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# Positive solutions to boundary value problems of a high-order fractional differential equation in a Banach space

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## Abstract

In this paper, by using the fixed-point theorem in the cone of strict-set-contraction operators, we study a class of higher-order boundary value problems of nonlinear fractional differential equation in a Banach space. The sufficient conditions for the existence of at least two positive solutions is obtained. In addition, an example to illustrate the main results is given.

**Keywords:** fractional differential equation; boundary value problem; measure of noncompactness; strict-set-contraction operators

## 1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. The fractional differential equations have been of great interest recently. This is because of both the intensive development of the theory of fractional calculus itself and its numerous applications in various fields of science and engineering including fluid flow, rheology, control, electrochemistry, electromagnetic, porous media and probability, *etc.* (see [1–4]).

In recent years, the existence and uniqueness of solutions of the initial and boundary value problems for fractional equations have been extensively studied (see [4–16] and the references therein). But there are few works that deal with the existence of solutions of nonlinear fractional differential equations in Banach spaces; see [17–24]. In [19], Hussein investigated the existence of pseudo solutions for the following nonlinear  $m$ -point boundary value problem of fractional type:

$$\begin{cases} D_{0+}^q u(t) + a(t)f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, & u(1) = \sum_{i=1}^{m-2} \zeta_i u(\eta_i) \end{cases}$$

in a reflexive Banach space  $E$ , where  $D_{0+}^q$  is the pseudo fractional differential operator of order  $n - 1 < q \leq n$ ,  $n \geq 2$ .

In [22], by the monotone iterative technique and the Mönch fixed point theorem, Lv *et al.* investigated the existence of a solution to the following Cauchy problem for the differ-

ential equation with fractional order in a real Banach space  $E$ :

$${}^C D^q u(t) = f(t, u(t)), \quad u(0) = u_0,$$

where  ${}^C D^q u(t)$  is the Caputo derivative of order  $0 < q < 1$ .

By means of Darbo's fixed point theorem, Su [23] established the existence result of solutions to the following boundary value problem of fractional differential equation on unbounded domain  $[0, +\infty)$ :

$$\begin{cases} D_{0+}^q u(t) = f(t, u(t)), & t \in [0, +\infty), 1 < q \leq 2, \\ u(0) = \theta, & D_{0+}^{q-1} u(\infty) = u_\infty \end{cases}$$

in a Banach space  $E$ .  $D_{0+}^q$  is the Riemann-Liouville fractional derivative.

Being directly inspired by [9, 15, 19, 22] but taking quite a different method from that in [15, 19–23], we discuss in this paper the following high-order boundary value problem (BVP for short) in a Banach space  $E$ :

$$\begin{cases} D_{0+}^q u(t) + f(t, u, u', \dots, u^{(n-2)}) = \theta, & \forall t \in (0, 1), q \in (n-1, n], \\ u^{(i)}(0) = 0, & 0 \leq i \leq n-3, \\ \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = \theta, & \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = \theta, \end{cases} \quad (1.1)$$

where  $\theta$  is the zero element of  $E$ ,  $n \geq 2$ ,  $\alpha, \beta, \gamma$  and  $\delta$  are nonnegative constants satisfying  $\rho^{-1} = \alpha\gamma + \alpha\delta + \beta\gamma > 0$ , and  $D_{0+}^q$  is the Caputo fractional derivative. Note that the nonlinear term  $f$  depends on  $u$  and its derivatives  $u', u'', \dots, u^{(n-2)}$ .

The paper is organized as follows. In Section 2 we give some basic definitions in Riemann-Liouville fractional calculus and the Kuratowski noncompactness. In Section 3 we present the expression and properties of Green's function associated with BVP (1.1), and by using the fixed-point theorem for strict-set-contraction operators and introducing a new cone  $\Omega$ , we obtain the existence of at least two positive solutions for BVP (1.1) under certain conditions on the nonlinearity. Moreover, an example illustrating our main result is given in Section 4.

## 2 Preliminaries and lemmas

For convenience of the reader, we present here some definitions and preliminaries which are used throughout the paper. These definitions and lemmas can be found in the recent literature such as [1, 5].

**Definition 2.1** ([1]) The Riemann-Liouville fractional integral of order  $q > 0$  of a function  $y(t)$  is given by

$$I_{0+}^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds,$$

provided that the right-hand side is defined pointwise.

**Definition 2.2** ([1]) The fractional derivative of order  $q > 0$  of a function  $y : \mathbf{R}_0^+ \rightarrow \mathbf{R}$  is given by

$$D_{0+}^q y(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-q-1} y(s) ds,$$

where  $n = [q] + 1$ ,  $[q]$  denotes the integer part of number  $q$ , provided that the right-hand side is defined pointwise. In particular, for  $q = n$ ,  $D_{0+}^q y(t) = y^{(n)}(t)$ .

**Lemma 2.3** ([1]) Let  $q > 0$ . Then the fractional differential equation

$$D_{0+}^q y(t) = 0$$

has the unique solution  $y(t) = c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n}$ ,  $c_i \in \mathbf{R}$ ,  $i = 1, 2, \dots, n$ , here  $n - 1 < q \leq n$ .

