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Oscillation results for second-order nonlinear neutral differential equations

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Abstract
We obtain several oscillation criteria for a class of second-order nonlinear neutral differential equations. New theorems extend a number of related results reported in the literature and can be used in cases where known theorems fail to apply. Two illustrative examples are provided.

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1 Introduction

In this paper, we are concerned with the oscillation of a class of nonlinear second-order neutral differential equations

$$\left(r(t)\left(\left(x(t)+p(t)x(t-\tau)\right)'\right)^{\gamma}\right)'+q(t)f\left(x(t),x\left(\sigma(t)\right)\right)=0, \tag{1}$$

where $t \geq t_0 > 0$, $\tau \geq 0$, and $\gamma \geq 1$ is a quotient of two odd positive integers. In what follows, it is always assumed that

(H$_1$) $r \in \mathrm{C}^1([t_0,+\infty),(0,+\infty))$;
(H$_2$) $p,q \in \mathrm{C}([t_0,+\infty),[0,+\infty))$ and $q(t)$ is not identically zero for large t;
(H$_3$) $f \in \mathrm{C}(\mathbb{R}^2,\mathbb{R})$ and $f(x,y)/y^{\gamma} \geq \kappa$ for all $y \neq 0$ and for some $\kappa > 0$;
(H$_4$) $\sigma \in \mathrm{C}^1([t_0,+\infty),\mathbb{R})$, $\sigma(t) \leq t$, $\sigma'(t) > 0$, and $\lim_{t\to+\infty}\sigma(t) = +\infty$.

By a solution of equation (1) we mean a continuous function $x(t)$ defined on an interval $[t_x,+\infty)$ such that $r(t)((x(t)+p(t)x(t-\tau))')^{\gamma}$ is continuously differentiable and $x(t)$ satisfies (1) for $t \geq t_x$. We consider only solutions satisfying $\sup\{|x(t)| : t \geq T \geq t_x\} > 0$ and tacitly assume that equation (1) possesses such solutions. A solution of (1) is called oscillatory if it has arbitrarily large zeros on $[t_x,+\infty)$; otherwise, it is called nonoscillatory. We say that equation (1) is oscillatory if all its continuable solutions are oscillatory.

During the past decades, a great deal of interest in oscillatory and nonoscillatory behavior of various classes of differential and functional differential equations has been shown. Many papers deal with the oscillation of neutral differential equations which are often encountered in applied problems in science and technology; see, for instance, Hale [1]. It is known that analysis of neutral differential equations is more difficult in comparison with that of ordinary differential equations, although certain similarities in the behavior of solutions of these two classes of equations are observed; see, for instance, the monographs [2–4], the papers [5–22] and the references cited there.

Oscillation results for (1) have been reported in [2, 4, 6, 8, 11, 14, 18–20]. A commonly used assumption is

$$\int_{t_0}^{+\infty} r^{-1/\gamma}(s)\,\mathrm{d}s = +\infty, \tag{2}$$

although several authors were concerned with the oscillation of equation (1) in the case where

$$\int_{t_0}^{+\infty} r^{-1/\gamma}(s)\,\mathrm{d}s < +\infty. \tag{3}$$

In particular, Xu and Meng [19, Theorem 2.3] established sufficient conditions for the oscillation of (1) assuming that

$$p'(t) \geq 0 \quad \text{and} \quad \lim_{t\to+\infty} p(t) = A. \tag{4}$$

Further results in this direction were obtained by Ye and Xu [20] under the assumptions that

$$p'(t) \geq 0 \quad \text{and} \quad \sigma(t) \leq t - \tau; \tag{5}$$

see also the paper by Han *et al.* [8] where inaccuracies in [20] were corrected and new oscillation criteria for (1) were obtained [8, Theorems 2.1 and 2.2]. We conclude this brief review of the literature by mentioning that Li *et al.* [13] and Sun *et al.* [18] extended the results obtained in [8] to Emden-Fowler neutral differential equations and neutral differential equations with mixed nonlinearities.

Our principal goal in this paper is to derive new oscillation criteria for equation (1) without requiring restrictive conditions (4) and (5). Developing further ideas from the paper by Hasanbulli and Rogovchenko [9] concerned with a particular case of equation (2) with $\gamma = 1$, we study the oscillation of (1) in the case where $\gamma \geq 1$.

