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Stability analysis of second-order differential systems with Erlang distribution random impulses

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Abstract

Differential systems with random impulses are a new kind of mathematical models. In this paper, we put forward a model of second-order impulsive differential systems with Erlang distribution random impulses. Sufficient conditions are obtained for oscillation in mean and p-moment stability of this model respectively. An example is presented to illustrate the efficiency of the results obtained.

MSC: 34A37; 34A12; 34A34

Keywords: linear differential system; random impulses; stability; oscillation; Erlang distribution

1 Introduction

It is recognized that the impulsive differential system is an effective model for many real world phenomena, thus it has been widely used in the study of physics, engineering, information and communications technology, *etc.* in the past years and a lot of valuable results have been obtained (see [1–15] and references therein).

For impulsive differential systems, most researchers concern about two kinds of impulse times: fixed impulse times and varying impulse times, which mean that the impulse time is some functions of the 'state x' [7–9]. However, the impulse phenomena sometimes happen at random times, and any solution of systems driven by this kind of impulses is a stochastic process, which is very different from those of differential systems with impulses at fixed moments and varying impulse times [11]. Thus, the randomness introduced in impulsive differential systems by this way has brought us new difficulties and problems in the study of impulsive differential systems. Some other kinds of randomness brought to a system can be seen in [16–18].

In fact, only few researchers have studied this kind of impulse (see [11–15] and references therein). Wu and Meng first introduced random impulsive ordinary differential equations and investigated the boundedness of solutions to these models by Lyapunov's direct method in [12]. In [13], Wu *et al.* discussed the existence and uniqueness in mean square of solutions to certain impulsive differential systems by employing the Cauchy-Schwarz inequality, Lipschitz condition and techniques in stochastic analysis. In [14], Anguraj *et al.* presented the existence and exponential stability of mild solutions of semilinear differential equations with random impulses. In [15], the existence and uniqueness of stochastic differential equations with random impulses were studied by Wu and Zhou via employing the Bihari inequality under non-Lipschitz conditions. In [11], Wu and Duan studied the oscillation, stability and boundedness of second-order differential systems with random impulses. But in their paper, the random impulses must be independent and follow the same exponential distribution which is a very strong condition. Inspired by their work, we generalized their results to a much more general distribution, called the Erlang distribution, which has been widely used to describe the waiting times. We will call this kind of impulse the 'random impulses' throughout this paper. And in this paper, we will discuss properties such as oscillation, stability of second-order differential systems with Erlang distribution random impulses.

The rest of this paper is organized as follows. In Section 2, we recall some preliminary definitions. In Section 3, we first give some useful and important lemmas, then establish our main oscillation and stability results. In Section 4, we give an example to illustrate the effectiveness of our results. Finally, the conclusion and the acknowledgements are mentioned in Section 5.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given complete probability space and τ_k's be non-negative random variables defined in $(\Omega, \mathcal{F}, \mathbb{P})$, $k = 1, 2, \ldots$. Furthermore, assume that τ_i and τ_j are independent with each other when $i \neq j$ for $i, j = 1, 2, \ldots$. $\mathbb{E}$ denotes the mathematical expectation. For the sake of simplicity, we denote $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_\tau = [\tau, +\infty)$, where $\tau \in \mathbb{R}$ is a constant. Consider the second-order linear differential systems with random impulses,

$$y''(t) + a(t)y'(t) + p(t)y(t) = 0, \quad t \in \mathbb{R}_\tau, t \neq \xi_k, \forall k \in \mathbb{N}, \tag{1}$$

and

$$\triangle y(\xi_k) = b_k y(\xi_k^-), \qquad \triangle y'(\xi_k) = b_k y'(\xi_k^-), \quad \forall k \in \mathbb{N}, \tag{2}$$

where $a, p \in (\mathbb{R}_\tau, \mathbb{R})$ are Lebesgue measurable and locally essentially bounded functions, $\xi_0 = t_0 \in \mathbb{R}_\tau$ and $\xi_k = \xi_{k-1} + \tau_k$ for all $k \in \mathbb{N}$, $\triangle y(\xi_k) \equiv y(\xi_k) - y(\xi_k^-)$, $\triangle y'(\xi_k) \equiv y'(\xi_k) - y'(\xi_k^-)$, and $y(\xi_k^-) = \lim_{t \to \xi_k^-} y(t)$.

