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Positive solutions for boundary value problems of fractional difference equations depending on parameters

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Abstract
We use the Krasnosel'skii fixed point theorem to obtain the sufficient conditions of the existence of two positive solutions for the boundary value problem of fractional difference equations depending on parameters.

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Keywords: discrete boundary value problem; fixed point theory; cone; positive solution

1 Introduction

In this paper, we consider the boundary value problems of fractional difference equations depending on parameters of the form

$$-\Delta^{\nu_j} y_j(t) = \lambda_j f_j\bigl(y_1(t+\nu_1-1),\dots,y_n(t+\nu_n-1)\bigr), \tag{1.1}$$

$$y_j(\nu_j-2) = \psi_j(y_j), \qquad y_j(\nu_j+b) = \phi_j(y_j), \tag{1.2}$$

where $t \in [0,b]_{\mathbb{N}_0} := \{0,1,\dots,b\}$, $\lambda_j > 0$, $1 < \nu_j \le 2$, $f_j : [0,+\infty) \times \cdots \times [0,+\infty) \to [0,+\infty)$ are continuous functions. For each j, we have that $\psi_j, \phi_j : \mathbb{R}^{b+3} \to \mathbb{R}$ $(j = 1,2,\dots,n)$ are given functions. We point out that fractional difference equations have been extensively studied in recent years. Systems of discrete fractional boundary value problems are also popular. In [1], the authors discussed the existence of positive solutions for coupled systems of nonlinear fractional difference equations:

$$\Delta^2 u(n-1) + \lambda a(n) f\bigl(u(n),v(n)\bigr) = 0,$$

$$\Delta^2 v(n-1) + \mu b(n) g\bigl(u(n),v(n)\bigr) = 0,$$

$$u(0) = \beta u(\eta), \qquad u(N) = \alpha u(\eta), \qquad v(0) = \beta v(\eta), \qquad v(N) = \alpha v(\eta),$$

where $\eta \in \{1,\dots,N-1\}$, $n \in \{1,\dots,N-1\}$, $N \ge 4$, $\alpha,\beta,\lambda,\mu > 0$. In [2], Goodrich studied the following pair of discrete fractional boundary value problems:

$$-\Delta^{\nu_1} y_1(t) = \lambda_1 a_1(t+\nu_1-1) f_1\bigl(y_1(t+\nu_1-1), y_2(t+\nu_2-1)\bigr),$$

$$-\Delta^{\nu_2} y_2(t) = \lambda_2 a_2(t+\nu_2-1) f_2\bigl(y_1(t+\nu_1-1), y_2(t+\nu_2-1)\bigr),$$

$$y_1(\nu_1 - 2) = \psi_1(y_1), \qquad y_2(\nu_2 - 2) = \psi_2(y_2),$$
$$y_1(\nu_1 + b) = \phi_1(y_1), \qquad y_2(\nu_2 + b) = \phi_2(y_2),$$

where $t \in [0,b]_{\mathbb{N}_0} := \{0,1,\dots,b\}$, $\lambda_1, \lambda_2 > 0$, $\nu_1, \nu_2 \in (1,2]$, it is the same as [1] when $\nu_1, \nu_2 = 2$. Goodrich obtained the existence of at least one positive solution to this problem by means of the Krasnosel'skii theorem for cones. We shall deduce the existence of at least two positive solutions to problem (1.1)-(1.2) in this paper. These results extend the results of [2].

The paper is organized as follows. In Section 2, we present basic definitions and demonstrate some lemmas in order to prove our main results. In Section 3, we establish some results for the existence of at least two solutions to problem (1.1)-(1.2), and we present an example to illustrate our main results.

2 Preliminaries

For the convenience of the reader, we give some definitions and fundamental facts of the discrete fractional calculus, which can be found in [3–6] and their references.

Definition 2.1 [3] We define

$$t^{(\nu)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}$$

for any t and ν, for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the gamma function and $t+1$ is not a pole, then $t^{(\nu)} = 0$.

Definition 2.2 [3] The νth fractional sum of a function f defined on the set $\mathbb{N}_a := \{a, a+1,\dots\}$, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t;a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{(\nu-1)} f(s),$$

where $t \in \{a+\nu, a+\nu+1,\dots\} =: \mathbb{N}_{a+\nu}$. We also define the νth fractional difference for $\nu > 0$ by

$$\Delta^{\nu} f(t) = \Delta^N \Delta^{\nu-N} f(t),$$

where $t \in \mathbb{N}_{a+\nu}$ and $0 \le N-1 < \nu \le N$.

Lemma 2.3 [3] *Let t and ν be any numbers for which $t^{(\nu)}$ and $t^{(\nu-1)}$ are defined. Then*

$$\Delta t^{(\nu)} = \nu t^{(\nu-1)}.$$

Lemma 2.4 [2] *Let $0 \le N-1 < \nu \le N$. Then*

$$\Delta^{-\nu} \Delta^{\nu} y(t) = y(t) + c_1 t^{(\nu-1)} + c_2 t^{(\nu-2)} + \cdots + c_N t^{(\nu-N)}$$

for some $c_i \in R$, with $1 \le i \le N$.