2 Oscillation criteria

In what follows, all functional inequalities are tacitly assumed to hold for all t large enough, unless mentioned otherwise. As usual, we use the notation $z(t) := x(t) + p(t)x(t-\tau)$ and $g_+(t) := \max\{g(t), 0\}$. Let

$$\mathbb{D} = \{(t,s) : t_0 \leq s \leq t < +\infty\} \quad \text{and} \quad \mathbb{D}_0 = \{(t,s) : t_0 \leq s < t < +\infty\}.$$

We say that a function $H \in \mathrm{C}(\mathbb{D}, [0,+\infty))$ belongs to a class $\mathcal{W}_\gamma$ if

(i) $H(t,t) = 0$ and $H(t,s) > 0$ for all $(t,s) \in \mathbb{D}_0$;

(ii) H has a nonpositive continuous partial derivative with respect to the second variable satisfying

$$\frac{\partial}{\partial s} H(t,s) = -h(t,s)\big(H(t,s)\big)^{\gamma/(\gamma+1)}$$

for a locally integrable function $h \in \mathcal{L}_{\mathrm{loc}}(\mathbb{D}, \mathbb{R})$.

In what follows, we assume that, for all $t \geq t_0$,

$$1 - p(t)\frac{R(t-\tau)}{R(t)} > 0, \tag{6}$$

where

$$R(t) := \int_t^{+\infty} r^{-1/\gamma}(s)\,\mathrm{d}s.$$

In order to establish our main theorems, we need the following auxiliary result. The first inequality is extracted from the paper by Jiang and Li [11, Lemma 5], whereas the second one is a variation of the well-known Young inequality [23].

Lemma 1 (i) *Let $\gamma \geq 1$ be a ratio of two odd integers. Then*

$$A^{1+1/\gamma} - |A-B|^{1+1/\gamma} \leq \frac{1}{\gamma}B^{1/\gamma}\big[(\gamma+1)A - B\big] \tag{7}$$

for all $AB \geq 0$.

(ii) *For any two numbers $C, D \geq 0$ and for any $q > 1$,*

$$C^q + (q-1)D^q - qCD^{q-1} \geq 0,$$

the equality holds if and only if $C = D$.

Theorem 2 *Assume that conditions* (H_1)-(H_4), (3), *and* (6) *are satisfied. Suppose also that there exist two functions $\rho_1, \rho_2 \in \mathrm{C}^1([t_0, +\infty), \mathbb{R})$ such that, for some $\beta \geq 1$ and for some $H \in \mathcal{W}_\gamma$,*

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\psi_1(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}} \frac{v_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma} h^{\gamma+1}(t,s)\right]\mathrm{d}s = +\infty \tag{8}$$

and

$$\limsup_{t\to+\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t \left[H(t,s)\psi_2(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}} v_2(s)r(s)h^{\gamma+1}(t,s)\right]\mathrm{d}s = +\infty, \tag{9}$$

where

$$\psi_1(t) := v_1(t)\left[\kappa q(t)\big(1 - p(\sigma(t))\big)^\gamma + \sigma'(t)\left(\frac{r(t)\rho_1(t)}{r^{1/(\gamma+1)}(\sigma(t))}\right)^{(\gamma+1)/\gamma} - \big(r(t)\rho_1(t)\big)'\right], \tag{10}$$

$$v_1(t) := \exp\left[-(\gamma+1)\int^t \sigma'(s)\left(\frac{r(s)\rho_1(s)}{r(\sigma(s))}\right)^{1/\gamma}\mathrm{d}s\right], \tag{11}$$

$$\psi_2(t) := v_2(t)\left[\kappa q(t)\left(1 - p(\sigma(t))\frac{R(\sigma(t)-\tau)}{R(\sigma(t))}\right)^\gamma + r(t)\rho_2^{(\gamma+1)/\gamma}(t) - \big(r(t)\rho_2(t)\big)'\right], \tag{12}$$

and

$$v_2(t) := \exp\left[-(\gamma+1)\int^t \rho_2^{1/\gamma}(s)\,\mathrm{d}s\right]. \tag{13}$$

Then equation (1) *is oscillatory.*

Proof Let $x(t)$ be a nonoscillatory solution of (1). Since γ is a quotient of two odd positive integers, $-x(t)$ is also a solution of (1). Hence, without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) \geq x(t) > 0$, and by virtue of

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\gamma}\bigr)' = -q(t)f\bigl(x(t), x\bigl(\sigma(t)\bigr)\bigr) \leq 0,$$

the function $(r(t)(z'(t))^{\gamma})'$ is nonincreasing for all $t \geq t_1$. Therefore, $z'(t)$ does not change sign eventually, that is, there exists a $t_2 \geq t_1$ such that either $z'(t) > 0$ or $z'(t) < 0$ for all $t \geq t_2$. We consider each of two cases separately.