**Lemma 2.4** ([5]) Let  $q > 0$ . Then the following equality holds for  $y \in L(0, 1)$ ,  $D_{0+}^q y \in L(0, 1)$ :

$$I_{0+}^q D_{0+}^q y(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_N t^{q-N}$$

for some  $c_i \in \mathbf{R}$ ,  $i = 1, 2, \dots, N$ , here  $N$  is the smallest integer greater than or equal to  $q$ .

Let the real Banach space  $E$  with the norm  $\|\cdot\|$  be partially ordered by a cone  $P$  of  $E$ , i.e.,  $u \leq v$  if and only if  $v - u \in P$ , and  $P$  is said to be normal if there exists a positive constant  $N$  such that  $\theta \leq u \leq v$  implies  $\|u\| \leq N\|v\|$ , where the smallest  $N$  is called the normal constant of  $P$ . For details on cone theory, see [25].

The basic space used in this paper is  $C[I, E]$ . For any  $u \in C[I, E]$ , evidently,  $(C[I, E], \|\cdot\|_C)$  is a Banach space with the norm  $\|u\|_C = \sup_{t \in I} |u(t)|$ , and  $P = \{u \in C[I, E] : u(t) \geq \theta \text{ for } t \in I\}$  is a cone of the Banach space  $C[I, E]$ . We use  $\alpha$ ,  $\alpha_C$  to denote the Kuratowski noncompactness measure of bounded sets in the spaces  $E$ ,  $C(I, E)$ , respectively. As for the definition of the Kuratowski noncompactness measure, we refer to Ref. [25].

**Definition 2.5** ([25], Strict-set contraction operator) Let  $E_1, E_2$  be real Banach spaces,  $S \subset E_1$ .  $T : S \rightarrow E_2$  is a continuous and bounded operator. If there exists a constant  $k$  such that  $\alpha(T(S)) \leq k\alpha(S)$ , then  $T$  is called a  $k$ -set contraction operator. When  $k < 1$ ,  $T$  is called a strict-set contraction operator.

**Lemma 2.6** ([25]) If  $D \subset C[I, E]$  is bounded and equicontinuous, then  $\alpha(D(t))$  is continuous on  $I$  and

$$\alpha_C(D) = \max_{t \in I} \alpha(D(t)), \quad \alpha\left(\left\{\int_I u(t) dt : u \in D\right\}\right) \leq \int_I \alpha(D(t)) dt,$$

where  $D(t) = \{u(t) : u \in D, t \in I\}$ .

**Lemma 2.7** ([25]) Let  $K$  be a cone in a Banach space  $E$ . Assume that  $\Omega_1, \Omega_2$  are open subsets of  $E$  with  $\theta \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$ . If  $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow K$  is a strict-set contraction operator such that either

- (i)  $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_1$  and  $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_2$ , or  
 (ii)  $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_1$  and  $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_2$ ,  
 then  $T$  has a fixed point in  $K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

In the paper, we always assume that the following three assumptions hold:

(H0)  $0 < \rho^* := \rho \int_0^1 \varphi(s) ds < +\infty$ , where  $\varphi(s)$  is defined as

$$\varphi(s) = \frac{1}{\Gamma(q-n+2)}(\beta + \alpha s)[\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n}.$$

(H1) There exist  $a \in C[I, R_+]$  and  $h \in C[R_+^{n-1}, R_+]$  such that

$$\begin{aligned} \|f(t, u_1, \dots, u_{n-1})\| &\leq a(t)h(\|u_1\|, \dots, \|u_{n-1}\|), \\ \forall t \in I, u_k \in P, k &= 1, \dots, n-1. \end{aligned} \tag{2.1}$$

(H2)  $f : I \times P_r^{n-1} \rightarrow P$  for any  $r > 0$ ,  $f$  is uniformly continuous on  $I \times P_r^{n-1}$  and there exist nonnegative constants  $L_k, k = 1, \dots, n-1$ , with

$$\frac{2}{\rho^*} \left( \sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1} \right) < 1 \tag{2.2}$$

such that

$$\alpha(f(t, D_1, D_2, \dots, D_{n-1})) \leq \sum_{k=1}^{n-1} L_k \alpha(D_k), \quad \forall t \in I, \text{ bounded } D_k \in P_r, \tag{2.3}$$

where  $P_r = \{u \in P : \|u\| \leq r\}$ .

### 3 Main results

**Lemma 3.1** Given  $y \in C[I, E]$ , then the unique solution of

$$\begin{cases} D_{0^+}^{q-n+2} x(t) + y(t) = 0, & 0 < t < 1, n-1 < q \leq n, n > 2, \\ \alpha x(0) - \beta x'(0) = 0, & \gamma x(1) + \delta x'(1) = 0 \end{cases} \tag{3.1}$$

is

$$x(t) = \int_0^1 G(t, s)y(s) ds, \tag{3.2}$$

where

$$G(t, s) = \begin{cases} G_1(t, s) = -\frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} + G_0(t, s), & 0 \leq s \leq t \leq 1, \\ G_0(t, s), & 0 \leq t \leq s \leq 1, \end{cases} \tag{3.3}$$

and

$$G_0(t, s) = \frac{\rho(\beta + \alpha t)}{\Gamma(q-n+2)}[\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n}.$$