Before giving the main results, we first introduce some definitions.

Definition 2.1 A stochastic process $y(t)$ is said to be a sample path solution to the system (1) with (2) satisfying the initial value condition,

$$y_{t_0} = y_0,$$

if for any sample value $t_1 < t_2 < \cdots < t_k < \cdots$, of $\{\xi_k\}_{k \geq 1}$, then $y(t)$ satisfies the following equations:

$$\begin{cases} y''(t) + a(t)y'(t) + p(t)y(t) = 0, & t \in \mathbb{R}_\tau, \xi_k \neq t, \forall k \in \mathbb{N}, \\ \triangle y(t_k) = b_k y(t_k^-), & \forall k \in \mathbb{N}, \\ \triangle y'(t_k) = b_k y'(t_k^-), & \forall k \in \mathbb{N}. \end{cases}$$

Definition 2.2 Let $p > 0$, then the system (1) with (2) is said to be

(i) p-moment stable if for any $\epsilon > 0$ and $t_0 \in \mathbb{R}_\tau$, there exists a $\delta = \delta(\epsilon, t_0) > 0$ such that

$$|y_0|^p < \delta \quad \text{implies} \quad \mathbb{E}|y(t)|^p < \epsilon \quad \text{for all } t \geq t_0;$$

(ii) uniformly p-moment stable if the δ in (i) is independent of t_0;

(iii) asymptotically p-moment stable if it is p-moment stable, and for any $\epsilon_1 > 0$, δ_1 and $t_0 \in \mathbb{R}_\tau$, there exists a $T = T(\epsilon_1, \delta_1, t_0)$ such that

$$|y_0|^p < \delta_1 \quad \text{implies} \quad \mathbb{E}|y(t)|^p < \epsilon_1 \quad \text{for all } t \geq t_0 + T;$$

(iv) uniformly asymptotically p-moment stable if it is uniformly p-moment stable, and the T in (iii) is independent of t_0.

Generally, two-moment stable is called stable in mean square.

Definition 2.3 A solution $y(t)$ to the system (1) with (2) is said to be non-oscillatory in mean if $\mathbb{E}y(t)$ is either eventually positive or eventually negative. Otherwise, it is called oscillatory.

Definition 2.4 The Erlang distribution is a continuous probability distribution with probability density function as follows:

$$f(x;k,\lambda) = \begin{cases} \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, & x > 0, \\ 0, & \text{otherwise}, \end{cases}$$

where $k \in \mathbb{N}$ is the shape parameter and $\lambda > 0$ is the rate parameter. In the following, we will denote the probability density function of the Erlang distribution by $\mathrm{Erlang}(k, \lambda)$.

Remark 2.1 Some properties of the Erlang distribution.

(i) The mean value of $\mathrm{Erlang}(k,\lambda)$ is $\frac{k}{\lambda}$, the variance of $\mathrm{Erlang}(k,\lambda)$ is $\frac{k}{\lambda^2}$;

(ii) $\mathrm{Erlang}(1,\lambda) = \mathrm{exponential}(\lambda)$;

(iii) If X, Y are independent, and $X \sim \mathrm{Erlang}(k_1,\lambda)$, $Y \sim \mathrm{Erlang}(k_2,\lambda)$, then $X + Y \sim \mathrm{Erlang}(k_1 + k_2, \lambda)$;

(iv) The cumulative distribution function (CDF) of the Erlang distribution is

$$F(x;k,\lambda) = 1 - \sum_{n=0}^{k-1} e^{-\lambda x} \frac{(\lambda x)^n}{n!}.$$

Consider the following auxiliary differential system:

$$x''(t) + a(t)x'(t) + p(t)x(t) = 0, \quad t \in \mathbb{R}_\tau. \tag{3}$$

A solution $x(t)$ to the system (3) means that $x(t)$ has the second-order derivative $x''(t)$ on $\mathbb{R}_\tau$ and satisfies the system (3) for all $t \in \mathbb{R}_\tau$.