Case 1. Assume first that $z'(t) > 0$ for all $t \geq t_2$. Equation (1) and condition (H_2) yield

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\gamma}\bigr)' + \kappa q(t)x^{\gamma}\bigl(\sigma(t)\bigr) \leq 0. \tag{14}$$

In view of (H_4), there exists a $t_3 \geq t_2$ such that, for all $t \geq t_3$,

$$x\bigl(\sigma(t)\bigr) \geq \bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)z\bigl(\sigma(t)\bigr), \tag{15}$$

and

$$z'\bigl(\sigma(t)\bigr) \geq \left(\frac{r(t)}{r(\sigma(t))}\right)^{1/\gamma} z'(t). \tag{16}$$

It follows from (14) and (15) that

$$\bigl(r(t)\bigl(z'(t)\bigr)^{\gamma}\bigr)' \leq -\kappa q(t)\bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)^{\gamma} z^{\gamma}\bigl(\sigma(t)\bigr). \tag{17}$$

Define a generalized Riccati substitution by

$$u_1(t) := v_1(t)r(t)\left[\left(\frac{z'(t)}{z(\sigma(t))}\right)^{\gamma} + \rho_1(t)\right], \quad t \geq t_3. \tag{18}$$

Differentiating (18) and using (16) and (17), one arrives at

$$\begin{aligned} u_1'(t) &= \frac{v_1'(t)}{v_1(t)}u_1(t) + v_1(t)\frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(\sigma(t))} \\ &\quad - \gamma v_1(t)r(t)\sigma'(t)\left(\frac{z'(t)}{z(\sigma(t))}\right)^{\gamma}\frac{z'(\sigma(t))}{z(\sigma(t))} + v_1(t)\bigl(r(t)\rho_1(t)\bigr)' \\ &\leq -(\gamma+1)\sigma'(t)\left(\frac{r(t)\rho_1(t)}{r(\sigma(t))}\right)^{1/\gamma}u_1(t) - v_1(t)\kappa q(t)\bigl(1 - p\bigl(\sigma(t)\bigr)\bigr)^{\gamma} \\ &\quad - \gamma\sigma'(t)v_1(t)r(t)\left(\frac{r(t)}{r(\sigma(t))}\right)^{1/\gamma}\left[\frac{u_1(t)}{v_1(t)r(t)} - \rho_1(t)\right]^{(\gamma+1)/\gamma} + v_1(t)\bigl(r(t)\rho_1(t)\bigr)'. \end{aligned} \tag{19}$$

Let

$$A := \frac{u_1(t)}{v_1(t)r(t)} \quad \text{and} \quad B := \rho_1(t).$$

By virtue of Lemma 1, part (i), we have the following estimate:

$$\begin{aligned}\left[\frac{u_1(t)}{v_1(t)r(t)} - \rho_1(t)\right]^{(\gamma+1)/\gamma} \geq{}& \left(\frac{u_1(t)}{v_1(t)r(t)}\right)^{(\gamma+1)/\gamma} \\ &- \frac{1}{\gamma}\rho_1^{1/\gamma}(t)\left[(\gamma+1)\frac{u_1(t)}{v_1(t)r(t)} - \rho_1(t)\right]. \end{aligned} \tag{20}$$

It follows now from (19) and (20) that

$$u_1'(t) \leq -\psi_1(t) - \gamma\sigma'(t)u_1(t)\left(\frac{u_1(t)}{v_1(t)r(\sigma(t))}\right)^{1/\gamma}, \tag{21}$$

where ψ_1 is defined by (10). Replacing in (21) t with s, multiplying both sides by $H(t,s)$ and integrating with respect to s from t_3 to t, we have, for some $\beta \geq 1$ and for any $t \geq t_3$,