*Proof* Deduced from Lemma 2.4, we have

$$x(t) = -I_{0+}^{q-n+2} y(t) + c_1 + c_2 t$$

for some  $c_1, c_2 \in \mathbf{R}$ . Then we get

$$x(t) = -\frac{1}{\Gamma(q-n+2)} \int_0^t (t-s)^{q-n+1} y(s) ds + c_1 + c_2 t,$$

$$x'(t) = -\frac{1}{\Gamma(q-n+1)} \int_0^t (t-s)^{q-n} y(s) ds + c_2.$$

By boundary conditions  $\alpha x(0) - \beta x'(0) = 0$ ,  $\gamma x(1) + \delta x'(1) = 0$ , and noting that  $\Gamma(q-n+2) = (q-n+1)\Gamma(q-n+1)$ , we have

$$c_1 = \frac{\rho\beta}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} y(s) ds$$

and

$$c_2 = \frac{\rho\alpha}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} y(s) ds.$$

Thus

$$\begin{aligned} x(t) &= -\int_0^t \frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} y(s) ds \\ &\quad + \frac{\rho\beta}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} y(s) ds \\ &\quad + \frac{\rho\alpha t}{\Gamma(q-n+2)} \int_0^1 [\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n} y(s) ds \\ &= -\int_0^t \frac{(t-s)^{q-n+1}}{\Gamma(q-n+2)} y(s) ds + \rho(\beta + \alpha t) \int_0^1 \frac{[\gamma(1-s) + \delta(q-n+1)](1-s)^{q-n}}{\Gamma(q-n+2)} y(s) ds \\ &= \int_0^1 G(t,s) y(s) ds, \end{aligned}$$

where  $G(t,s)$  is Green's function defined by (3.3). This completes the proof. □

Moreover, there is one paper [10] in which the following statement was shown.

**Lemma 3.2** ([10]) *The function  $G(t,s)$  defined by Lemma 2.3 has the following properties:*

- (i)  $G(t,s)$  is continuous on  $[0,1] \times [0,1]$ ;
- (ii) if  $\beta > \frac{n-q}{q-n+1} \alpha$ , then  $0 < G(t,s) \leq G(s,s)$  for any  $t, s \in [0,1]$ .

**Lemma 3.3** *If  $\beta > \frac{n-q}{q-n+1} \alpha$ , then the function  $G(t,s)$  satisfies:*

$$\lambda\varphi(s) \leq G(t,s) \leq \rho\varphi(s) \quad \text{for } t, s \in [0,1],$$

where

$$\lambda := \min \left\{ \frac{4\rho\alpha\gamma\delta[\beta(q-n+1) + \alpha(q-n)]}{\zeta}, \frac{4\rho\alpha\beta\gamma\delta(q-n+1)}{\zeta} \right\},$$

$$\zeta := [\alpha\gamma - \beta\gamma + \alpha\delta(q-n+1)]^2 + 4\alpha\beta\gamma[\gamma + \delta(q-n+1)].$$

*Proof* According to (3.3) and Lemma 3.2, we have

$$G^*(s) \leq G(t, s) \leq \rho\varphi(s),$$

where

$$G^*(s) = \begin{cases} G_1(1, s), & 0 \leq s < \frac{\beta\gamma - \alpha\delta(q-n)}{\alpha\delta + \beta\gamma}, \\ G_0(0, s), & \frac{\beta\gamma - \alpha\delta(q-n)}{\alpha\delta + \beta\gamma} \leq s < 1. \end{cases}$$

Since

$$\begin{aligned} \inf_{0 < s < 1} \frac{G_1(1, s)}{\varphi(s)} &= \inf_{0 < s < 1} \frac{-\rho^{-1}(1-s) + (\beta + \alpha)[\gamma(1-s) + \delta(q-n+1)]}{\rho^{-1}(\beta + \alpha s)[\gamma(1-s) + \delta(q-n+1)]} \\ &\geq \frac{\rho\delta[\beta(q-n+1) + \alpha(q-n)]}{(\beta + \alpha s)[\gamma(1-s) + \delta(q-n+1)]} \\ &\geq \frac{4\rho\alpha\gamma\delta[\beta(q-n+1) + \alpha(q-n)]}{[\alpha\gamma - \beta\gamma + \alpha\delta(q-n+1)]^2 + 4\alpha\beta\gamma[\gamma + \delta(q-n+1)]} \\ &= \frac{4\rho\alpha\gamma\delta[\beta(q-n+1) + \alpha(q-n)]}{\zeta}, \end{aligned}$$

and

$$\begin{aligned} \inf_{0 < s < 1} \frac{G_0(0, s)}{\varphi(s)} &= \inf_{0 < s < 1} \frac{\rho[\beta\gamma(1-s) + \beta\delta(q-n+1)]}{(\beta + \alpha s)[\gamma(1-s) + \delta(q-n+1)]} \\ &\geq \frac{4\rho^{-1}\alpha\beta\gamma\delta(q-n+1)}{[\alpha\gamma - \beta\gamma + \alpha\delta(q-n+1)]^2 + 4\alpha\beta\gamma[\gamma + \delta(q-n+1)]} \\ &= \frac{4\rho\alpha\beta\gamma\delta(q-n+1)}{\zeta}, \end{aligned}$$

then we get

$$\lambda\varphi(s) \leq G(t, s) \leq G(s, s) \leq \rho\varphi(s). \quad \square$$

