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Existence result for a nonlinear fractional differential equation with integral boundary conditions at resonance

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Abstract

In this paper, we establish the existence result for nonlinear fractional differential equations with integral boundary conditions at resonance by means of coincidence degree theory. As applications, three examples are presented to illustrate the main results.

1 Introduction

In this paper, we prove the existence of solutions for fractional differential equations at resonance with integral boundary conditions given by

$$\begin{cases} D_{0+}^{\alpha} u(t) = f(t, u(t), D_{0+}^{\alpha-1} u(t), \dots, D_{0+}^{\alpha-(n-1)} u(t)), & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \\ u(1) = (\alpha + m) \int_0^1 t^m u(t) dt, \end{cases} \quad (1.1)$$

where $0 < t < 1$, $n - 1 < \alpha < n$, $m \in \mathbb{N}$, D_{0+}^{α} is the Riemann-Liouville fractional derivative, and $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function.

During the last few years, fractional differential equations have been studied extensively due to their significant applications in various sciences such as physics, mechanics, chemistry, phenomena arising in engineering. See [1–3] and the references therein. Many researchers paid attention to the existence of boundary value problems for nonlinear fractional differential equations, see [4–8]. The authors obtained the results by using the classical tools for such problems including the Leray-Schauder nonlinear alternative theorem, *etc.* Recently, there have been some papers devoted to the theory of fractional differential equations with integral boundary conditions, see [9–11].

In [9], Cabada and Wang studied the integral boundary value problem for nonlinear fractional differential equations:

$$\begin{cases} {}^C D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = u''(0) = 0, \\ u(1) = \lambda \int_0^1 u(s) ds, \end{cases}$$

where $2 < \alpha < 3$, $0 < \lambda < 2$, ${}^C D_{0+}^\alpha$ is the Caputo fractional derivative and $f : [0, 1] \times [0, +\infty) \rightarrow [0, +\infty)$ is a continuous function. The author established sufficient conditions for the existence of nonlinear fractional differential equations with an integral boundary problem.

In [10], Yuan discussed an $(n - 1, 1)$ -type semipositone integral boundary value problem for the coupled system of fractional differential equations

$$\begin{cases} D_{0+}^\alpha u(t) + \lambda f(t, v(t)) = 0, & 0 < t < 1, \\ D_{0+}^\alpha v(t) + \lambda g(t, u(t)) = 0, & 0 < t < 1, \\ u^{(j)}(0) = v^{(j)}(0) = 0, & 0 \leq j \leq n - 2, \\ u(1) = \mu \int_0^1 u(s) ds, & v(1) = \mu \int_0^1 v(s) ds, \end{cases}$$

where λ, μ are parameters and $\lambda > 0$, $0 < \mu < \alpha$, $\alpha \in (n - 1, n]$, $n \geq 3$, D_{0+}^α is the Riemann-Liouville fractional derivative, and $f, g : (0, 1) \times [0, +\infty) \rightarrow \mathbb{R}$ are sign-changing continuous functions. Some existence results were given by using the Leray-Schauder nonlinear alternative theorem and the Krasnoselskii fixed point theorem.

In [11], Feng *et al.* considered a higher-order nonlinear nonlinear fractional differential equations with integral boundary conditions given by

$$\begin{cases} D_{0+}^\alpha x(t) + g(t)f(t, x(t)) = 0, & 0 < t < 1, \\ x(0) = x'(0) = \dots = x^{(n-2)}(0) = 0, & 0 < t < 1, \\ x(1) = \int_0^1 h(t)x(t) dt, \end{cases}$$

where $0 < t < 1$, $n - 1 < \alpha \leq n$, $n \geq 3$, D_{0+}^α is the Riemann-Liouville fractional derivative, and $g \in C((0, 1), [0, +\infty))$ which may be singular at $t = 0$ or (and) $t = 1$, $h \in L^1[0, 1]$ is non-negative, and $f \in C([0, 1] \times [0, +\infty), [0, +\infty))$ is a continuous function. By means of the Guo-Krasnoselskii fixed point theorem, some results on the existence of positive solutions are obtained.