The following condition will be needed by all main results in Section 3.

(H) Let τ_k follow $\mathrm{Erlang}(m_k, \lambda)$, where $m_k \in \mathbb{N}$, $k = 1, 2, \ldots$, and $m_i \leq m_j$, for any $i \leq j$ and let τ_i be independent with τ_j when $i \neq j$.

Remark 2.2 From Remark 2.1(i), we know that the assumption $m_i \leq m_j$ $(i \leq j)$ means that the impulses interval τ_k is non-decreasing in k in the sense of expectation.

3 Main results

In this section, some results on p-moment stability and oscillation in mean of the second-order linear differential systems (1) with random impulses (2) are presented. Inspired by [19], we obtain the following lemma, which guarantees $\lim_{k\to+\infty}\xi_k = +\infty$ with probability 1.

Lemma 3.1 *Assume that the condition* (H) *holds, then* $\lim_{k\to+\infty}\xi_k = +\infty$ *with probability* 1.

Proof For a given non-negative random variable τ and a constant $c > 0$, we define

$$\tau^{(c)} := \begin{cases} \tau, & \text{if } \tau \leq c, \\ 0, & \text{otherwise.} \end{cases}$$

By Kolmogorov's three series theorem and Kolmogorov's zero-or-one laws [20], $\sum_{k=1}^{+\infty}\tau_k = +\infty$ almost surely if and only if at least one of the series $\sum_{k=1}^{+\infty}\mathbb{P}(\tau_k > c)$ and $\sum_{k=1}^{+\infty}\mathbb{E}(\tau_k^{(c)})$ diverges for some $c > 0$ or any $c > 0$.

From the CDF of the Erlang distribution, we obtain that $\mathbb{P}(\tau_k > c) = 1 - \mathbb{P}(\tau_k \leq c) = 1 - F(c; m_k, \lambda)$ is increasing in k, together with $\mathbb{P}(\tau_1 > c) > 0$, we conclude that $\sum_{k=1}^{+\infty}\mathbb{P}(\tau_k > c)$ diverges. Thus, we know that $\sum_{k=1}^{+\infty}\tau_k = +\infty$ almost surely. The proof is complete. □

Lemma 3.2 (see [11]) $y(t)$ *is a solution of the system* (1) *with* (2) *if and only if*

$$y(t) = \sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}[1 + b_i] \cdot \mathcal{X}_{[\xi_k,\xi_{k+1})}(t)\right)x(t),$$

where $x(t)$ *is a solution of the system* (3) *with the same initial conditions of the system* (1) *with* (2), *and* $\mathcal{X}$ *is the index function, i.e.*,

$$\mathcal{X}_{[\xi_k,\xi_{k+1})}(t) = \begin{cases} 1, & \textit{if } \xi_k \leq t < \xi_{k+1}, \\ 0, & \textit{otherwise.} \end{cases}$$

Here and in the sequel, we assume that a product equals unity if the number of factors is equal to zero.

Theorem 3.1 *Let the condition* (H) *hold. Further assume that there are a finite number of* b_k *such that* $b_k < -1$. *If there exists a* $T \in \mathbb{R}_\tau$ *such that*

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1 + b_i) \cdot \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)$$

does not change its sign for all $t \geq T$, *then all solutions of the system* (1) *with* (2) *are oscillatory in mean if and only if all solutions of the system* (3) *are oscillatory. Here and in the following,* $z_0 \equiv t - t_0$.