**Lemma 3.4** Let  $u(t) = I_{0^+}^{n-2}x(t)$ ,  $x \in C[I, E]$ . Then problem (1.1) can be transformed into the following modified problem:

$$\begin{cases} D_{0^+}^{q-n+2}x(t) + f(s, I_{0^+}^{n-2}x(s), \dots, I_{0^+}^1x(s), x(s)) = \theta, \\ \alpha x(0) - \beta x'(0) = \theta, \quad \gamma x(1) + \delta x'(1) = \theta, \end{cases} \quad (3.4)$$

where  $0 < t < 1$ ,  $n-1 < q \leq n$ ,  $n \geq 2$ . Moreover, if  $x \in C[I, E]$  is a solution of problem (3.3) and  $x \geq \theta$ ,  $x \not\equiv \theta$ , then the function  $u(t) = I_{0^+}^{n-2}x(s)$  is a positive solution of (1.1).

To obtain a positive solution, we construct a cone  $\Omega$  by

$$\Omega = \left\{ x(t) \in P : x(t) \geq \frac{\lambda}{\rho} x(s), \forall t, s \in I \right\}, \quad (3.5)$$

where  $P = \{x \in C[I, E] : x(t) \geq \theta, t \in I\}$ .

Define an integral operator  $T : \Omega \rightarrow E$  by

$$(Tx)(t) = \int_0^1 G(t, s) f(s, I_0^{n-2} x(s), I_0^{n-3} x(s), \dots, I_0^1 x(s), x(s)) ds, \quad 0 \leq t \leq 1. \quad (3.6)$$

**Lemma 3.5** Assume that (H0)-(H2) hold. Then  $T : \Omega \rightarrow \Omega$  is a strict-set contraction operator.

*Proof* From Lemma 3.3 and (3.6), we obtain

$$(Tx)(t) \geq \lambda \int_0^1 \varphi(s) f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) ds.$$

On the other hand,

$$\begin{aligned} (Tx)(s) &= \int_0^1 G(s, s) f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) ds \\ &\leq \rho \int_0^1 \varphi(s) f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) ds \\ &= \frac{\rho}{\lambda} \int_0^1 \lambda \varphi(s) f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) ds \leq \frac{\rho}{\lambda} (Tx)(t). \end{aligned}$$

Then  $(Tx)(t) \geq \frac{\lambda}{\rho} (Tx)(s)$ , which implies  $(Tx) \in \Omega$ , i.e.,  $T(\Omega) \subset \Omega$ .

Next we prove that  $T$  is continuous on  $\Omega$ . Let  $\{x_j\}, \{x\} \subset \Omega$  and  $\|x_j - x\|_{\Omega} \rightarrow 0$  ( $j \rightarrow \infty$ ). Hence  $\{x_j\}$  is a bounded subset of  $\Omega$ . Thus, there exists  $r > 0$  such that  $r = \sup_j \|x_j\|_{\Omega} < \infty$  and  $\|x\|_{\Omega} \leq r$ .

According to the properties of  $f$ , for  $\forall \varepsilon > 0$ , there exists  $J > 0$  such that

$$\|f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) - f(s, I_0^{n-2} x(s), \dots, I_0^1 x(s), x(s))\| \leq \frac{\varepsilon}{\rho^*}$$

for  $j \geq J, \forall t \in I$ .

Then

$$\begin{aligned} \|(Tx_j)(t) - (Tx)(t)\| &\leq \int_0^1 G(t, s) \|f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) \\ &\quad - f(s, I_0^{n-2} x(s), \dots, I_0^1 x(s), x(s))\| ds \\ &\leq \rho \int_0^1 \varphi(s) ds \cdot \|f(s, I_0^{n-2} x_j(s), \dots, I_0^1 x_j(s), x_j(s)) \\ &\quad - f(s, I_0^{n-2} x(s), \dots, I_0^1 x(s), x(s))\| < \varepsilon. \end{aligned}$$

Therefore,  $\forall \varepsilon > 0$  for any  $t \in I$  and  $j \geq J$ . We get

$$\|(Tx_j)(t) - (Tx)(t)\| \rightarrow 0.$$

This implies  $T$  is continuous on  $\Omega$ .

By the properties of continuity of  $G(t, s)$ , it is easy to see that  $T$  is equicontinuous on  $I$ .

Finally, we are going to show that  $T$  is a strict-set contraction operator. Let  $D \subset \Omega$  be bounded. Then, by condition (H1), Lemma 3.1 implies  $\alpha_C(TD) = \max_{t \in I} \alpha((TD)(t))$ . It follows from (3.6) that

$$\begin{aligned} \alpha((TD)(t)) &= \alpha\left(\left\{\int_0^1 G(t, s)f(s, (T_1x)(s), \dots, (T_{n-2}x)(s), x(s)) ds \in D\right\}\right) \\ &\leq \int_0^1 \alpha(\{G(t, s)f(s, (T_1x)(s), \dots, (T_{n-2}x)(s), x(s)) ds \in D\}) \\ &\leq \rho \int_0^1 \varphi(s)\alpha(f(I \times (I_0^{n-2}D)(I) \times \dots \times (I_0^1D)(I) \times D(I))) ds \\ &\leq \rho^* \left(\sum_{k=1}^{n-2} L_k \alpha((I_0^{n-1-k}D)(I)) + L_{n-1} \alpha(D(I))\right), \end{aligned}$$

which implies

$$\alpha_C(TD) \leq \rho^* \left(\sum_{k=1}^{n-2} L_k \alpha((I_0^{n-1-k}D)(I)) + L_{n-1} \alpha(D(I))\right). \tag{3.7}$$