At the same time, we notice that these problems happen to be at resonance if $\lambda = 2$ in [9], $\mu = \alpha$ in [10] or $\int_0^1 h(t)t^{\alpha-1} dt = 1$ in [11]. Though there has been an increasing interest in the study of the existence of solutions for fractional differential equations at resonance (see [12–17]), the corresponding theorem and applications to the fractional differential equations with integral conditions at resonance have not been investigated now. Motivated by the works above, we discuss the integral boundary value problems of fractional differential equations (1.1), which happen to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} D_{0+}^\alpha u(t) = 0, & 0 < t < 1, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0, \\ u(1) = (\alpha + m) \int_0^1 t^m u(t) dt, \end{cases}$$

has $u(t) = ct^{\alpha-1}$, $c \in \mathbb{R}$ as a nontrivial solution.

The rest of this paper is organized as follows. In Section 2, we give some necessary notations, definitions and lemmas. In Section 3, we study the existence of solutions of (1.1)

by the coincidence degree theory due to Mawhin [18]. Finally, three examples are given to illustrate our results in Section 4.

2 Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and properties can be found in the literature. For more details, see [1–3].

Definition 2.1 [1] The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Definition 2.2 [1] The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n-1 < \alpha \leq n$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1 [1] Let $n-1 < \alpha \leq n$, $u \in C(0,1) \cap L^1(0,1)$, then

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \dots + C_n t^{\alpha-n},$$

where $C_i \in \mathbb{R}$, $i = 1, 2, \dots, n$.

Lemma 2.2 [1] If $\alpha > 0$, $m \in \mathbb{N}$ and $D = d/dx$. If the fractional derivatives $D_{0+}^{\alpha} u(t)$ and $D_{0+}^{\alpha+m} u(t)$ exist, then

$$D^m D_{0+}^{\alpha} u(t) = D_{0+}^{\alpha+m} u(t).$$

Lemma 2.3 [1] The relation

$$I_{a+}^{\alpha} I_{a+}^{\beta} f(x) = I_{a+}^{\alpha+\beta} f(x)$$

is valid in the case $\beta > 0$, $\alpha + \beta > 0$, $f(x) \in L_1(a, b)$.

Lemma 2.4 The relation

$$\int_0^r I_{0+}^{\alpha} f(x) dx = I_{0+}^{\alpha+1} f(r)$$

is valid in the case $\alpha > 0$, $f(x) \in L_1(0, r)$, $r \in \mathbb{R}^+$.

Proof

$$\begin{aligned} \int_0^r I_{0+}^\alpha f(x) dx &= \frac{1}{\Gamma(\alpha)} \int_0^r \int_0^t (t-s)^{\alpha-1} f(s) ds dt = \frac{1}{\Gamma(\alpha)} \int_0^r \int_s^r (t-s)^{\alpha-1} f(s) dt ds \\ &= \frac{1}{\Gamma(\alpha)} \int_0^r f(s) \int_s^r (t-s)^{\alpha-1} dt ds = \frac{1}{\Gamma(\alpha)} \int_0^r f(s) \frac{(r-s)^\alpha}{\alpha} ds \\ &= I_{0+}^{\alpha+1} f(r). \end{aligned} \quad \square$$

Now let us recall some notation about the coincidence degree continuation theorem. Let Y, Z be real Banach spaces, $L : \text{dom } L \subset Y \rightarrow Z$ be a Fredholm map of index zero and $P : Y \rightarrow Y, Q : Z \rightarrow Z$ be continuous projectors such that $\ker L = \text{Im } P, \text{Im } L = \ker Q$ and $Y = \ker L \oplus \ker P, Z = \text{Im } L \oplus \text{Im } Q$. It follows that $L|_{\text{dom } L \cap \ker P} : \text{dom } L \cap \ker P \rightarrow \text{Im } L$ is invertible. We denote the inverse of this map by K_P . If Ω is an open bounded subset of Y , the map N will be called L -compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N = K_P(I - Q)N : \overline{\Omega} \rightarrow Y$ is compact.