Lemma 2.5 [3] *Let $1 < \nu \le 2$ and $h : [\nu-1, \nu+b-1]_{\mathbb{N}_{\nu-1}} \to \mathbb{R}$ be given. The unique solution of the FBVP*

$$-\Delta^{\nu} y(t) = h(t+\nu-1),$$
$$y(\nu-2) = 0 = y(\nu+b)$$

is given by

$$y(t) = \sum_{s=0}^{b} G(t,s)h(s+\nu-1),$$

where $G : [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0} \to R$ is defined by

$$G(t,s) = \frac{1}{\Gamma(\nu)} \begin{cases} \frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{(\nu+b)^{(\nu-1)}} - (t-s-1)^{(\nu-1)}, & 0 \le s < t-\nu+1 \le b, \\ \frac{t^{(\nu-1)}(\nu+b-s-1)^{(\nu-1)}}{(\nu+b)^{(\nu-1)}}, & 0 \le t-\nu+1 \le s \le b. \end{cases} \tag{2.1}$$

Lemma 2.6 [2] *The Green's function $G(t,s)$ given in Lemma 2.5 satisfies:*
(i) *$G(t,s) \ge 0$ for each $(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$;*
(ii) *$\max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = G(s+\nu-1,s)$ for each $s \in [0,b]_{\mathbb{N}_0}$;*
(iii) *there exists a number $\gamma \in (0,1)$ such that*

$$\min_{\frac{\nu+b}{4} \le t \le \frac{3(\nu+b)}{4}} G(t,s) \ge \gamma \max_{t\in[\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}}} G(t,s) = \gamma G(s+\nu-1,s)$$

for $s \in [0,b]_{\mathbb{N}_0}$.

First of all, we let $\mathcal{B}_j$ represent the Banach space of all maps from $[\nu_j-2,\dots,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}$ into $\mathbb{R}$ when equipped with the usual maximum norm $\|\cdot\|$. Then, we put $\chi := \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots \times \mathcal{B}_n$. By equipping χ with the norm

$$\big\|(y_1,\dots,y_n)\big\| = \|y_1\| + \cdots + \|y_n\|,$$

it follows that $(\chi, \|\cdot\|)$ is a Banach space.
Now consider the operator $S : \chi \to \chi$ defined by

$$S(y_1,\dots,y_n)(t_1,\dots,t_n) = \big(S_1(y_1,\dots,y_n)(t_1),\dots,S_n(y_1,\dots,y_n)(t_n)\big), \tag{2.2}$$

where we define $S_j : \chi \to \mathcal{B}_j$ by

$$\begin{aligned} S_j(y_1,\dots,y_n)(t_j) &= \alpha_j(t_j)\psi_j(y_j) + \beta_j(t_j)\phi_j(y_j) \\ &\quad + \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\big(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\big), \end{aligned} \tag{2.3}$$

where

$$\alpha_j(t_j) := \frac{1}{\Gamma(\nu_j-1)}\left[t_j^{(\nu_j-2)} - \frac{1}{b+2} t_j^{(\nu_j-1)}\right], \qquad \beta_j(t_j) := \frac{t_j^{(\nu_j-1)}}{(\nu_j+b)^{(\nu_j-1)}}.$$

Theorem 2.7 *Let $f_j : [0,+\infty) \times \cdots \times [0,+\infty) \to [0,+\infty)$ and $\phi_j, \psi_j \in C([\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}, \mathbb{R})$ be given for $j = 1,\dots,n$, where $C([\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}, \mathbb{R})$ stands for the continuous functions on $[\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}$. Then $y(t) = (y_1,\dots,y_n) \in \chi$ is a solution of discrete FBVP (1.1)-(1.2) if and only if $y(t)$ is a fixed point of S.*

Proof From Lemma 2.4, we find that a general solution to problem (1.1)-(1.2) is

$$y_j(t_j) = -\Delta^{-\nu_j}\lambda_j f_j\big(y_1(t_j + \nu_1 - 1),\dots,y_n(t_j + \nu_n - 1)\big) + c_{1j}t_j^{(\nu_j-1)} + c_{2j}t_j^{(\nu_j-2)}.$$

From boundary condition (1.2), we get

$$\begin{aligned} y_j(\nu_j - 2) &= -\Delta^{-\nu_j}\lambda_j f_j\big(y_1(t_j + \nu_1 - 1),\dots,y_n(t_j + \nu_n - 1)\big)|_{t_j=\nu_j-2} \\ &\quad + c_{1j}(\nu_j - 2)^{(\nu_j-1)} + c_{2j}(\nu_j - 2)^{(\nu_j-2)} \\ &= -\frac{1}{\Gamma(\nu_j)}\sum_{s=0}^{t_j-\nu_j}(t_j - s - 1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s + \nu_1 - 1),\dots,y_n(s + \nu_n - 1)\big)|_{t_j=\nu_j-2} \\ &\quad + c_{2j}\Gamma(\nu_j - 1) \\ &= c_{2j}\Gamma(\nu_j - 1) = \psi_j(y_j), \end{aligned}$$

so

$$c_{2j} = \frac{\psi_j(y_j)}{\Gamma(\nu_j - 1)}.$$

On the other hand, applying boundary condition (1.2) to $y_j(t)$ implies that

$$\begin{aligned} y_j(\nu_j + b) &= -\Delta^{-\nu_j}\lambda_j f_j\big(y_1(t_j + \nu_1 - 1),\dots,y_n(t_j + \nu_n - 1)\big)|_{t_j=\nu_j+b} \\ &\quad + c_{1j}(\nu_j + b)^{(\nu_j-1)} + c_{2j}(\nu_j + b)^{(\nu_j-2)} \\ &= -\frac{1}{\Gamma(\nu_j)}\sum_{s=0}^{b}(t_j - s - 1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s + \nu_1 - 1),\dots,y_n(s + \nu_n - 1)\big) \\ &\quad + c_{1j}(\nu_j + b)^{(\nu_j-1)} + \frac{\psi_j(y_j)}{\Gamma(\nu_j - 1)}(\nu_j + b)^{(\nu_j-2)} \\ &= \phi_j(y_j), \end{aligned}$$

so

$$\begin{aligned} c_{1j} &= \frac{\phi_j(y_j)}{(\nu_j + b)^{(\nu_j-1)}} - \frac{\psi_j(y_j)(\nu_j + b)^{(\nu_j-2)}}{\Gamma(\nu_j - 1)(\nu_j + b)^{(\nu_j-1)}} \\ &\quad + \frac{1}{\Gamma(\nu_j)(\nu_j + b)^{(\nu_j-1)}}\sum_{s=0}^{b}(\nu_j + b - s - 1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s + \nu_1 - 1),\dots,y_n(s + \nu_n - 1)\big) \\ &= \frac{\phi_j(y_j)}{(\nu_j + b)^{(\nu_j-1)}} - \frac{\psi_j(y_j)}{(b + 2)\Gamma(\nu_j - 1)} \\ &\quad + \frac{1}{\Gamma(\nu_j)(\nu_j + b)^{(\nu_j-1)}}\sum_{s=0}^{b}(\nu_j + b - s - 1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s + \nu_1 - 1),\dots,y_n(s + \nu_n - 1)\big). \end{aligned}$$