Obviously,

$$\begin{aligned} \alpha(I_0^{n-1-k}D)(I) &= \alpha\left(\left\{\int_0^s \frac{(s-v)^{n-2-k}}{(n-2-k)!} x(s) ds : v \in [0, s], s \in I, k = 1, \dots, n-2\right\}\right) \\ &\leq \frac{1}{(n-2-k)!} \alpha(D(I)). \end{aligned} \tag{3.8}$$

Using a similar method as in the proof of Theorem 2.1.1 in [25], we have

$$\alpha(D(I)) \leq 2\alpha_C(D). \tag{3.9}$$

Therefore, it follows from (3.7), (3.8) and (3.9) that

$$\alpha_C(TD) \leq \frac{2}{\rho^*} \left(\sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1}\right) \alpha_C(D).$$

Noticing that (2.3), we obtain that  $T$  is a strict-set contraction operator. This completes the proof. □

Now we are in a position to give the main result of this work.

**Theorem 3.6** *Let the cone  $P$  be normal and conditions (H0)-(H2) hold. In addition, assume that the following conditions are satisfied:*



(H3) There exist  $u^* \in P \setminus \{\theta\}$ ,  $c_1 \in C[I, R_+]$  and  $h_1 \in C[P^{n-1}, R_+]$  such that

$$f(t, u_1, \dots, u_{n-1}) \geq c_1(t)h_1(u_1, \dots, u_{n-1})u^*, \quad \forall t \in I, u_k \in P,$$

and

$$h_1^\infty = \lim_{\sum_{k=1}^{n-1} \|u_k\| \rightarrow \infty} \frac{h_1(u_1, \dots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_k\|} > \Delta^{-1}, \quad \text{as } u_k \in P,$$

where  $\Delta := N^{-2} \rho^{-1} (\lambda^2 \int_0^1 \varphi(s)c_1(s) ds \|u^*\|)$ .

(H4) There exist  $u_* \in P \setminus \{\theta\}$ ,  $c_2 \in C[I, R_+]$  and  $h_2 \in C[P^{n-1}, R_+]$  such that

$$f(t, u_1, \dots, u_{n-2}) \geq c_2(t)h_2(u_1, \dots, u_{n-1})u_*, \quad \forall t \in I, u_k \in P,$$

and

$$h_1^0 = \lim_{\sum_{k=1}^{n-1} \|u_k\| \rightarrow 0} \frac{h_2(u_1, \dots, u_{n-1})}{\sum_{k=1}^{n-1} \|u_i\|} > \nabla^{-1}, \quad \text{as } u_k \in P,$$

where  $\nabla := N^{-2} \rho^{-1} (\lambda^2 \int_0^1 \varphi(s)c_2(s) ds \|u_*\|)$ .

(H5) There exists  $\xi > 0$  such that

$$\rho N M_\xi \int_0^1 \varphi(s)a(s) ds < \xi,$$

where  $M_\xi = \max_{u_k \in P_\xi} \{h(\|u_1\|, \dots, \|u_{n-1}\|)\}$ .

Then problem (1.1) has at least two different positive solutions.

*Proof* Consider condition (H3), there exists  $r_1 > 0$  such that

$$h_1(u_1, \dots, u_{n-1}) \geq (h_1^\infty - \varepsilon_1) \sum_{k=1}^{n-1} \|u_k\|, \quad \forall u_k \in P, \sum_{k=1}^{n-1} \|u_k\| \geq r_1,$$

where  $\varepsilon_1 > 0$  satisfies  $(h_1^\infty - \varepsilon_1)\Delta \geq 1$ .

Therefore,

$$f(t, u_1, \dots, u_{n-1}) \geq (h_1^\infty - \varepsilon_1) \sum_{k=1}^{n-1} \|u_k\| \cdot c_1(t)u^*, \quad \forall u_k \in P, \sum_{k=1}^{n-1} \|u_k\| \geq r_1.$$

Take

$$R > \max\{N\rho\lambda^{-1}r_1, \xi\}.$$

Then, for  $t \in I$ ,  $x \in \Omega$ ,  $\|x\|_\Omega = R$ , we have by (3.5)

$$\|x(t)\| \geq \lambda(\rho N)^{-1} \|x\|_\Omega \geq \lambda(\rho N)^{-1} R > r_1.$$