$$\begin{aligned}&\int_{t_3}^{t} H(t,s)\psi_1(s)\,\mathrm{d}s + \int_{t_3}^{t} h(t,s)\bigl(H(t,s)\bigr)^{\gamma/(\gamma+1)}u_1(s)\,\mathrm{d}s \\ &\quad + \frac{\gamma}{\beta}\int_{t_3}^{t} H(t,s)\sigma'(s)u_1(s)\left(\frac{u_1(s)}{v_1(s)r(\sigma(s))}\right)^{1/\gamma}\mathrm{d}s \\ &\leq H(t,t_3)u_1(t_3) - \frac{\gamma(\beta-1)}{\beta}\int_{t_3}^{t} H(t,s)\sigma'(s)u_1(s)\left(\frac{u_1(s)}{v_1(s)r(\sigma(s))}\right)^{1/\gamma}\mathrm{d}s. \end{aligned} \tag{22}$$

Let $q := 1 + 1/\gamma$,

$$C := \left(\frac{\gamma}{\beta}\right)^{\gamma/(\gamma+1)}\left(\frac{(H(t,s)\sigma'(s))^{\gamma}}{v_1(s)r(\sigma(s))}\right)^{1/(\gamma+1)}u_1(s),$$

and

$$D := -\left(\frac{\gamma\beta^{\gamma}}{(\gamma+1)^{\gamma+1}}\right)^{\gamma/(\gamma+1)}\left(\frac{v_1(s)r(\sigma(s))}{(\sigma'(s))^{\gamma}}h^{\gamma+1}(t,s)\right)^{\gamma/(\gamma+1)}.$$

Application of Lemma 1, part (ii), yields

$$\begin{aligned}&h(t,s)\bigl(H(t,s)\bigr)^{\gamma/(\gamma+1)}u_1(s) + \frac{\gamma}{\beta}H(t,s)\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))} \\ &\quad \geq -\frac{\beta^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{v_1(s)r(\sigma(s))}{(\sigma'(s))^{\gamma}}h^{\gamma+1}(t,s). \end{aligned}$$

Hence, by the latter inequality and (22), we have

$$\begin{aligned}&\int_{t_3}^{t}\left[H(t,s)\psi_1(s) - \frac{\beta^{\gamma}}{(\gamma+1)^{\gamma+1}}\frac{v_1(s)r(\sigma(s))}{(\sigma'(s))^{\gamma}}h^{\gamma+1}(t,s)\right]\mathrm{d}s \\ &\quad \leq H(t,t_3)u_1(t_3) - \frac{\gamma(\beta-1)}{\beta}\int_{t_3}^{t} H(t,s)\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s. \end{aligned} \tag{23}$$

Using monotonicity of H, we conclude that, for all $t \geq t_3$,

$$\begin{aligned}&\int_{t_3}^{t}\left[H(t,s)\psi_1(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s\\ &\quad \leq H(t,t_3)\left|u_1(t_3)\right| \leq H(t,t_0)\left|u_1(t_3)\right|.\end{aligned}$$

Thus,

$$\begin{aligned}&\int_{t_0}^{t}\left[H(t,s)\psi_1(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s\\ &\quad \leq H(t,t_0)\left[\left|u_1(t_3)\right| + \int_{t_0}^{t_3}\left|\psi_1(s)\right|\mathrm{d}s\right],\end{aligned}$$

and

$$\begin{aligned}&\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\psi_1(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s\\ &\quad \leq \left|u_1(t_3)\right| + \int_{t_0}^{t_3}\left|\psi_1(s)\right|\mathrm{d}s < +\infty,\end{aligned}$$

which contradicts (8).

Case 2. Assume now that $z'(t) < 0$ for all $t \geq t_2$. It follows from the inequality $(r(t)(z'(t))^\gamma)' \leq 0$ that, for all $s \geq t \geq t_2$,

$$z'(s) \leq \left(\frac{r(t)}{r(s)}\right)^{1/\gamma} z'(t).$$

Integrating this inequality from t to l, $l \geq t \geq t_2$, we have

$$z(l) \leq z(t) + r^{1/\gamma}(t)z'(t)\int_t^l \frac{1}{r^{1/\gamma}(s)}\,\mathrm{d}s.$$

Passing to the limit as $l \to +\infty$, we conclude that

$$z(t) \geq -R(t)r^{1/\gamma}(t)z'(t),$$

which yields

$$\left(\frac{z(t)}{R(t)}\right)' \geq 0.$$

Hence, we have

$$x(t) = z(t) - p(t)x(t-\tau) \geq z(t) - p(t)z(t-\tau) \geq \left(1 - p(t)\frac{R(t-\tau)}{R(t)}\right)z(t).$$