Finally, we can get that

$$\begin{aligned}
y_j(t_j) &= -\frac{1}{\Gamma(\nu_j)}\sum_{s=0}^{t_j-\nu_j}(t_j-s-1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\big)\\
&\quad+\left[\frac{\phi_j(y_j)}{(\nu_j+b)^{(\nu_j-1)}}-\frac{\psi_j(y_j)}{(b+2)\Gamma(\nu_j-1)}\right]t_j^{(\nu_j-1)}\\
&\quad+\frac{t_j^{(\nu_j-1)}}{\Gamma(\nu_j)(\nu_j+b)^{(\nu_j-1)}}\sum_{s=0}^{b}(\nu_j+b-s-1)^{(\nu_j-1)}\lambda_j f_j\big(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\big)\\
&\quad+\frac{\psi_j(y_j)}{\Gamma(\nu_j-1)}t_j^{(\nu_j-2)}\\
&=\psi_j(y_j)\left[-\frac{t_j^{(\nu_j-1)}}{(b+2)\Gamma(\nu_j-1)}+\frac{t_j^{(\nu_j-2)}}{\Gamma(\nu_j-1)}\right]+\phi_j(y_j)\cdot\frac{t_j^{(\nu_j-1)}}{(\nu_j+b)^{(\nu_j-1)}}\\
&\quad+\sum_{s=0}^{t_j-\nu_j}\left\{\left(\frac{t_j^{(\nu_j-1)}(\nu_j+b-s-1)^{(\nu_j-1)}}{\Gamma(\nu_j)(\nu_j+b)^{(\nu_j-1)}}-\frac{(t_j-s-1)^{(\nu_j-1)}}{\Gamma(\nu_j)}\right)\right.\\
&\quad\left.\times\lambda_j f_j\big(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\big)\right\}\\
&\quad+\sum_{s=t_j-\nu_j+1}^{b}\frac{t_j^{(\nu_j-1)}(\nu_j+b-s-1)^{(\nu_j-1)}}{\Gamma(\nu_j)(\nu_j+b)^{(\nu_j-1)}}\lambda_j f_j\big(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\big)\\
&=\psi_j\alpha_j(t_j)+\phi_j(y_j)\beta_j(t_j)+\lambda_j\sum_{s=0}^{b}G_j(t_j,s)f_j\big(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\big).
\end{aligned}$$

The opposite direction is obvious, so it is omitted. Consequently, we get that $y_j(t)$ is a solution of (1.1)-(1.2) if and only if $(y_1,\ldots,y_n)\in\chi$ is a fixed point of S, as desired. □

Lemma 2.8 *The function $\alpha_j(t_j)$ is strictly decreasing in t_j for $t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}$. In addition,*

$$\min_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\alpha_j(t_j)=0,\qquad \max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\alpha_j(t_j)=1.$$

On the other hand, the function $\beta_j(t_j)$ is strictly increasing in t_j for $t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}$. In addition,

$$\min_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\beta_j(t_j)=0,\qquad \max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\beta_j(t_j)=1.$$

Proof Note that for every $t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}$,

$$\begin{aligned}
\Delta_{t_j}\alpha_j(t_j)&=\Delta_{t_j}\left[\frac{t_j^{(\nu_j-2)}}{\Gamma(\nu_j-1)}-\frac{t_j^{(\nu_j-1)}}{(b+2)\Gamma(\nu_j-1)}\right]\\
&=\frac{1}{\Gamma(\nu_j-1)}\left[(\nu_j-2)t^{(\nu_j-3)}-(\nu_j-1)\frac{t_j^{(\nu_j-2)}}{b+2}\right]<0.
\end{aligned}$$

So, the first claim about $\alpha_j(t_j)$ holds. On the other hand,

$$\begin{aligned}
\alpha_j(\nu_j-2) &= \frac{(\nu_j-2)^{(\nu_j-2)}}{\Gamma(\nu_j-1)} - \frac{(\nu_j-2)^{(\nu_j-1)}}{(b+2)\Gamma(\nu_j-1)} = 1, \\
\alpha_j(\nu_j+b) &= \frac{(\nu_j+b)^{(\nu_j-2)}}{\Gamma(\nu_j-1)} - \frac{(\nu_j+b)^{(\nu_j-1)}}{(b+2)}\Gamma(\nu_j-1) \\
&= \frac{\Gamma(\nu_j+b+1)}{\Gamma(\nu_j+b+1-\nu_j+2)\Gamma(\nu_j-1)} - \frac{\Gamma(\nu_j+b+1)}{\Gamma(\nu_j+b+1-\nu_j+1)(b+2)\Gamma(\nu_j-1)} \\
&= \frac{\Gamma(\nu_j+b+1)}{\Gamma(b+3)\Gamma(\nu_j-1)} - \frac{\Gamma(\nu_j+b+1)}{\Gamma(b+2)(b+2)\Gamma(\nu_j-1)} \\
&= 0.
\end{aligned}$$