Thus

$$\begin{aligned}
 (Tx)(t) &= \int_0^1 G(t,s)f(s, I_0^{n-2}x(s), \dots, I_0^1x(s), x(s)) ds \\
 &\geq \int_0^1 \lambda\varphi(s)f(s, I_0^{n-2}x(s), \dots, I_0^1x(s), x(s)) ds \\
 &\geq (h_1^\infty - \varepsilon_1)\lambda \int_0^1 \varphi(s) \left( \sum_{k=1}^{n-2} \|I_0^{n-1-k}x(s)\| + \|x(s)\| \right) c_1(s) ds \cdot u^* \\
 &\geq (h_1^\infty - \varepsilon_1)\lambda \int_0^1 \varphi(s)c_1(s)\|x(s)\| ds \cdot u^* \\
 &\geq (h_1^\infty - \varepsilon_1)\lambda^2(\rho N)^{-1}\|x\|_\Omega \left( \int_0^1 \varphi(s)c_1(s) ds \right) \cdot u^* \\
 &= (h_1^\infty - \varepsilon_1) \frac{\lambda^2}{\rho N^2} \left( \int_0^1 \varphi(s)c_1(s) ds \|u^*\| \right) \cdot \frac{N\|x\|_\Omega}{\|u^*\|} u^* \\
 &\geq \frac{N\|x\|_\Omega}{\|u^*\|} \cdot u^*, \tag{3.10}
 \end{aligned}$$

and consequently,

$$\|Tx\|_\Omega \geq \|x\|_\Omega, \quad \forall x \in \Omega, \|x\|_\Omega = R. \tag{3.11}$$

Similarly, by condition (H4), there exists  $r_2 > 0$  such that

$$h_2(u_1, \dots, u_{n-1}) \geq (h_1^0 - \varepsilon_2) \sum_{k=1}^{n-1} \|u_k\|, \quad \forall u_k \in P, 0 < \sum_{k=1}^{n-1} \|u_k\| \leq r_2,$$

where  $\varepsilon_2 > 0$  satisfies  $(h_1^0 - \varepsilon_2)\nabla \geq 1$ .

Therefore,

$$f(t, u_1, \dots, u_{n-1}) \geq (h_1^0 - \varepsilon_2) \sum_{k=1}^{n-1} \|u_k\| \cdot c_2(t)u_*, \quad \forall u_k \in P, 0 < \sum_{k=1}^{n-1} \|u_k\| \leq r_2.$$

Choose

$$0 < r < \min \left\{ \left( \sum_{k=0}^{n-2} \frac{1}{k!} \right)^{-1} r_2, \xi \right\}.$$

Then, for  $t \in I_\tau$ ,  $x \in \Omega$ ,  $\|x\|_\Omega = r$ , similar to (3.10), we have

$$\begin{aligned}
 (Tx)(t) &\geq (h_1^0 - \varepsilon_2)\lambda \int_0^1 \varphi(s)c_2(s)\|x(s)\| ds \cdot u_* \\
 &\geq (h_1^0 - \varepsilon_2)\lambda^2(\rho N)^{-1}\|x\|_\Omega \left( \int_0^1 \varphi(s)c_2(s) ds \right) \cdot u_* \\
 &= (h_1^0 - \varepsilon_2) \frac{\lambda^2}{\rho N^2} \left( \int_0^1 \varphi(s)c_2(s) ds \|u_*\| \right) \cdot \frac{N\|x\|_\Omega}{\|u_*\|} u_* \\
 &\geq \frac{N\|x\|_\Omega}{\|u_*\|} \cdot u_*,
 \end{aligned}$$

which implies

$$\|Tx\|_{\Omega} \geq \|x(s)\|_{\Omega}, \quad \forall x \in \Omega, \|x\|_{\Omega} = r. \tag{3.12}$$

On the other hand, according to Lemma 3.3 and (3.6), we get

$$(Tx)(t) \leq \rho \int_0^1 \varphi(s)f(s, I_0^{n-2}x(s), \dots, I_0^1x(s), x(s)) ds. \tag{3.13}$$

By condition (H1), for  $t \in I, x \in \Omega, \|x\|_{\Omega} = \xi$ , we have

$$\|f(t, u_1, \dots, u_{n-1})\| \leq a(t)h(\|u_1\|, \dots, \|u_{n-1}\|) \leq M_{\xi}a(t).$$

Therefore,

$$\|(Tx)(t)\|_{\Omega} \leq \rho NM_{\xi} \cdot \int_0^1 \varphi(s)a(s) ds < \xi = \|x\|_{\Omega}. \tag{3.14}$$

Applying Lemma 2.7 to (3.12), (3.13) and (3.14) yields that  $T$  has a fixed point  $x_1 \in \bar{\Omega}_{r,\xi}$ ,  $r \leq \|x_1\| \leq \xi$ , and a fixed point  $x_2 \in \bar{\Omega}_{\xi,R}$ ,  $\xi \leq \|x_2\| \leq R$ . Noticing (3.13), we get  $\|x_1\| \neq \xi$  and  $\|x_2\| \neq \xi$ . This and Lemma 3.4 complete the proof.  $\square$

**Theorem 3.7** *Let the cone  $P$  be normal and conditions (H0) ~ (H3) hold. In addition, assume that the following condition is satisfied:*

(H6)

$$\frac{h(\|u_1\|, \dots, \|u_{n-1}\|)}{\sum_{k=1}^{n-1} \|u_k\|} \rightarrow 0, \quad \text{as } u_k \in P, \sum_{k=1}^{n-1} \|u_k\| \rightarrow 0^+. \tag{3.15}$$

Then problem (1.1) has at least one positive solution.