It follows from (1) and the latter inequality that there exists a $t_4 \geq t_2$ such that

$$\bigl(r(t)\bigl(z'(t)\bigr)^\gamma\bigr)' + \kappa q(t)\left(1 - p\bigl(\sigma(t)\bigr)\frac{R(\sigma(t)-\tau)}{R(\sigma(t))}\right)^\gamma z^\gamma\bigl(\sigma(t)\bigr) \leq 0. \tag{24}$$

For $t \ge t_4$, define a generalized Riccati substitution by

$$u_2(t) := v_2(t)r(t)\left[\left(\frac{z'(t)}{z(t)}\right)^{\gamma} + \rho_2(t)\right]. \tag{25}$$

Differentiating (25), we have

$$\begin{aligned} u_2'(t) = {} & \frac{v_2'(t)}{v_2(t)}u_2(t) + v_2(t)\frac{(r(t)(z'(t))^{\gamma})'}{z^{\gamma}(t)} \\ & - \gamma v_2(t)r(t)\left[\frac{u_2(t)}{v_2(t)r(t)} - \rho_2(t)\right]^{(\gamma+1)/\gamma} + v_2(t)\bigl(r(t)\rho_2(t)\bigr)'. \end{aligned} \tag{26}$$

Letting in Lemma 1, part (i),

$$A := \frac{u_2(t)}{v_2(t)r(t)} \quad \text{and} \quad B := \rho_2(t),$$

we have

$$\begin{aligned} \left[\frac{u_2(t)}{v_2(t)r(t)} - \rho_2(t)\right]^{(\gamma+1)/\gamma} \ge {} & \left(\frac{u_2(t)}{v_2(t)r(t)}\right)^{(\gamma+1)/\gamma} \\ & - \rho_2^{1/\gamma}(t)\left[\frac{\gamma+1}{\gamma}\frac{u_2(t)}{v_2(t)r(t)} - \frac{1}{\gamma}\rho_2(t)\right]. \end{aligned}$$

It follows from (24) and (26) that

$$u_2'(t) \le -\psi_2(t) - \gamma\left(\frac{u_2^{\gamma+1}(t)}{v_2(t)r(t)}\right)^{1/\gamma}, \tag{27}$$

where ψ_2 is defined by (12). Replacing in (27) t with s, multiplying both sides by $H(t,s)$ and integrating with respect to s from t_4 to t, we conclude that, for some $\beta \ge 1$ and for all $t \ge t_4$,

$$\begin{aligned} & \int_{t_4}^{t} H(t,s)\psi_2(s)\,\mathrm{d}s + \int_{t_4}^{t} h(t,s)\bigl(H(t,s)\bigr)^{\gamma/(\gamma+1)}u_2(s)\,\mathrm{d}s \\ & \quad + \frac{\gamma}{\beta}\int_{t_4}^{t} H(t,s)\left(\frac{u_2^{\gamma+1}(s)}{v_2(s)r(s)}\right)^{1/\gamma}\mathrm{d}s \\ & \le H(t,t_4)u_2(t_4) - \frac{\gamma(\beta-1)}{\beta}\int_{t_4}^{t} H(t,s)\left(\frac{u_2^{\gamma+1}(s)}{v_2(s)r(s)}\right)^{1/\gamma}\mathrm{d}s. \end{aligned} \tag{28}$$

Letting in Lemma 1, part (ii),

$$C := \left(\frac{\gamma}{\beta}\right)^{\gamma/(\gamma+1)}\left(\frac{H^{\gamma}(t,s)}{v_2(s)r(s)}\right)^{1/(\gamma+1)}u_2(s)$$

and

$$D := -\left(\frac{\gamma\beta^{\gamma}}{(\gamma+1)^{\gamma+1}}\right)^{\gamma/(\gamma+1)}\bigl(v_2(s)r(s)h^{\gamma+1}(t,s)\bigr)^{\gamma/(\gamma+1)},$$

we conclude that

$$h(t,s)\big(H(t,s)\big)^{\gamma/(\gamma+1)}u_2(s)+\frac{\gamma}{\beta}H(t,s)\left(\frac{u_2^{\gamma+1}(s)}{\nu_2(s)r(s)}\right)^{1/\gamma}\geq-\frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\nu_2(s)r(s)h^{\gamma+1}(t,s).$$