Proof Let $y(t)$ be any sample path solution of the system (1) with (2), then it follows from Lemma 3.2 that

$$y(t) = \sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \mathcal{X}_{[\xi_k,\xi_{k+1})}(t)\right)x(t).$$

Since there are a finite number of b_k such that $b_k < -1$, there exists a K such that $\prod_{i=1}^{k}(1+b_i)$ are either non-negative or non-positive for all $k \geq K$. Hence, by either monotone convergence or Tonelli's theorem [20],

$$\begin{aligned}
\mathbb{E}y(t) &= \mathbb{E}\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \mathcal{X}_{[\xi_k,\xi_{k+1})}(t)\right)x(t)\\
&= \sum_{k=0}^{+\infty}\mathbb{E}\left(\prod_{i=1}^{k}(1+b_i)\cdot \mathcal{X}_{[\xi_k,\xi_{k+1})}(t)\right)x(t)\\
&= \sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \mathbb{E}\mathcal{X}_{[\xi_k,\xi_{k+1})}(t)\right)x(t).
\end{aligned}$$

Further,

$$\begin{aligned}
\mathbb{E}\mathcal{X}_{[\xi_k,\xi_{k+1})}(t) &= \mathbb{P}(\tau_1+\tau_2+\cdots+\tau_k \leq z_0 < \tau_1+\tau_2+\cdots+\tau_{k+1})\\
&= \mathbb{P}(\tau_1+\tau_2+\cdots+\tau_k \leq z_0) - \mathbb{P}(\tau_1+\tau_2+\cdots+\tau_{k+1} \leq z_0)\\
&= \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{e^{-\lambda z_0}(\lambda z_0)^n}{n!}.
\end{aligned}$$

So,

$$\begin{aligned}
\mathbb{E}y(t) &= \sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{e^{-\lambda z_0}(\lambda z_0)^n}{n!}\right)x(t)\\
&= e^{-\lambda z_0}x(t)\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right).
\end{aligned}$$

Since

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)$$

does not change its sign for all $t \geq T$,

$$e^{-\lambda z_0}x(t)\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot \sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)$$

does not change its sign for all $t \geq T$, too. Hence, $\mathbb{E}y(t)$ has the same sign as $x(t)$ for all $t \geq T$. That is, all solutions of the system (1) with (2) are oscillatory in mean if and only if all solutions of the system (3) are oscillatory. The proof is complete. □

Theorem 3.2 *Let the condition* (H) *hold and further assume that there are a finite number of b_k such that $b_k < -1$. Then all solutions of the system* (1) *with* (2) *are oscillatory in mean if and only if all solutions of the system* (3) *are oscillatory.*

Proof According to Theorem 3.1, we only need to prove that there exists a $T \in \mathbb{R}_\tau$ such that

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right) \tag{4}$$

does not change its sign for all $t \geq T$.

In the following, we will discuss the sign of (4) in two cases respectively.

Case I. Assume that there are a finite number of b_k such that $b_k < -1$ and no $b_i = -1$. Then there exists a finite set $\hat{N} = \{k_i : k_i \in \mathbb{N}$ satisfying $k_1 < k_2 < \cdots < k_n$, where n is finite$\}$ such that $b_k > -1$ for all $k \in \mathbb{N}\backslash\hat{N}$ and $b_k < -1$ for all $k \in \hat{N}$.

(a) If n is odd,

$$\prod_{i=1}^{k}(1+b_i) < 0 \quad \text{for all } k \geq k_n.$$

So,

$$\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+\cdots+m_k}^{m_1+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!} < 0 \quad \text{for all } k \geq k_n \text{ and } t > t_0. \tag{5}$$