Theorem 2.1 *Let L be a Fredholm operator of index zero and N be L -compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

- (1) $Lx \neq \lambda Nx$ for each $(x, \lambda) \in [(\text{dom } L \setminus \ker L) \cap \partial \Omega] \times (0, 1)$;
- (2) $Nx \notin \text{Im } L$ for each $x \in \ker L \cap \partial \Omega$;
- (3) $\text{deg}(JQN|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Z \rightarrow Z$ is a continuous projection as above with $\text{Im } L = \ker Q$ and $J : \text{Im } Q \rightarrow \ker L$ is any isomorphism.

Then the equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

3 Main results

In this paper, we always assume the following conditions.

$$(H_0) \quad \frac{1}{\Gamma(\alpha+1)} \neq (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)! \Gamma(\alpha+2+i)}.$$

In this section, we prove the existence results for (1.1). We use the Banach space $E = C[0, 1]$ with the norm $\|u\|_\infty = \max_{0 \leq t \leq 1} |u(t)|$. For $\alpha > 0, n = [\alpha] + 1$, we define a linear space

$$Y = \{u | u, D_{0+}^{\alpha-i} u \in E, i = 1, 2, \dots, n-1\}. \tag{3.1}$$

By means of the line functional analysis theory, we can prove that Y is a Banach space with the norm

$$\|u\|_Y = \|D_{0+}^{\alpha-1} u\|_\infty + \dots + \|D_{0+}^{\alpha-(n-1)} u\|_\infty + \|u\|_\infty.$$

Define L to be the linear operator from Y to E with

$$\text{dom}(L) = \left\{ u \in Y \mid u(0) = u'(0) = \dots = u^{n-2}(0) = 0; u(1) = (\alpha + m) \int_0^1 t^m u(t) dt \right\}$$

and

$$Lu = D_{0+}^\alpha u, \quad u \in \text{dom}(L). \tag{3.2}$$

Define $N : Y \rightarrow E$ by

$$Nu(t) = f(t, D_{0+}^{\alpha-1}u(t), \dots, D_{0+}^{\alpha-(n-1)}u(t)). \tag{3.3}$$

Then problem (1.1) can be written by $Lu = Nu$.

Lemma 3.1 *The mapping $L : \text{dom}(L) \subset Z$ is a Fredholm operator of index zero.*

Proof It is clear that

$$\ker(L) = \{ct^{\alpha-1}, c \in \mathbb{R}\} \cong \mathbb{R}^1.$$

Let $x \in \text{Im} L$, so there exists a function $u \in \text{dom} L$ which satisfies $Lu = x$. By Lemma 2.1, we have

$$u(t) = I_{0+}^{\alpha}x(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \dots + c_nt^{\alpha-n}.$$

By $u(0) = u'(0) = \dots = u^{(n-2)}(0) = 0$, we can obtain $c_2 = \dots = c_n = 0$. Hence

$$u(t) = I_{0+}^{\alpha}x(t) + c_1t^{\alpha-1}.$$

By $u(1) = (\alpha + m) \int_0^1 t^m u(t) dt$, we have

$$\begin{aligned} I_{0+}^{\alpha}x(1) + c_1 &= u(1) = (\alpha + m) \int_0^1 t^m [I_{0+}^{\alpha}x(t) + c_1t^{\alpha-1}] dt \\ &= (\alpha + m) \int_0^1 t^m I_{0+}^{\alpha}x(t) dt + c_1 \int_0^1 (\alpha + m)t^{\alpha+m-1} dt \\ &= (\alpha + m) \int_0^1 t^m I_{0+}^{\alpha}x(t) dt + c_1. \end{aligned}$$