Using the latter inequality and (28), we have

$$\begin{aligned}&\int_{t_4}^{t}\left[H(t,s)\psi_2(s)-\frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\nu_2(s)r(s)h^{\gamma+1}(t,s)\right]\mathrm{d}s\\&\quad\leq H(t,t_4)u_2(t_4)-\frac{\gamma(\beta-1)}{\beta}\int_{t_4}^{t}H(t,s)\left(\frac{u_2^{\gamma+1}(s)}{\nu_2(s)r(s)}\right)^{1/\gamma}\mathrm{d}s.\end{aligned}\tag{29}$$

Proceeding as in the proof of Case 1, we obtain contradiction with our assumption (9). Therefore, equation (1) is oscillatory. □

Theorem 3 *Assume that conditions* (H_1)-(H_4), (3), *and* (6) *are satisfied. Suppose also that there exist functions* $H\in\mathcal{W}_\gamma$, $\rho_1,\rho_2\in C^1([t_0,+\infty),\mathbb{R})$, $\phi_1,\phi_2\in C([t_0,+\infty),\mathbb{R})$ *such that, for all* $T\geq t_0$ *and for some* $\beta>1$,

$$0<\inf_{s\geq t_0}\left[\liminf_{t\to+\infty}\frac{H(t,s)}{H(t,t_0)}\right]\leq+\infty,\tag{30}$$

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\psi_1(s)-\frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s\geq\phi_1(T),\tag{31}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\psi_2(s)-\frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\nu_2(s)r(s)h^{\gamma+1}(t,s)\right]\mathrm{d}s\geq\phi_2(T),\tag{32}$$

where ψ_1, ψ_2, ν_1, *and* ν_2 *are as in Theorem* 2. *If*

$$\limsup_{t\to+\infty}\int_{t_0}^t\frac{\sigma'(s)(\phi_{1+}(s))^{(\gamma+1)/\gamma}}{\nu_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\mathrm{d}s=+\infty\tag{33}$$

and

$$\limsup_{t\to+\infty}\int_{t_0}^t\frac{(\phi_{2+}(s))^{(\gamma+1)/\gamma}}{\nu_2^{1/\gamma}(s)r^{1/\gamma}(s)}\mathrm{d}s=+\infty,\tag{34}$$

equation (1) *is oscillatory.*

Proof Without loss of generality, assume again that (1) possesses a nonoscillatory solution $x(t)$ such that $x(t)>0$, $x(t-\tau)>0$, and $x(\sigma(t))>0$ on $[t_1,+\infty)$ for some $t_1\geq t_0$. From the proof of Theorem 2, we know that there exists a $t_2\geq t_1$ such that either $z'(t)>0$ or $z'(t)<0$ for all $t\geq t_2$.

Case 1. Assume first that $z'(t)>0$ for all $t\geq t_2$. Proceeding as in the proof of Theorem 2, we arrive at inequality (23), which yields, for all $t>t_3$ and for some $\beta>1$,

$$\phi_1(t_3)\leq\limsup_{t\to+\infty}\frac{1}{H(t,t_3)}\int_{t_3}^t\left[H(t,s)\psi_1(s)-\frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{\nu_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s$$

$$\leq u_1(t_3) - \frac{\gamma(\beta-1)}{\beta} \liminf_{t\to+\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s.$$

The latter inequality implies that, for all $t > t_3$ and for some $\beta > 1$,

$$\phi_1(t_3) + \frac{\gamma(\beta-1)}{\beta} \liminf_{t\to+\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s \leq u_1(t_3).$$

Consequently,

$$\phi_1(t_3) \leq u_1(t_3), \tag{35}$$

and

$$\liminf_{t\to+\infty} \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s \leq \frac{\beta}{\gamma(\beta-1)} \bigl(u_1(t_3) - \phi_1(t_3)\bigr) < +\infty. \tag{36}$$

Assume now that

$$\int_{t_3}^{+\infty} \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s = +\infty. \tag{37}$$

Condition (30) implies existence of a $\vartheta > 0$ such that

$$\liminf_{t\to+\infty} \frac{H(t,s)}{H(t,t_0)} > \vartheta. \tag{38}$$

It follows from (37) that, for any positive constant η, there exists a $t_5 > t_3$ such that, for all $t \geq t_5$,

$$\int_{t_3}^{t} \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s \geq \frac{\eta}{\vartheta}. \tag{39}$$