For any fixed k, $0 \leq k < k_n$,

$$\begin{aligned}
&\prod_{i=1}^{k_n}|1+b_i|\frac{(\lambda z_0)^{m_1+\cdots+m_{k_n}+j-1}}{(m_1+\cdots+m_{k_n}+j-1)!}\\
&\quad\geq \prod_{i=k+1}^{k_n}|1+b_i|\frac{(\lambda z_0)^{m_{k+1}+\cdots+m_{k_n}}}{(m_1+\cdots+m_{k_n}+j-1)!}\cdot\prod_{i=1}^{k}|1+b_i|\frac{(\lambda z_0)^{m_1+\cdots+m_k+j-1}}{(m_1+\cdots+m_k+j-1)!}\\
&\quad\geq \prod_{i=k+1}^{k_n}|1+b_i|\frac{(\lambda z_0)^{m_{k+1}+\cdots+m_{k_n}}}{(m_1+\cdots+m_{k_n}+m_{k+1}-1)!}\cdot\prod_{i=1}^{k}|1+b_i|\frac{(\lambda z_0)^{m_1+\cdots+m_k+j-1}}{(m_1+\cdots+m_k+j-1)!}
\end{aligned}$$

holds for $j = 1, 2, \ldots, m_{k+1}$.

In fact,

$$\prod_{i=k+1}^{k_n}|1+b_i|\frac{(\lambda z_0)^{m_{k+1}+\cdots+m_{k_n}}}{(m_1+\cdots+m_{k_n}+m_{k+1}-1)!} = \beta_k\cdot z_0^{m_{k+1}+\cdots+m_{k_n}} \geq k_n - 1$$

holds for all $t \geq t_0 + T_k$, where $T_k = \max\{1, \frac{k_n-1}{\beta_k}\}$, and

$$\beta_k = \prod_{i=k+1}^{k_n}|1+b_i|\frac{\lambda^{m_{k+1}+\cdots+m_{k_n}}}{(m_1+\cdots+m_{k_n}+m_{k+1}-1)!}$$

is a positive constant.

Thus,

$$\prod_{i=1}^{k_n}|1+b_i|\frac{(\lambda z_0)^{m_1+\cdots+m_{k_n}+j-1}}{(m_1+\cdots+m_{k_n}+j-1)!} \geq (k_n-1)\prod_{i=1}^{k}|1+b_i|\frac{(\lambda z_0)^{m_1+\cdots+m_k+j-1}}{(m_1+\cdots+m_k+j-1)!} \tag{6}$$

holds for $j=1,2,\ldots,m_{k+1}$. From (6), we obtain that

$$\left|\prod_{i=1}^{k_n}(1+b_i)\cdot\sum_{n=m_1+\cdots+m_{k_n}}^{m_1+\cdots+m_{k_n+1}-1}\frac{(\lambda z_0)^n}{n!}\right| \geq (k_n-1)\left|\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+\cdots+m_k}^{m_1+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right|. \tag{7}$$

From (5) and (7), it follows that

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)<0$$

for $t\geq t_0+T$, where $T=\max_{0\leq k<k_n}T_k$.

(b) If n is even, similar to the procedure of (a) in Case I, we can prove that

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)>0$$

holds for all $t\geq t_0+T$.

From (a), (b), we know that

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)$$

does not change its sign for all $t\geq t_0+T$.

Case II. Assume there are a finite number of b_k such that $b_k<-1$ and at least a $b_i=-1$. Then let $m=\min\{i\in\mathbb{N}:b_i=-1\}$, and let $b_k>-1$ for all $k\in\{1,2,\ldots,m-1\}\backslash\hat{N}$ and $b_i<-1$ for all $i\in\hat{N}$, where $\hat{N}=\{k_1,k_2,\ldots,k_n\}$ satisfying $k_1<k_2<\cdots<k_n$. Without loss of generality, we assume $k_n<m$. Then

$$\prod_{i=1}^{k}(1+b_i)=0 \quad \text{for all } k\geq m.$$

Thus,

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right) = \sum_{k=0}^{m-1}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right).$$

Similar to the proof of Case I, we can prove that

$$\sum_{k=0}^{m-1}\left(\prod_{i=1}^{k}(1+b_i)\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)$$

does not change its sign for all $t \geq t_0 + T$.