Taking into account Lemma 2.2 and using integration by parts, we obtain

$$\begin{aligned} I_{0+}^{\alpha}x(1) &= (\alpha + m) \int_0^1 t^m I_{0+}^{\alpha}x(t) dt = (\alpha + m) \int_0^1 t^m dI_{0+}^{\alpha+1}x(t) \\ &= (\alpha + m) \left[t^m I_{0+}^{\alpha+1}x(t) \Big|_0^1 - m \int_0^1 t^{m-1} I_{0+}^{\alpha+1}x(t) dt \right] \\ &= (\alpha + m) \left[I_{0+}^{\alpha+1}x(1) - m \int_0^1 t^{m-1} dI_{0+}^{\alpha+2}x(t) \right] \\ &\dots \\ &= (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i}x(1). \end{aligned}$$

On the other hand, suppose that $x \in E$ satisfies $I_{0+}^{\alpha}x(1) - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i}x(1) = 0$. Let $u(t) = I_{0+}^{\alpha}x(t)$, we can easily prove $u(t) \in \text{dom}(L)$.

Thus, we conclude that

$$\text{Im}(L) = \left\{ x \in E \mid I_{0+}^{\alpha} x(1) - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i} x(1) = 0 \right\}.$$

Consider the linear operators $Q : Z \rightarrow \mathbb{R}$ defined by

$$Qx(t) = \sigma \cdot \left[I_{0+}^{\alpha} x(1) - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i} x(1) \right],$$

where $\sigma = \left[\frac{1}{\alpha \Gamma(\alpha)} - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} \frac{1}{\Gamma(\alpha+2+i)} \right]^{-1}$ and $Qx(t) \cong \mathbb{R}$.

Take $x(t) \in E$, then

$$\begin{aligned} Q(Qx(t)) &= \sigma \cdot \left[I_{0+}^{\alpha} Qx(t) \Big|_{t=1} - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i} Qx(t) \Big|_{t=1} \right] \\ &= \sigma \frac{1}{\sigma} Qx(t) = Qx(t). \end{aligned}$$

We can see $Q^2 = Q$.

For $x(t) \in Z$, $x(t) = x(t) - Qx(t) + Qx(t)$, one has $Q[x(t) - Qx(t)] = 0$.

Therefore

$$x(t) - Qx(t) \in \text{Ker}(Q) = \text{Im}(L), \quad Qx(t) \in \text{Im}(Q).$$

That is to say,

$$Z = \text{Im}(L) + \text{Im}(Q).$$

If $u \in \text{Im}(L) \cap \text{Im}(Q)$, so we have $u = c \in \mathbb{R}$.

Because of $u = c \in \text{Im}(L)$, then

$$I_{0+}^{\alpha} c \Big|_{t=1} - (\alpha + m) \sum_{i=0}^m (-1)^i \frac{m!}{(m-i)!} I_{0+}^{\alpha+1+i} c \Big|_{t=1} = 0.$$

As a result $c \cdot \sigma = 0$, we get $c = 0$.

Then

$$Z = \text{Im}(L) \oplus \text{Im}(Q).$$

Note that $\text{Ind } L = \dim \ker L - \text{codim } \text{Im } L = 0$. Then L is a Fredholm mapping of index zero. \square

We can define the operators $P : Y \rightarrow Y$, where

$$Pu = \frac{1}{\Gamma(\alpha)} D_{0+}^{\alpha-1} u(0) t^{\alpha-1}.$$

For $u \in Y$,

$$P(Pu) = P\left(\frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1}\right) = \frac{1}{\Gamma(\alpha)}D_{0+}^{\alpha-1}u(0)t^{\alpha-1} = Pu.$$

So we have $P^2 = P$.

Note that

$$\text{Ker } P = \{u \in Y \mid D_{0+}^{\alpha-1}u(0) = 0\}.$$

Since $u = u - Pu + Pu$, it is easy to say $u - Pu \in \text{Ker}(P)$, $Pu \in \text{Ker}(L)$. And $Y = \text{Ker}(P) + \text{Ker}(L)$.