It follows that

$$\max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}} \alpha_j(t_j) = 1, \qquad \min_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}} \alpha_j(t_j) = 0.$$

It may be shown in a similar way that $\beta_j(t)$ satisfies the properties given in the statement of this lemma. We omit the details. □

Corollary 1 *Let $I_j = [\frac{b+\nu_j}{4}, \frac{3(b+\nu_j)}{4}]$. There are constants $M_{\alpha_j}, M_{\beta_j} \in (0,1)$ such that $\min_{t_j\in I_j} \alpha_j(t_j) = M_{\alpha_j}\|\alpha_j\|$, $\min_{t_j\in I_j} \beta_j(t_j) = M_{\beta_j}\|\beta_j\|$ for $j = 1,2,\ldots,n$, where $\|\cdot\|$ is the usual maximum norm.*

Theorem 2.9 [7] *Let $\mathcal{B}$ be a Banach space, and let $P \subseteq \mathcal{B}$ be a cone in $\mathcal{B}$. Assume that S_1 and S_2 are open subsets of $\mathcal{B}$ with $0 \in S_1 \subset \overline{S}_1 \subset S_2$. Assume, further, that*

$$T : P \cap (\overline{S}_2 \setminus S_1) \to P$$

is a completely continuous operator. If either
(1) $\|Tu\| \le \|u\|$, $u \in P \cap \partial S_1$, $\|Tu\| \ge \|u\|$, $u \in P \cap \partial S_2$; *or*
(2) $\|Tu\| \ge \|u\|$, $u \in P \cap \partial S_1$, $\|Tu\| \le \|u\|$, $u \in P \cap \partial S_2$.
Then T has a fixed point in $P \cap (\overline{S}_2 \setminus S_1)$.

3 Main results

In this section, we present the theorems for the existence of at least two positive solutions to problem (1.1)-(1.2). In the sequel, we let

$$A_j = \max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}} \sum_{s=0}^{b} G_j(t_j,s), \tag{3.1}$$

$$B_j = \min_{t_j\in[\frac{\nu_j+b}{4},\frac{3(\nu_j+b)}{4}]} \sum_{s=0}^{b} G_j(t_j,s). \tag{3.2}$$

We now present the conditions that we presume in the sequel.

(L_1) $\lim_{(y_1+\cdots+y_n)\to 0^+} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = \infty$ for $t_j \in [\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}$, $j = 1,2,\ldots,n$.

(L$_2$) $\lim_{(y_1+\cdots+y_n)\to\infty} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = \infty$ for $t_j \in [\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}$, $j = 1, 2, \ldots, n$.
(L$_3$) $\lim_{(y_1+\cdots+y_n)\to 0^+} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = 0$ for $t_j \in [\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}$, $j = 1, 2, \ldots, n$.
(L$_4$) $\lim_{(y_1+\cdots+y_n)\to\infty} \frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} = 0$ for $t_j \in [\nu_j - 2, \nu_j + b]_{\mathbb{N}_{\nu_j-2}}$, $j = 1, 2, \ldots, n$.
(G$_1$) The functionals ψ_j, ϕ_j are linear. In particular, we assume that

$$\psi_j(y_j) = \sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} y_j(i), \qquad \phi_j(y_j) = \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} y_j(k)$$

for $c^j_{i-\nu_j+2}, d^j_{k-\nu_j+2} \in \mathbb{R}$, $j = 1, \ldots, n$.
(G$_2$) For $j = 1, 2, \ldots, n$, there are

$$\sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} G_j(i,s) \geq 0, \qquad \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} G_j(k,s) \geq 0$$

for each $s \in [0, b]_{\mathbb{N}_0}$, and in addition

$$\sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} + \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2} \leq \frac{1}{2n}. \tag{3.3}$$

(G$_3$) We have that each of $\psi_j(\alpha_j)$, $\psi_j(\beta_j)$, $\phi_j(\alpha_j)$, and $\phi_j(\beta_j)$ is nonnegative for $j = 1, \ldots, n$.

Let $I := [\frac{\nu_1+b}{4}, \frac{3(\nu_1+b)}{4}] \times \cdots \times [\frac{\nu_n+b}{4}, \frac{3(\nu_n+b)}{4}]$. In the sequel, we shall also make use of the cone

$$\begin{aligned} \kappa := \Big\{ &(y_1, \ldots, y_n) \in \chi : y_1, \ldots, y_n \geq 0, \psi_j(y_j) \geq 0, \phi_j(y_j) \geq 0, j = 1, \ldots, n, \\ &\min_{(t_1,\ldots,t_n)\in I} \big[y_1(t_1) + \cdots + y_n(t_n)\big] \geq \gamma \big\| (y_1, \ldots, y_n) \big\| \Big\}, \end{aligned} \tag{3.4}$$

where $\gamma \in (0, 1)$ is a constant defined by

$$\gamma = \min\{\gamma_1, \ldots, \gamma_n, M_{\alpha_1}, \ldots, M_{\alpha_n}, M_{\beta_1}, \ldots, M_{\beta_n}\}, \tag{3.5}$$

where M_{α_j}, M_{β_j} come from Corollary 1 and γ_j is associated by Lemma 2.6(iii) to $G_j(t_j, s)$, $j = 1, 2, \ldots, n$.