Using integration by parts and (39), we have, for all $t \geq t_5$,

$$\begin{aligned}
&\frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \frac{\sigma'(s) u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s) r^{1/\gamma}(\sigma(s))} \,\mathrm{d}s \\
&\quad = \frac{1}{H(t,t_3)} \int_{t_3}^{t} H(t,s) \,\mathrm{d}\left[\int_{t_3}^{s} \frac{\sigma'(\xi) u_1^{(\gamma+1)/\gamma}(\xi)}{v_1^{1/\gamma}(\xi) r^{1/\gamma}(\sigma(\xi))} \,\mathrm{d}\xi\right] \\
&\quad = \frac{1}{H(t,t_3)} \int_{t_3}^{t} \left[\int_{t_3}^{s} \frac{\sigma'(\xi) u_1^{(\gamma+1)/\gamma}(\xi)}{v_1^{1/\gamma}(\xi) r^{1/\gamma}(\sigma(\xi))} \,\mathrm{d}\xi\right] \left[-\frac{\partial H(t,s)}{\partial s}\right] \mathrm{d}s \\
&\quad \geq \frac{\eta}{\vartheta} \frac{1}{H(t,t_3)} \int_{t_5}^{t} \left[-\frac{\partial H(t,s)}{\partial s}\right] \mathrm{d}s = \frac{\eta}{\vartheta} \frac{H(t,t_5)}{H(t,t_3)} \geq \frac{\eta}{\vartheta} \frac{H(t,t_5)}{H(t,t_0)}.
\end{aligned}$$

By virtue of (38), there exists a $t_6 \geq t_5$ such that, for all $t \geq t_6$,

$$\frac{H(t,t_5)}{H(t,t_0)} \geq \vartheta,$$

which implies that

$$\frac{1}{H(t,t_3)}\int_{t_3}^{t} H(t,s)\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s \ge \eta, \quad t \ge t_6.$$

Since η is an arbitrary positive constant,

$$\liminf_{t\to+\infty}\frac{1}{H(t,t_3)}\int_{t_3}^{t} H(t,s)\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s = +\infty,$$

but the latter contradicts (36). Consequently,

$$\int_{t_3}^{+\infty}\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s < +\infty,$$

and, by virtue of (35),

$$\int_{t_3}^{+\infty}\frac{\sigma'(s)\phi_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s \le \int_{t_3}^{+\infty}\frac{\sigma'(s)u_1^{(\gamma+1)/\gamma}(s)}{v_1^{1/\gamma}(s)r^{1/\gamma}(\sigma(s))}\,\mathrm{d}s < +\infty,$$

which contradicts (33).

Case 2. Assume now that $z'(t) < 0$ for $t \ge t_2$. It has been established in Theorem 2 that (29) holds. Using (29) and proceeding as in Case 1 above, we arrive at the desired conclusion. □

As an immediate consequence of Theorem 3, we have the following result.

Theorem 4 *Let ψ_1, ψ_2, v_1, and v_2 be as in Theorem* 3, *and assume that conditions* (H_1)-(H_4), (3), *and* (6) *are satisfied. Suppose also that there exist functions $H \in \mathcal{W}_\gamma$, $\rho_1, \rho_2 \in$ $C^1([t_0,+\infty),\mathbb{R})$, $\phi_1, \phi_2 \in C([t_0,+\infty),\mathbb{R})$ such that* (30), (33), *and* (34) *hold. If, for all $T \ge t_0$ and for some $\beta > 1$,*

$$\liminf_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\psi_1(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}\frac{v_1(s)r(\sigma(s))}{(\sigma'(s))^\gamma}h^{\gamma+1}(t,s)\right]\mathrm{d}s \ge \phi_1(T) \tag{40}$$

and

$$\liminf_{t\to+\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\psi_2(s) - \frac{\beta^\gamma}{(\gamma+1)^{\gamma+1}}v_2(s)r(s)h^{\gamma+1}(t,s)\right]\mathrm{d}s \ge \phi_2(T), \tag{41}$$

equation (1) *is oscillatory.*

3 Examples

Efficient oscillation tests can be easily derived from Theorems 2-4 with different choices of the functions H, ρ_1, ρ_2, ϕ_1, and ϕ_2. In this section, we illustrate possible applications with two examples.