*Proof* By (H3), we can choose  $R > N\rho\lambda^{-1}r_1$ . As in the proof of Theorem 3.6, it is easy to see that (3.11) holds. On the other hand, considering (3.15), there exists  $r_3 > 0$  such that

$$h(\|u_1\|, \dots, \|u_{n-1}\|) \leq \varepsilon_3 \sum_{k=1}^{n-1} \|u_k\|, \quad \text{for } t \in I, u_k \in P, 0 < \sum_{k=1}^{n-1} \|u_k\| \leq r_3,$$

where  $\varepsilon_3 > 0$  satisfies

$$\varepsilon_3 = \left( N\rho \sum_{k=1}^{n-1} \frac{1}{k!} \int_0^1 \varphi(s)a(s) ds \right)^{-1}.$$

Choose  $0 < r^* < \min\{(\sum_{k=0}^{n-2} \frac{1}{k!})^{-1}r_3, R\}$ . For  $t \in I, x \in \Omega, \|x\|_{\Omega} = r^*$ , it follows from (3.5) that

$$\begin{aligned} 0 < (\rho N)^{-1}\lambda r^* &\leq \|x\| < r_3, \\ 0 < (\rho N)^{-1}\lambda r^* &\leq \sum_{k=1}^{n-1} \|I_0^{n-1-k}x(t)\| \leq \sum_{k=1}^{n-1} \frac{1}{k!} \|x\| < r_3. \end{aligned} \tag{3.16}$$

Then, for  $t \in I, x \in \Omega, \|x\|_{\Omega} = r^*$ , we have

$$\begin{aligned} \|(Tx)(t)\| &\leq \rho \int_0^1 \varphi(s) \|f(s, I_{0^+}^{n-2}x(s), \dots, I_{0^+}^1x(s), x(s))\| ds \\ &\leq \rho \int_0^1 \varphi(s) a(s) h(\|I_{0^+}^{n-2}x(s)\|, \dots, \|I_{0^+}^1x(s)\|, \|x(s)\|) ds \\ &\leq \varepsilon_3 \rho \int_0^1 \varphi(s) a(s) \left( \sum_{k=1}^{n-2} \|I_{0^+}^{n-1-k}x(s)\| + \|x(s)\| \right) ds \\ &\leq N \varepsilon_3 \rho \sum_{k=0}^{n-1} \frac{1}{k!} r^* \int_0^1 \varphi(s) a(s) ds = r^*, \end{aligned}$$

and consequently,

$$\|(Tx)(t)\|_{\Omega} \leq \|x\|_{\Omega}, \quad \forall x \in \Omega, \|x\| \leq r^*. \tag{3.17}$$

Since  $0 < r^* < R$ , applying Lemma 2.7 to (3.11) and (3.17) yields that  $T$  has a fixed point  $x_0 \in \bar{\Omega}_{r^*, R}, r^* \leq \|x_0\| \leq R$ . This and Lemma 3.5 complete the proof.  $\square$

#### 4 An example

Consider the following system of scalar differential equations of fractional order:

$$\begin{cases} -D^{\frac{5}{2}} u_k(t) = \frac{1}{24} u_k(t) + \frac{1+t}{324k^2} \{ [2u_{k+2}(t) + 2u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_{2j}(t)]^3 \\ \quad + \sqrt{3u_{2k}(t) + 3u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_j(t)} \}, \quad t \in I, \\ u_k(0) = u'_k(0) = 0, \quad \frac{1}{4} u_k(0) - \frac{1}{2} u'_k(0) = 0, \\ u_k(1) + \frac{1}{2} u'_k(1) = 0, \quad k = 1, 2, 3, \dots \end{cases} \tag{4.1}$$

**Conclusion** Problem (4.1) has at least two positive solutions.

*Proof* Let  $E = l^1 = \{u = (u_1, u_2, \dots, u_k, \dots) : \sum_{k=1}^{\infty} |u_k| < \infty\}$  with the norm  $\|u\| = \sum_{k=1}^{\infty} |u_k|$ , and  $P = \{(u_1, \dots, u_k, \dots) : u_k \geq 0, k = 1, 2, 3, \dots\}$ . Then  $P$  is a normal cone in  $E$  with a normal constant  $N = 1$ , and system (4.1) can be regarded as a boundary value problem of the form (1.1). In this situation,  $q = \frac{5}{2}, n = 3, \alpha = \frac{1}{4}, \beta = \frac{1}{2}, \gamma = 1, \delta = \frac{1}{2}, u = (u_1, \dots, u_n, \dots), f = (f_1, f_2, \dots, f_n, \dots)$ , in which

$$\begin{aligned} f_k(t, u_k, u'_k) &= \frac{1}{24} u_k(t) + \frac{1+t}{324k^2} \left\{ \left[ 2u_{k+2}(t) + 2u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_{2j}(t) \right]^3 \right. \\ &\quad \left. + \sqrt{3u_{2k}(t) + 3u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_j(t)} \right\}. \end{aligned} \tag{4.2}$$

By calculating, we have  $\rho = \frac{4}{5}, \lambda \approx 0.1515$ , and

$$\frac{1}{2} = \beta > \frac{n-q}{q-n+1} \alpha = \frac{1}{4},$$

and

$$\rho^* = \rho \int_0^1 \varphi(s) ds = \frac{1}{5\sqrt{\pi}} \int_0^1 (2+s) \left( \sqrt{1-s} + \frac{1}{4\sqrt{1-s}} \right) ds = \frac{44}{75\sqrt{\pi}} \approx 0.3310,$$

which implies that condition (H0) is satisfied. Observing the inequality  $\sum_{k=1}^{\infty} \frac{1}{k^2} < 2$ , we get, by (4.2),

$$\|f(t, u, v)\| = \sum_{k=1}^{\infty} |f_k(t, u_k, v_k)| \leq \frac{1+t}{3} \left( \frac{1}{8} \|u\| + \frac{1}{2} (\|u\| + \|v\|)^3 + \frac{1}{27} \sqrt{\|u\| + \|v\|} \right). \quad (4.3)$$