If $u \in \text{Ker}(L) \cap \text{Ker}(P)$, so $u = c_1 t^{\alpha-1}$, we can derive $D^{\alpha-1}c_1 t^{\alpha-1}|_{t=0} = 0$, so $c_1 = 0$. Then

$$Y = \text{Ker}(L) \oplus \text{Ker}(P).$$

For $u \in Y$,

$$\begin{aligned} \|Pu\|_Y &= \|D_{0+}^{\alpha-1}u\|_{\infty} + \dots + \|D_{0+}^{\alpha-(n-1)}u\|_{\infty} + \|u\|_{\infty} \\ &= \frac{1}{\Gamma(\alpha)}|D_{0+}^{\alpha-1}u(0)| \cdot (\|t^{\alpha-1}\|_{\infty} + \|D_{0+}^{\alpha-1}t^{\alpha-1}\|_{\infty} + \dots + \|D_{0+}^{\alpha-(n-1)}t^{\alpha-1}\|_{\infty}) \\ &= \left(\sum_{i=1}^{n-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)|D_{0+}^{\alpha-1}u(0)| \\ &= a|D_{0+}^{\alpha-1}u(0)|, \end{aligned} \tag{3.4}$$

where $a = \frac{1}{\Gamma(\alpha)} + \sum_{i=1}^{n-1} \frac{1}{\Gamma(i)}$.

We define $K_p : \text{Im } L \rightarrow \text{dom } L \cap \text{ker } P$ by $K_p x = I_{0+}^{\alpha}x$. For $x \in \text{Im}(L)$, we have $LK_p x = D_{0+}^{\alpha}I_{0+}^{\alpha}x = x$.

And for $u \in \text{dom}(L) \cap \text{Ker}(P)$, we have $D_{0+}^{\alpha-1}u(0) = 0$. So the coefficients c_1, \dots, c_n in the expressions

$$u = I_{0+}^{\alpha}D_{0+}^{\alpha}u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$$

are all equal to zero. Thus we obtain

$$K_p Lu = I_{0+}^{\alpha}D_{0+}^{\alpha}u = u,$$

which shows that $K_p = (L_{\text{dom}(L) \cap \text{Ker}(P)})^{-1}$.

Again, for each $x \in \text{Im}(L)$,

$$\begin{aligned} \|K_p x\|_Y &= \|I_{0+}^{\alpha}x\|_Y = \|I_{0+}^{\alpha}x\|_{\infty} + \|D_{0+}^{\alpha-1}I_{0+}^{\alpha}x\|_{\infty} + \dots + \|D_{0+}^{\alpha-(n-1)}I_{0+}^{\alpha}x\|_{\infty} \\ &\leq \left(\sum_{i=1}^{n-1} \frac{1}{\Gamma(i+1)} + \frac{1}{\Gamma(\alpha+1)}\right)\|x\|_{\infty} \\ &\leq \left(\sum_{i=1}^{n-1} \frac{1}{\Gamma(i)} + \frac{1}{\Gamma(\alpha)}\right)\|x\|_{\infty} \\ &= a\|x\|_{\infty}. \end{aligned} \tag{3.5}$$

Lemma 3.2 [17] $F \subset C^\mu[0,1]$ is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here uniformly bounded means that there exists $M > 0$ such that for every $u \in F$,

$$\|u\|_{C^\mu} = \|D_{0+}^\mu u\|_\infty + \dots + \|D_{0+}^{\mu-(N-1)} u\|_\infty + \|u\|_\infty < M,$$

and equicontinuous means that $\forall \epsilon > 0, \exists \delta > 0$ such that

$$|u(t_1) - u(t_2)| < \epsilon \quad (\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, \forall u \in F)$$

and

$$|D_{0+}^{\alpha-i} u(t_1) - D_{0+}^{\alpha-i} u(t_2)| < \epsilon \quad (\forall t_1, t_2 \in [0,1], |t_1 - t_2| < \delta, \forall u \in F, i = 0, 1, \dots, N-1).$$

By Lemma 3.2 and standard arguments, the following lemma holds.