In summary, by Case I, Case II and Theorem 3.1, all solutions of the system (1) with (2) are oscillatory in mean if and only if all solutions of the system (3) are oscillatory. The proof is complete. □

Remark 3.1 Theorem 3.2 is a generalization of Theorem 2 in [11] since the condition (H) can degenerate to the condition (C) in [11].

Theorem 3.3 *Let the condition* (H) *hold. If there exists a constant* $\alpha > 0$ *such that*

$$\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}|1+b_i|^p\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)\leq \alpha e^{\lambda z_0}$$

holds for all $t \geq t_0$, *then the system* (1) *with* (2) *is* (*uniformly, asymptotically, uniformly asymptotically, etc.*) *p-moment stable if and only if the system* (3) *is stable correspondingly.*

Proof Let $y(t)$ be any solution of the system (1) with (2). Similar to the proof of Theorem 3.1, we obtain that

$$\mathbb{E}\left|y(t)\right|^p = e^{-\lambda z_0}\cdot\sum_{k=0}^{+\infty}\left(\prod_{i=1}^{k}|1+b_i|^p\cdot\sum_{n=m_1+m_2+\cdots+m_k}^{m_1+m_2+\cdots+m_{k+1}-1}\frac{(\lambda z_0)^n}{n!}\right)\cdot\left|x(t)\right|^p.$$

By assumption, we obtain that

$$\mathbb{E}\left|y(t)\right|^p \leq \alpha\left|x(t)\right|^p. \tag{8}$$

So, if the trivial solution of the system (3) is stable, then for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x_0| < \sqrt[p]{\delta} \quad \text{implies} \quad \left|x(t)\right| < \sqrt[p]{\frac{\epsilon}{\alpha}} \quad \text{for all } t \geq t_0.$$

From $x_0 = y_0$ and (8), we obtain that

$$|y_0|^p < \delta \quad \text{implies} \quad \mathbb{E}\left|y(t)\right|^p < \epsilon \quad \text{for all } t \geq t_0,$$

which means that the trivial solution of the system (1) with (2) is p-moment stable.

The remaining proof is similar to the proof above, so we omit it. The proof is complete. □

Remark 3.2 If $b_k(\tau_k) \equiv b_k$ is finite for all $k = 1, 2, \ldots$, when the condition (H) degenerates to the condition (C) in [11], Theorem 3.3 degenerates to Theorem 3 in [11].

4 Example

Let the condition (H) hold. Consider the second-order linear differential systems with random impulses as follows:

$$y''(t) + t^2 y'(t) + y(t) = 0, \quad t \in \mathbb{R}_+, t \neq \xi_k, \forall k \in \mathbb{N}, \tag{9}$$

and

$$\triangle y(\xi_k) = b_k y(\xi_k^-), \qquad \triangle y'(\xi_k) = b_k y'(\xi_k^-), \quad \forall k \in \mathbb{N}, \tag{10}$$

and the auxiliary differential equation

$$x''(t) + t^2 x'(t) + x(t) = 0. \tag{11}$$

By the classic Lyapunov's theory, the system (11) is stable, and if we let $m_k = 1$ for $k = 1, 2, \ldots$, $\lambda = 1$, $b_1 = 1$, $b_2 = -1$, b_k be arbitrary for $k \geq 3$, $p = 2$, $\alpha = 4$, then by Theorem 3.3, we obtain that the system (9) with (10) is stable in mean square. It can be seen that for this example, the most important impulsive effects are the first two impulsive functions b_1 and b_2.

5 Conclusion

In this paper, we first put forward a model of second-order impulsive differential systems with Erlang distribution random impulses. Then we obtain sufficient conditions for oscillation in mean and p-moment stability of this model respectively. Finally, an example is presented to illustrate the efficiency of the results obtained.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
SZ completed the proof and wrote the initial draft. JS provided the problem and gave some suggestions for amendment. SZ then finalized the manuscript. Correspondence was mainly done by JS. All authors read and approved the final manuscript.

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