Hence (H1) is satisfied for  $a(t) = \frac{1+t}{3}$ , and

$$h(x, y) = \frac{1}{8}x + \frac{1}{2}(x+y)^3 + \frac{1}{27}\sqrt{x+y}.$$

Now, we check condition (H2). Obviously,  $f : I \times P_r^2 \rightarrow P$  for any  $r > 0$ , and  $f$  is uniformly continuous on  $I \times P_r^2$ . Let  $f = f^{(1)} + f^{(2)}$ , where  $f^{(1)} = (f_1^{(1)}, \dots, f_k^{(1)}, \dots)$  and  $f^{(2)} = (f_1^{(2)}, \dots, f_k^{(2)}, \dots)$ , in which

$$f_k^{(1)}(t, u, v) = \frac{1+t}{324k^2} \left\{ \left[ 2u_{k+2}(t) + 2u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_{2j}(t) \right]^3 + \sqrt{3u_{2k}(t) + 3u'_{k+1}(t) + \sum_{j=1}^{\infty} u_j(t) + \sum_{j=1}^{\infty} u'_j(t)} \right\} \quad (k = 1, 2, 3, \dots), \quad (4.4)$$

and

$$f_k^{(2)}(t, u, v) = \frac{1}{24}u_k(t) \quad (k = 1, 2, 3, \dots). \quad (4.5)$$

For any  $t \in I$  and bounded subsets  $D_1, D_2 \subset E$ , by (4.4), (4.5), we know

$$\alpha(f^{(2)}(I, D_1, D_2)) \leq \frac{1}{24}\alpha(D_1), \quad \forall t \in I, D_1, D_2 \subset E, \quad (4.6)$$

and

$$\begin{aligned} 0 &\leq \|f^{(1)}(t, u, v)\| = \sum_{k=1}^{\infty} |f_k^{(1)}(t, u_k, v_k)| \\ &\leq (\|u\| + \|v\|)^3 + \frac{2}{27}\sqrt{\|u\| + \|v\|}, \quad \forall t \in I, u, v \in E. \end{aligned}$$

Similar to the proof of [25, Example 2.12], we have

$$\alpha(f^{(1)}(t, D_1, D_2)) = 0, \quad \forall t \in I, \text{ bounded sets } D_1, D_2 \subset E. \quad (4.7)$$

It follows from (4.6) and (4.7) that

$$\alpha(f(I, D_1, D_2)) \leq \frac{1}{24}\alpha(D_1), \quad \forall t \in I, D_1, D_2 \subset E,$$

and

$$\frac{2}{\rho^*} \left( \sum_{k=1}^{n-2} \frac{L_k}{(n-2-k)!} + L_{n-1} \right) = \frac{24}{29\sqrt{\pi}} \approx 0.2518 < 1,$$

i.e., condition (H2) holds for  $L_1 = \frac{1}{24}$ ,  $L_2 = 0$ .

On the other hand, by (4.2), we have

$$f_k(t, u, v) \geq \frac{1+t}{324k^2} (\|u\| + \|v\|)^3, \quad \forall t \in I_\tau, u, v \in P \ (k = 1, 2, 3, \dots),$$

and

$$f_k(t, u, v) \geq \frac{1+t}{324k^2} \sqrt{\|u\| + \|v\|}, \quad \forall t \in I_\tau, u, v \in P \ (k = 1, 2, 3, \dots).$$

Hence condition (H3) is satisfied for

$$c_1(t) = \frac{1+t}{324k^2}, \quad h_{1,k}(u, v) = (\|u\| + \|v\|)^3, \quad \text{and} \quad u^* = \left( 1, \dots, \frac{1}{k^2}, \dots \right),$$

in this situation,

$$h_{1,k}^\infty = \lim_{\|u\| + \|v\| \rightarrow \infty} \frac{(\|u\| + \|v\|)^3}{\|u\| + \|v\|} = \infty > \Delta^{-1}.$$

And condition (H4) is also satisfied for

$$c_2(t) = \frac{1+t}{324k^2}, \quad h_{2,k}(u, v) = \sqrt{\|u\| + \|v\|}, \quad \text{and} \quad u_* = \left( 1, \dots, \frac{1}{k^2}, \dots \right),$$

in this situation,

$$h_{2,k}^\infty = \lim_{\|u\| + \|v\| \rightarrow 0} \frac{\sqrt{\|u\| + \|v\|}}{\|u\| + \|v\|} = \infty > \nabla^{-1}.$$

Finally, choose  $\xi = 1$ . It is easy to check that condition (H5) is satisfied. In this case,  $M_\xi \approx 4.1774$ , and so

$$\rho N M_\xi \int_0^1 \varphi(s) a(s) ds \approx 0.9604 < \xi = 1.$$

From Theorem 3.6, the conclusion follows and the proof is complete. □

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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