Lemma 3.3 $K_p(I - Q)N : Y \rightarrow Y$ is completely continuous.

For convenience, we use the following notations: $\rho_0 = aA + 2a\|\varphi\|_\infty$; $\rho_i = 2a\|\psi_i\|_\infty, i = 1, 2, \dots, n$. To obtain our main results, we need the following conditions.

(H₁) There exist functions $\varphi, \psi_i, \in E, i = 1, 2, \dots, n$, such that for all $t \in [0,1]$,

$$|f(t, x_1, x_2, \dots, x_n)| \leq \varphi + \psi_1|x_1| + \psi_2|x_2| + \dots + \psi_n|x_n|.$$

(H₂) There exists a constant $A > 0$ such that for every $y \in \mathbb{R}$, if $|D_{0+}^{\alpha-1} u(t)| > A$, for all $t \in [0,1]$, then

$$QNu(t) \neq 0.$$

(H₃) There exists a constant $B > 0$ such that, for each $c \in \mathbb{R}$ satisfying $|c| > B$, we have at least one of the following:

$$cQN(ct^{\alpha-1}) > 0,$$

or

$$cQN(ct^{\alpha-1}) < 0.$$

(H₄) $\sum_{i=1}^n \rho_i < 1$.

Lemma 3.4 $\Omega_1 = \{u \in \text{dom}(L) \setminus \text{Ker}(L) | Lu = \lambda Nu, \lambda \in [0,1]\}$ is bounded.

Proof For $u \in \Omega_1, Lu = \lambda Nu$, thus $\lambda \neq 0$. And $Lu = \lambda Nu \in \text{Im}(L) = \text{Ker}(Q)$, that is,

$$\lambda QNu(t) = 0.$$

By (H₂), there exists a constant $t_0 \in [0,1]$ such that $|D_{0+}^{\alpha-1} u(t_0)| \leq A$.

Again, for $u \in \Omega_1$, then $(I - P)u \in \text{dom}(L) \setminus \text{Ker}(L)$ and $LPu = 0$, thus from (3.5), we have

$$\|(I - P)u\|_Y = \|K_p L(I - P)u\|_Y = \|K_p Lu\|_Y \leq a \|Lu\|_\infty \leq a \|Nu\|_\infty.$$

Now, by Lemma 2.2,

$$\begin{aligned} |D_{0+}^{\alpha-1}u(0)| &\leq |D_{0+}^{\alpha-1}u(t_0)| + \left| \int_0^{t_0} D_{0+}^\alpha u(s) ds \right| \leq |D_{0+}^{\alpha-1}u(t_0)| + |t_0| \max_{0 \leq t \leq t_0} |D_{0+}^\alpha u(t)| \\ &\leq |D_{0+}^{\alpha-1}u(t_0)| + \|D_{0+}^\alpha u(t)\|_\infty \leq A + \|Lu\|_\infty \leq A + \|Nu\|_\infty, \end{aligned}$$

that is,

$$|D_{0+}^{\alpha-1}u(0)| \leq A + \|Nu\|_\infty. \tag{3.6}$$

From (3.4), we have

$$\begin{aligned} \|u\|_Y &= \|Pu + (I - P)u\|_Y \leq \|Pu\|_Y + \|(I - P)u\|_Y \\ &\leq a |D_{0+}^{\alpha-1}u(0)| + a \|Nu\|_\infty \\ &\leq a(A + \|Nu\|_\infty) + a \|Nu\|_\infty \\ &\leq aA + 2a \|Nu\|_\infty. \end{aligned}$$