Lemma 3.1 *Let S be the operator defined as in* (2.2). *Then $S : \kappa \to \kappa$.*

Proof By means of (G$_1$), we get

$$\begin{aligned} \psi_j\big(S_j(y_1, \ldots, y_n)\big) &= \sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} \big(S_j(y_1, \ldots, y_n)\big)(i) \\ &= \sum_{i=\nu_j-2}^{\nu_j+b} \sum_{l=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} c^j_{l-\nu_j+2} y_j(l) \alpha_j(i) \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{i=\nu_j-2}^{\nu_j+b} \sum_{k=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} d^j_{k-\nu_j+2} y_j(k) \beta_j(i) \\
&+ \lambda_j \sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} \sum_{s=0}^{b} G_j(i,s) f_j\bigl(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\bigr) \\
=\; & \psi_j(\alpha_j)\psi_j(y_j) + \psi_j(\beta_j)\phi_j(y_j) \\
&+ \lambda_j \psi_j \left(\sum_{s=0}^{b} G_j(t,s) f_j\bigl(y_1(s+\nu_1-1)\bigr),\dots,y_n(s+\nu_n-1) \right)
\end{aligned}
$$

for $(y_1,\dots,y_n) \in \chi$.

By assumptions (G_2) and (G_3) together with the nonnegativity of $f_j(y_1,\dots,y_n)$ and the fact that $(y_1,\dots,y_n) \in \chi$, we can get $\psi_j(S_j(y_1,\dots,y_n)) \geq 0$. By means of the same method, we obtain $\phi_j(S_j(y_1,\dots,y_n)) \geq 0$ for $j = 1,2,\dots,n$.

On the other hand, we show that

$$
\min_{(t_1,\dots,t_n)\in I} \bigl[S_1(y_1,\dots,y_n)(t_1) + \cdots + S_n(y_1,\dots,y_n)(t_n) \bigr] \geq \gamma \bigl\| S(y_1,\dots,y_n) \bigr\|
$$

for $(y_1,\dots,y_n) \in \chi$. In fact, by Lemma 2.6(iii), we have

$$
\begin{aligned}
&\min_{t_j \in [\frac{b+\nu_j}{4}, \frac{3(\nu_j+b)}{4}]} S_j(y_1,\dots,y_n)(t_j) \\
&\quad\geq \min_{t_j \in [\frac{b+\nu_j}{4}, \frac{3(\nu_j+b)}{4}]} \alpha_j(t_j)\psi_j(y_j) + \min_{t_j \in [\frac{b+\nu_j}{4}, \frac{3(\nu_j+b)}{4}]} \beta_j(t_j)\phi_j(y_j) \\
&\quad\quad+ \min_{t_j \in [\frac{b+\nu_j}{4}, \frac{3(\nu_j+b)}{4}]} \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\bigl(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\bigr) \\
&\quad\geq M_{\alpha_j} \|\alpha_j\| \psi_j(y_j) + M_{\beta_j} \|\beta_j\| \phi_j(y_j) \\
&\quad\quad+ \lambda_j \sum_{s=0}^{b} \gamma_j G_j(s+\nu_j-1,s) f_j\bigl(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\bigr) \\
&\quad\geq \widetilde{\gamma}_j \|\alpha_j\| \psi_j(y_j) + \widetilde{\gamma}_j \|\beta_j\| \phi_j(y_j) \\
&\quad\quad+ \widetilde{\gamma}_j \max_{t_j \in [\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}} \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\bigl(y_1(s+\nu_1-1),\dots,y(s+\nu_n-1)\bigr) \\
&\quad\geq \widetilde{\gamma}_j \max_{t_j \in [\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}} \Biggl[\alpha_j(t_j)\psi_j(y_j) + \beta_j\phi_j(y_j) \\
&\quad\quad+ \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\bigl(y_1(s+\nu_1-1),\dots,y(s+\nu_n-1)\bigr) \Biggr] \\
&\quad= \widetilde{\gamma}_j \bigl\| S_j(y_1,\dots,y_n) \bigr\|,
\end{aligned}
$$

where $\widetilde{\gamma}_j = \{M_{\alpha_j}, M_{\beta_j}, \gamma_j\}$, $j = 1,2,\dots,n$.

Let $\gamma = \min\{\gamma_1, \ldots, \gamma_n, M_{\alpha_1}, \ldots, M_{\alpha_n}, M_{\beta_1}, \ldots, M_{\beta_n}\}$. Then we obtain

$$\begin{aligned}
&\min_{(t_1,\ldots,t_n)\in I}\bigl[S_1(y_1,\ldots,y_n)(t_1) + \cdots + S_n(y_1,\ldots,y_n)(t_n)\bigr] \\
&\quad\geq \min_{(t_1,\ldots,t_n)\in I} S_1(y_1,\ldots,y_n)(t_1) + \cdots + \min_{(t_1,\ldots,t_n)\in I} S_n(y_1,\ldots,y_n)(t_n) \\
&\quad\geq \widetilde{\gamma}_1\bigl\|S_1(y_1,\ldots,y_n)\bigr\| + \cdots + \widetilde{\gamma}_n\bigl\|S_n(y_1,\ldots,y_n)\bigr\| \\
&\quad\geq \gamma\bigl\{\bigl\|S_1(y_1,\ldots,y_n)\bigr\| + \cdots + \bigl\|S_n(y_1,\ldots,y_n)\bigr\|\bigr\} \\
&\quad= \gamma\bigl\|S(y_1,\ldots,y_n)\bigr\|
\end{aligned}$$

for $(y_1, \ldots, y_n) \in \chi$.