Example 5 For $t \geq 1$, consider the second-order nonlinear neutral delay differential equation

$$\left(t^2\left(x(t) + \frac{t}{2t+1}x(t-1)\right)'\right)' + \left(2 + x^4(t)\right)x\left(\frac{t}{2}\right) = 0. \tag{42}$$

Here, $r(t) = t^2$, $p(t) = t/(2t+1)$, $\tau = 1$, $q(t) = 1$, $f(x(t), x(\sigma(t))) = (2 + x^4(t))x(t/2)$, whereas $R(t) = 1/t$.

Let $\gamma = 1$, $\kappa = 1$, $H(t,s) = (t-s)^2$, $\rho_1(t) = -1/(2t)$, $\rho_2(t) = -1/t$. Then $h^2(t,s) = 4$, $\nu_1(t) = \nu_2(t) = t^2$, $\psi_1(t) = t^2((t+2)/(2t+2)+1)$, $\psi_2(t) = t^2(3-(t^2/((2t+2)(t-2))))$, and a straightforward computation shows that all assumptions of Theorem 2 are satisfied. Hence, equation (42) is oscillatory.

Example 6 For $t \geq 1$, consider the second-order neutral delay differential equation

$$\left(e^t\left(x(t) + \frac{1}{3}x\left(t - \frac{\pi}{4}\right)\right)'\right)' + \frac{32\sqrt{65}}{3}e^t x\left(t - \frac{\arcsin(\sqrt{65}/65)}{8}\right) = 0. \tag{43}$$

Here, $r(t) = e^t$, $p(t) = 1/3$, $\tau = \pi/4$, $q(t) = 32\sqrt{65}e^t/3$, $R(t) = e^{-t}$, and $f(x(t), x(\sigma(t))) = x(t - (\arcsin(\sqrt{65}/65))/8)$.

Let $\gamma = 1$, $\kappa = 1$, $H(t,s) = (t-s)^2$, $\rho_1(t) = \rho_2(t) = 0$. Then $h^2(t,s) = 4$, $\nu_1(t) = \nu_2(t) = 1$, $\psi_1(t) = (64\sqrt{65}/9)e^t$, $\psi_2(t) = (32\sqrt{65}/3)(1-(1/3)e^{\pi/4})e^t$. It is not difficult to verify that all assumptions of Theorem 2 hold. Hence, equation (43) is oscillatory. In fact, one such solution is $x(t) = \sin 8t$.

4 Conclusions

Most oscillation results reported in the literature for neutral differential equation (1) and its particular cases have been obtained under the assumption (2) which significantly simplifies the analysis of the behavior of $z(t) = x(t) + p(t)x(t-\tau)$ for a nonoscillatory solution $x(t)$ of (1). In this paper, using a refinement of the integral averaging technique, we have established new oscillation criteria for second-order neutral delay differential equation (1) assuming that (3) holds.

We stress that the study of oscillatory properties of equation (1) in the case (3) brings additional difficulties. In particular, in order to deal with the case when $z'(t) < 0$ (which is simply eliminated if condition (2) holds), we have to impose an additional assumption $p(t) < R(t)/R(t-\tau) \leq 1$. In fact, it is well known (see, *e.g.*, [6, 14]) that if $x(t)$ is an eventually positive solution of (1), then

$$x(t) \geq \left(1 - p(t)\right)z(t). \tag{44}$$

One of the principal difficulties one encounters lies in the fact that (44) does not hold when (3) is satisfied, *cf.* [8]. Since the sign of the derivative $z'(t)$ is not known, our criteria for the oscillation of (1) include a pair of assumptions as, for instance, (8) and (9). On the other hand, we point out that, contrary to [8, 13, 18, 19], we do not need in our oscillation theorems quite restrictive conditions (4) and (5), which, in a certain sense, is a significant improvement compared to the results in the cited papers. However, this improvement has been achieved at the cost of imposing condition (6).

Therefore, two interesting problems for future research can be formulated as follows.

(P1) *Is it possible to establish oscillation criteria for* (1) *without requiring conditions* (4), (5), *and* (6)*?*

(P2) *Suggest a different method to investigate* (1) *in the case where* $\gamma < 1$ (*and thus inequality* (7) *does not hold*).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All three authors contributed equally to this work and are listed in alphabetical order. They all read and approved the final version of the manuscript.

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