By (H_1) , we have

$$\begin{aligned} \|u\|_Y &\leq aA + 2a \|Nu\|_\infty \\ &\leq aA + 2a \|f(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), \dots, D_{0+}^{\alpha-(n-1)}u(t))\|_\infty \\ &\leq aA + 2a (\|\varphi\|_\infty + \|\psi_1\|_\infty \|u\|_\infty + \|\psi_2\|_\infty \|D_{0+}^{\alpha-1}u\|_\infty + \dots + \|\psi_n\|_\infty \|D_{0+}^{\alpha-(n-1)}u\|_\infty) \\ &= \rho_0 + \rho_1 \|u\|_\infty + \rho_2 \|D_{0+}^{\alpha-1}u\|_\infty + \rho_3 \|D_{0+}^{\alpha-2}u\|_\infty + \dots + \rho_n \|D_{0+}^{\alpha-(n-1)}u\|_\infty. \end{aligned}$$

By the definition $\|u\|_Y$, we can derive

$$\|u\|_\infty \leq \frac{1}{1 - \rho_1} (\rho_0 + \rho_2 \|D_{0+}^{\alpha-1}u\|_\infty + \rho_3 \|D_{0+}^{\alpha-2}u\|_\infty + \dots + \rho_n \|D_{0+}^{\alpha-(n-1)}u\|_\infty).$$

Hence, we have

$$\|D_{0+}^{\alpha-1}u\|_\infty \leq \frac{1}{1 - \rho_1 - \rho_2} (\rho_0 + \rho_3 \|D_{0+}^{\alpha-2}u\|_\infty + \dots + \rho_n \|D_{0+}^{\alpha-(n-1)}u\|_\infty),$$

⋮

$$\|D_{0+}^{\alpha-(n-1)}u\|_\infty \leq \frac{\rho_0}{1 - \rho_1 - \rho_2 - \dots - \rho_n} := M.$$

It is easy to see $\|D_{0+}^{\alpha-1}v\|_\infty, \dots, \|D_{0+}^{\alpha-(n-1)}v\|_\infty$ and $\|u\|_\infty$ are bounded. We can conclude that $\|u\|_Y$ is bounded. □

Lemma 3.5 $\Omega_2 = \{u \in \text{Ker}(L) : Nu \in \text{Im}(L)\}$ is bounded.

Proof Let $u \in \text{Ker}(L)$, so we have $u = ct^{\alpha-1}$, $c \in \mathbb{R}$, $t \in [0, 1]$, and $Nu \in \text{Im}(L) = \text{Ker}(Q)$, thus

$$\lambda QNu(t) = 0.$$

By (H_2) , there exist two constant $t_0 \in [0, 1]$ such that

$$|D_{0+}^{\alpha-1}u(t_0)| = |D_{0+}^{\alpha-1}ct^{\alpha-1}|_{t_0} = |c\Gamma(\alpha)| \leq A.$$

It follows that $|c| \leq \frac{A}{\Gamma(\alpha)}$. Here, Ω_2 is bounded. □

Lemma 3.6 $\Omega_3 = \{u \in \text{Ker}(L) : \lambda u + (1 - \lambda)QNu = (0, 0), \lambda \in [0, 1]\}$ is bounded.

Proof Let $u \in \text{Ker}(L)$, so we have $u = ct^{\alpha-1}$, $c \in \mathbb{R}$, $t \in [0, 1]$.

Thus

$$\lambda ct^{\alpha-1} + (1 - \lambda)QN(t^{\alpha-1}) = 0.$$

If $\lambda = 0$, then $|c| \leq \frac{A}{\Gamma(\alpha)}$. If $\lambda = 1$, we have $c = 0$.

It follows that

$$\lambda c^2 t^{\alpha-1} + (1 - \lambda)cQN(t^{\alpha-1}) = 0.$$

Thus from the first part of (H_3) , then $|c| \leq B$. Here, Ω_3 is bounded. □

Remark 3.1 If the other parts of (H_3) hold, then the set $\Omega'_3 = \{u \in \text{Ker}(L) : -\lambda u + (1 - \lambda)QNu = 0, \lambda \in [0, 1]\}$ is bounded.