Finally, by the definitions S_j $(j = 1, 2, \ldots, n)$, it is clear that

$$S_j(y_1,\ldots,y_n)(t_j) \geq 0, \quad j = 1,2,\ldots n, (y_1,\ldots,y_n) \in \chi.$$

So, we conclude that $S : \kappa \to \kappa$. This completes the proof. □

Lemma 3.2 *Suppose that conditions* (G_1)-(G_3) *hold, and there exist two different positive numbers* r_1, r_2, $r_1 < r_2$, *such that*

$$\max_{0\leq(y_1+\cdots+y_n)\leq r_1} f_j(y_1,\ldots,y_n) \leq \frac{r_1}{2n\lambda_j A_j}, \tag{3.6}$$

$$\min_{\gamma r_2\leq(y_1+\cdots+y_n)\leq r_2} f_j(y_1,\ldots,y_n) \geq \frac{r_2}{n\lambda_j B_j}. \tag{3.7}$$

Then the operator S has a fixed point $(\overline{y}_1, \ldots, \overline{y}_n) \in \kappa$ *such that*

$$r_1 \leq \bigl\|(\overline{y}_1,\ldots,\overline{y}_n)\bigr\| \leq r_2.$$

Proof Let $\kappa_\xi = \{(y_1,\ldots,y_n) \in \kappa, \|(y_1,\ldots,y_n)\| < \xi\}$. Then, for any $(y_1,\ldots,y_n) \in \kappa$ and $\|(y_1,\ldots,y_n)\| = r_1$, we have

$$\begin{aligned}
\bigl\|S_j(y_1,\ldots,y_n)\bigr\| &= \max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\Biggl|\alpha_j(t_j)\psi_j(y_j) + \beta_j(t_j)\phi_j(y_j) \\
&\quad + \lambda_j\sum_{s=0}^{b} G_j(t_j,s)f_j\bigl(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\bigr)\Biggr| \\
&\leq \sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2}y_j(i) + \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2}y_j(k) \\
&\quad + \lambda_j\max_{t_j\in[\nu_j-2,\nu_j+b]_{\mathbb{N}_{\nu_j-2}}}\sum_{s=0}^{b} G_j(t_j,s)f_j\bigl(y_1(s+\nu_1-1),\ldots,y_n(s+\nu_n-1)\bigr) \\
&\leq r_1\Biggl[\sum_{i=\nu_j-2}^{\nu_j+b} c^j_{i-\nu_j+2} + \sum_{k=\nu_j-2}^{\nu_j+b} d^j_{k-\nu_j+2}\Biggr] + \lambda_j A_j\frac{r_1}{2n\lambda_j A_j} \\
&\leq \frac{r_1}{2n} + \frac{r_1}{2n} = \frac{r_1}{n} = \frac{1}{n}\bigl\|(y_1,\ldots,y_n)\bigr\|.
\end{aligned}$$

That is,

$$\begin{aligned}
&\left\|S(y_1,\dots,y_n)(t_1,\dots,t_n)\right\| \\
&\quad = \left\|\left(S_1(y_1,\dots,y_n)(t_1),\dots,S_n(y_1,\dots,y_n)(t_n)\right)\right\| \\
&\quad = \left\|S_1(y_1,\dots,y_n)\right\| + \cdots + \left\|S_n(y_1,\dots,y_n)\right\| \\
&\quad \le \frac{1}{n}\left\|(y_1,\dots,y_n)\right\| + \cdots + \frac{1}{n}\left\|(y_1,\dots,y_n)\right\| \\
&\quad = \left\|(y_1,\dots,y_n)\right\|
\end{aligned}$$

for $(y_1,\dots,y_n) \in \partial\kappa_{r_1}$.

On the other hand, for any $(y_1,\dots,y_n) \in \kappa$ and $\|(y_1,\dots,y_n)\| = r_2$, we have

$$\begin{aligned}
\left\|S_j(y_1,\dots,y_n)\right\| &\ge \alpha_j(t_j)\psi_j(y_j) + \beta_j(t_j)\phi_j(y_j) \\
&\qquad + \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\left(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\right) \\
&\ge \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\left(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\right) \\
&\ge \min_{t_j\in[\frac{\nu_j+b}{4},\frac{3(\nu_j+b)}{4}]} \lambda_j \sum_{s=0}^{b} G_j(t_j,s) f_j\left(y_1(s+\nu_1-1),\dots,y_n(s+\nu_n-1)\right) \\
&\ge \min_{t_j\in[\frac{\nu_j+b}{4},\frac{3(\nu_j+b)}{4}]} \lambda_j \sum_{s=0}^{b} G_j(t_j,s) \frac{r_2}{n\lambda_j B_j} \\
&= \frac{r_2}{n} = \frac{1}{n}\left\|(y_1,\dots,y_n)\right\|.
\end{aligned}$$

That is,

$$\begin{aligned}
&\left\|S(y_1,\dots,y_n)(t_1,\dots,t_n)\right\| \\
&\quad = \left\|\left(S_1(y_1,\dots,y_n)(t_1),\dots,S_n(y_1,\dots,y_n)(t_n)\right)\right\| \\
&\quad = \left\|S_1(y_1,\dots,y_n)\right\| + \cdots + \left\|S_n(y_1,\dots,y_n)\right\| \\
&\quad \ge \frac{1}{n}\left\|(y_1,\dots,y_n)\right\| + \cdots + \frac{1}{n}\left\|(y_1,\dots,y_n)\right\| \\
&\quad = \left\|(y_1,\dots,y_n)\right\|
\end{aligned}$$

for $(y_1,\dots,y_n) \in \partial\kappa_{r_2}$.

By the use of Theorem 2.9, there exists $(\overline{y}_1,\dots,\overline{y}_n) \in \kappa$ such that $S(\overline{y}_1,\dots,\overline{y}_n) = (\overline{y}_1,\dots,\overline{y}_n)$, the proof is complete. □

Theorem 3.3 *Suppose that conditions* (L_1), (L_2) *and* (G_1)-(G_3) *hold. Then, for every* $\lambda_j \in (0,\lambda_j^*)$, *problem* (1.1)-(1.2) *has at least two positive solutions, where*

$$\lambda_j^* = \frac{1}{2nA_j} \sup_{r>0} \frac{r}{\max_{0\le y_1+\cdots+y_n\le r} f_j(y_1,\dots,y_n)}. \tag{3.8}$$

Proof Define the function

$$p_j(r) = \frac{r}{2nA_j \max_{0\leq y_1+\cdots+y_n\leq r} f_j(y_1,\ldots,y_n)}, \quad j=1,\ldots,n,$$

we have that $p_j \in C((0,\infty),(0,\infty))$. In view of ($L_1$), we see that $\lim_{r\to 0}\frac{r}{f_j(r)} = 0$, that is, $\lim_{r\to 0}\frac{r}{2nA_jf_j(r)} = 0$, and

$$0 < p_j(r) = \frac{r}{2nA_j \max_{0\leq y_1+\cdots+y_n\leq r} f_j(y_1,\ldots,y_n)} \leq \frac{r}{2nA_jf_j(r)},$$

so $\lim_{r\to 0} p_j(r) = 0$.

