup_{t\to+\infty}\frac{1}{H(t,T_1)}\int_{T_1}^t\left[H(t,s)\psi_*(s)-\frac{\beta}{4}\nu_*(s)r(s)h^2(t,s)\right]\mathrm{d}s\\&\quad=\limsup_{t\to+\infty}\frac{1}{t^2}\int_{T_1}^t\left[\frac{(t-s)^2}{s}-\beta s\right]\mathrm{d}s=+\infty.\end{aligned}$$

Hence, by Theorem 2.1, equation (3.1) is oscillatory. As a matter of fact, one such solution is $x(t) = \sin(\ln t)$.

Example 3.2 For $t \geq 1$, consider a second-order delay damped differential equation

$$x''(t) - x'(t) + \sqrt{2}x\left(t - \frac{7\pi}{4}\right) = 0, \tag{3.2}$$

where $r(t) = 1$, $p(t) = -1$, $q(t) = \sqrt{2}$, $f(x) = x$, and $\tau(t) = t - 7\pi/4$. Letting $\mu = 1$, $\rho_1(t) = -1/2$, and $H(t,s) = (t-s)^2$, then $\nu_*(t) = 1$, $h^2(t,s) = 4$, and so $\psi_*(t) > 3/4$ and

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \left[H(t,s)\psi_*(s) - \frac{\beta}{4}\nu_*(s)r(s)h^2(t,s) \right] \mathrm{d}s \\
&\quad \geq \limsup_{t\to+\infty} \frac{1}{t^2} \int_{T_1}^{t} \left[\frac{3(t-s)^2}{4} - \beta \right] \mathrm{d}s = +\infty.
\end{aligned}$$

Hence, by Theorem 2.1, equation (3.2) is oscillatory. As a matter of fact, one such solution is $x(t) = \sin t$.

Example 3.3 For $t \geq 1$, consider a second-order advanced damped differential equation

$$x''(t) + x'(t) + x(t+1) = 0, \tag{3.3}$$

where $r(t) = 1$, $p(t) = 1$, $q(t) = 1$, $f(x) = x$, and $\tau(t) = t + 1$. Letting $\mu = 1$, $\rho_2(t) = 1/2$, and $H(t,s) = (t-s)^2$, then $\upsilon(t) = 1$, $h^2(t,s) = 4$, and so $\varphi_*(t) = \mathrm{e}^{-1} - 1/4$ and

$$\begin{aligned}
&\limsup_{t\to+\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \left[H(t,s)\varphi_*(s) - \frac{\beta}{4}\upsilon(s)r(s)h^2(t,s) \right] \mathrm{d}s \\
&\quad = \limsup_{t\to+\infty} \frac{1}{t^2} \int_{T_1}^{t} \left[\left(\mathrm{e}^{-1} - \frac{1}{4} \right)(t-s)^2 - \beta \right] \mathrm{d}s = +\infty.
\end{aligned}$$

Hence, by Theorem 2.3, equation (3.3) is oscillatory.

Remark 3.1 In this paper, we present some new oscillation criteria for the differential equation with a linear damping term (1.1). Our theorems can be applied to the cases where $p \geq 0$, $p \leq 0$, or p is an oscillatory function. Furthermore, the main results can be applied to the cases where the deviating argument τ is delayed or advanced. On the other hand, we do not need to require the assumption that $\tau'(t) > 0$ for $t \geq t_0$. Hence, the results obtained supplement and improve those reported in [8, 15].

Remark 3.2 Note that when $\tau(t) \equiv t$, Theorems 2.1 and 2.2 include [20, Theorem 17] and [20, Theorem 19], respectively. On the basis of assumption (2.24), Theorems 2.3 and 2.4 include [20, Theorem 17] and [20, Theorem 19], respectively.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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