Theorem 3.1 Suppose (H_1) - (H_3) hold, then problem (1.1) has at least one solution in Y .

Proof Let Ω be a bounded open set of Y such that $\bigcup_{i=1}^3 \overline{\Omega}_i \subset \Omega$. It follows from Lemma 3.3 that N is L -compact on Ω . By Lemma 3.4, Lemma 3.5 and Lemma 3.6, we get:

- (1) $Lu \neq \lambda Nu$ for every $(u, \nu) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$,
- (2) $Nu \notin \text{Im } L$ for every $u \in \text{Ker } L \cap \partial\Omega$,
- (3) Let $H((u, \nu), \lambda) = \pm\lambda I(u, \nu) + (1 - \lambda)JQN(u, \nu)$, where I is the identical operator. Via the homotopy property of degree, we obtain that

$$\begin{aligned} \deg(JQN|_{\text{ker } L}, \Omega \cap \text{ker } L, 0) &= \deg(H(\cdot, 0), \Omega \cap \text{ker } L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{ker } L, 0) \\ &= \deg(\pm I, \Omega \cap \text{ker } L, 0) \neq 0. \end{aligned}$$

Applying Theorem 2.1, we conclude that $Lu = Nu$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$. □

4 Some examples

To illustrate how our main results can be used in practice, we present three examples.

Example 4.1 Let us consider the following boundary value problem:

$$\begin{cases} D_{0^+}^{2.5}u(t) = \frac{t}{2} + e^{-u^2(t)} + \frac{1}{6}|D_{0^+}^{1.5}u(t)| + \cos(D_{0^+}^{0.5}u(t)), & 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u(1) = \frac{5}{2} \int_0^1 u(t) dt. \end{cases} \quad (4.1)$$

By the calculation, we can have $A = B = 7$ and (H_0) - (H_4) are all satisfied. Hence, we obtain that (4.1) has at least one solution.

Example 4.2 Consider the following integral boundary value problem:

$$\begin{cases} D_{0^+}^{3.6}u(t) = t^2 + \arctan u(t) + \frac{1}{10}D_{0^+}^{2.6}u(t) + \cos(D_{0^+}^{0.6}u(t)) + \sin(D_{0^+}^{1.6}u(t)), & 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \frac{23}{5} \int_0^1 t \cdot u(t) dt. \end{cases} \quad (4.2)$$

It is easy to see that (H_0) - (H_4) hold. By a simple calculation, we can have $A = 48$, $B = 14$. Thus it follows that problem (4.2) has at least one solution.

Example 4.3 We consider the boundary value problem

$$\begin{cases} D_{0^+}^{1.2}u(t) = \cos(t+1)^2 + \frac{1}{8} \sin u(t) + \frac{1}{9}D_{0^+}^{0.2}u(t), & 0 < t < 1, \\ u(0) = 0, \\ u(1) = \frac{16}{5} \int_0^1 t^2 u(t) dt. \end{cases} \quad (4.3)$$

Corresponding to problem (1.1), we have that $\alpha = 1.2$, $m = 2$. By a simple calculation, we can get $A = 11$, $B = 12$, then (H_0) - (H_4) hold. By Theorem 3.1, we obtain that (4.3) has at least one solution.

5 Conclusion

In this paper, we have obtained the existence of solutions for fractional differential equations with integral boundary conditions at resonance. By using the coincidence degree theory, we have found the existence results. Though the technique applied to establish the existence results for the problem is a standard one, yet its results are new in the context of integral boundary conditions. As applications, three examples are presented to illustrate the main results. In the future, we will consider the uniqueness of solutions for the fractional differential equations at resonance, and also can make further research on fractional differential equations with a singular integral boundary value problem at resonance.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the manuscript. The authors read and approved the final manuscript.

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