In view of (L_2), we see further that $\lim_{r\to\infty} p_j(r) = 0$. Thus, there exists $r_0 > 0$ such that $p_j(r_0) = \max_{r>0} p_j(r) = \lambda_j^*$, $j = 1,\ldots,n$. For any $\lambda_j \in (0,\lambda_j^*)$, by the intermediate value theorem, there exist two points $b_1 \in (0,r_0)$, $b_2 \in (r_0,\infty)$ such that $p_j(b_1) = p_j(b_2) = \lambda_j$. Thus, we have

$$f_j(y_1,\ldots,y_n) \leq \frac{b_1}{2n\lambda_jA_j}, \quad y_1+\cdots+y_n \in [0,b_1];$$
$$f_j(y_1,\ldots,y_n) \leq \frac{b_2}{2n\lambda_jA_j}, \quad y_1+\cdots+y_n \in [0,b_2].$$

On the other hand, in view of (L_1) and (L_2), we see that there exist $c_1 \in (0,b_1)$, $c_2 \in (b_2,\infty)$ such that

$$\frac{f_j(y_1,\ldots,y_n)}{y_1+\cdots+y_n} \geq \frac{1}{n\lambda_j\gamma B_j}, \quad y_1+\cdots+y_n \in (0,b_1] \cup [b_2\gamma,\infty).$$

That is,

$$f_j(y_1,\ldots,y_n) \geq \frac{c_1}{n\lambda_jB_j}, \quad y_1+\cdots+y_n \in [\gamma c_1,c_1],$$
$$f_j(y_1,\ldots,y_n) \geq \frac{c_2}{n\lambda_jB_j}, \quad y_1+\cdots+y_n \in [\gamma c_2,c_2],$$

where γ is defined by (3.5). An application of Lemma 3.2 leads to two distinct solutions of (1.1)-(1.2) which satisfy

$$c_1 \leq \|(\bar{y}_1,\ldots,\bar{y}_n)\| \leq b_1, \qquad b_2 \leq \|(\bar{y}'_1,\ldots,\bar{y}'_n)\| \leq c_2.$$

The proof is complete. □

Theorem 3.4 *Suppose that* (L_3), (L_4) *and* (G_1)-(G_3) *hold. Then, for any* $\lambda_j \geq \lambda_j^{**}$, *equation* (1.1)-(1.2) *has at least two positive solutions, where*

$$\lambda_j^{**} = \frac{1}{nB_j}\inf_{r>0}\frac{r}{\min_{\gamma r\leq y_1+\cdots+y_n\leq r} f_j(y_1,\ldots,y_n)}, \tag{3.9}$$

and γ *is defined by* (3.5).

The proof is similar to Theorem 3.3 and hence omitted.

We now present an example to illustrate the sorts of boundary conditions that can be treated by Theorem 3.3.

Example 3.1 Consider the following boundary value problems:

$$\begin{cases} \Delta^{\frac{13}{10}} y(t) = -\lambda_1 f_1(y_1(t+\frac{3}{10}), y_2(t+\frac{1}{2})), \\ \Delta^{\frac{3}{2}} y(t) = -\lambda_2 f_2(y_1(t+\frac{3}{10}), y_2(t+\frac{1}{2})), \end{cases} \tag{3.10}$$

$$\begin{aligned} y_1\left(-\frac{7}{10}\right) &= \frac{1}{12} y_1\left(\frac{13}{10}\right) - \frac{1}{25} y_1\left(\frac{53}{10}\right), \\ y_1\left(\frac{213}{10}\right) &= \frac{1}{30} y_1\left(\frac{83}{10}\right) - \frac{1}{100} y_1\left(\frac{73}{10}\right), \end{aligned} \tag{3.11}$$

$$\begin{aligned} y_2\left(-\frac{1}{2}\right) &= \frac{1}{40} y_2\left(\frac{3}{2}\right) - \frac{1}{150} y_2\left(\frac{15}{2}\right), \\ y_2\left(\frac{43}{2}\right) &= \frac{1}{17} y_2\left(\frac{5}{2}\right) - \frac{1}{30} y_2\left(\frac{11}{2}\right), \end{aligned} \tag{3.12}$$

where $b = 20$, $\nu_1 = \frac{13}{10}$, $\nu_2 = \frac{3}{2}$, we take

$$\begin{aligned} &f_1(y_1, y_2) = (y_1 + y_2)^{\frac{1}{2}} + (y_1 + y_2)^2, \qquad f_2(y_1, y_2) = (y_1 + y_2)^{\frac{1}{2}} + \frac{1}{64}(y_1 + y_2)^{\frac{3}{2}}, \\ &\psi_1(y_1) = \frac{1}{12} y_1\left(\frac{13}{10}\right) - \frac{1}{25} y_1\left(\frac{53}{10}\right), \qquad \phi_1(y_1) = \frac{1}{30} y_1\left(\frac{83}{10}\right) - \frac{1}{100} y_1\left(\frac{73}{10}\right), \\ &\psi_2(y_2) = \frac{1}{40} y_2\left(\frac{3}{2}\right) - \frac{1}{150} y_2\left(\frac{15}{2}\right), \qquad \phi_2(y_2) = \frac{1}{17} y_2\left(\frac{5}{2}\right) - \frac{1}{30} y_2\left(\frac{11}{2}\right), \end{aligned}$$

$f_1, f_2 : [0, +\infty) \times [0, +\infty) \to [0, +\infty)$, and y_1 is defined on the time scale $\{-\frac{7}{10}, \frac{3}{10}, \ldots, \frac{213}{10}\}$, y_2 is defined on the time scale $\{-\frac{1}{2}, \frac{1}{2}, \ldots, \frac{43}{2}\}$.

It is easy to get that (F_1), (F_2) hold. On the other hand, (G_1) holds. Now, we see that (G_2), (G_3) hold. In fact,

$$\begin{aligned} &\sum_{i=\nu_1-2}^{\nu_1+b} c^1_{i-\nu_1+2} + \sum_{k=\nu_1-2}^{\nu_1+b} d^1_{k-\nu_1+2} = \frac{1}{12} - \frac{1}{25} + \frac{1}{30} - \frac{1}{100} = \frac{1}{15} < \frac{1}{4}, \\ &\sum_{i=\nu_2-2}^{\nu_2+b} c^2_{i-\nu_2+2} + \sum_{k=\nu_2-2}^{\nu_2+b} d^2_{k-\nu_2+2} = \frac{1}{40} - \frac{1}{150} + \frac{1}{17} - \frac{1}{30} = \frac{149}{3{,}400} < \frac{1}{4}. \end{aligned}$$

In addition,

$$\begin{aligned} &\sum_{i=\nu_1-2}^{\nu_1+b} c^1_{i-\nu_1+2} G_1(i,s) \\ &\quad = \frac{1}{12} G_1(\nu_1, s) - \frac{1}{25} G_1(\nu_1+4, s) \\ &\quad = \frac{1}{12} \frac{\nu_1^{(\nu_1-1)}(\nu_1+b-s-1)^{(\nu_1-1)}}{\Gamma(\nu_1)(\nu_1+b)^{(\nu_1-1)}} - \frac{1}{25} G(\nu_1+4, s) \end{aligned}$$

$$\begin{aligned}
&\geq \frac{1}{12}\frac{\nu_1^{(\nu_1-1)}(\nu_1+b-s-1)^{(\nu_1-1)}}{\Gamma(\nu_1)(\nu_1+b)^{(\nu_1-1)}} - \frac{1}{25}\frac{(\nu_1+4)^{(\nu_1-1)}(\nu_1+b-s-1)^{(\nu_1-1)}}{\Gamma(\nu_1)(\nu_1+b)^{(\nu_1-1)}} \\
&= \frac{(\nu_1+b-s-1)^{(\nu_1-1)}}{\Gamma(\nu_1)(\nu_1+b)^{(\nu_1-1)}}\left[\frac{1}{12}\nu_1^{(\nu_1-1)} - \frac{1}{25}(\nu_1+4)^{(\nu_1-1)}\right] \\
&= \frac{\nu_1(\nu_1+b-s-1)^{(\nu_1-1)}}{(\nu_1+b)^{(\nu_1-1)}}\cdot\frac{16}{625} > 0.
\end{aligned}$$

By using a similar method, we get

$$\sum_{k=\nu_1-2}^{\nu_1+b} d^1_{k-\nu_1+2}G_1(k,s) \geq 0, \qquad \sum_{i=\nu_2-2}^{\nu_2+b} c^2_{i-\nu_2+2}G_2(i,s) \geq 0, \qquad \sum_{k=\nu_2-2}^{\nu_2+b} d^2_{k-\nu_2+2}G_2(k,s) \geq 0.$$

Hence, (G_2) holds.

Finally, we numerically calculate that

$$\begin{aligned}
\psi_1(\alpha_1) &= \frac{1}{12}\alpha_1(\nu_1) - \frac{1}{25}\alpha_1(\nu_1+4) \\
&= \frac{1}{12\Gamma(\nu_1-1)}\left[\nu_1^{(\nu_1-2)} - \frac{1}{b+2}\nu_1^{(\nu_1-1)}\right] \\
&\quad - \frac{1}{25\Gamma(\nu_1-1)}\left[(\nu_1+4)^{(\nu_1-2)} - \frac{1}{b+2}(\nu_1+4)^{(\nu_1-1)}\right] \\
&\approx 0.012.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\psi_1(\beta_1) \approx 0.012, \qquad \phi_1(\alpha_1) \approx 0.00091, \qquad \phi_1(\beta_1) \approx 0.018; \\
&\psi_2(\alpha_2) \approx 0.00769, \qquad \psi_2(\beta_2) \approx 0.00315, \qquad \phi_2(\alpha_2) \approx 0.0104, \\
&\phi_2(\beta_2) \approx 0.0038.
\end{aligned}$$

We obtain that each of $\psi_j(\alpha_j)$, $\psi_j(\beta_j)$, $\phi_j(\alpha_j)$ and $\phi_j(\beta_j)$ is nonnegative for $j = 1, 2$. So, condition (G_3) holds. Namely, f_1, f_2 and $\psi_1, \psi_2, \phi_1, \phi_2$ satisfy the conditions of Theorem 3.3.

A computation shows that $\lambda_1^* \approx 5.33 \times 10^{-3}$, $\lambda_2^* \approx 1.357 \times 10^{-2}$. Then, for every $\lambda_j \in (0, \lambda_j^*)$ $(j = 1, 2)$, problem (3.10)-(3.12) has at least two positive solutions.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SK conceived of the study, and participated in its design and coordination. XZ drafted the manuscript. HC participated in the design of the study and the sequence correction. All authors read and approved the final manuscript.

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