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An extension of generalized Apostol-Euler polynomials

Si Chen, Yi Cai and Qiu-Ming Luo*

*Correspondence:
luomath2007@163.com
Department of Mathematics, Chongqing Normal University, Chongqing Higher Education Mega Center, Huxi Campus, Chongqing, 401331, People's Republic of China

Abstract

Recently, Tremblay, Gaboury and Fugère introduced a class of the generalized Bernoulli polynomials (see Tremblay in Appl. Math. Let. 24:1888-1893, 2011). In this paper, we introduce and investigate an extension of the generalized Apostol-Euler polynomials. We state some properties for these polynomials and obtain some relationships between the polynomials and Apostol-Bernoulli polynomials, Stirling numbers of the second kind, Jacobi polynomials, Laguerre polynomials, Hermite polynomials and generalized Bernoulli polynomials.

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1 Introduction, definitions and motivation

The generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$ and the generalized Euler polynomials $E_n^{(\alpha)}(x)$ of order $\alpha \in \mathbb{Z}$ are defined by the following generating functions (see, [1–3, Vol. 3, p.253 et seq.] and [4, Section 2.8]):

$$\left(\frac{t}{e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \bigl(|t| < 2\pi\bigr) \tag{1.1}$$

and

$$\left(\frac{2}{e^t + 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x)\frac{t^n}{n!} \quad \bigl(|t| < \pi\bigr). \tag{1.2}$$

Recently, Luo and Srivastava introduced the generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ and the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ as follows.

Definition 1.1 (Luo and Srivastava [5]) The generalized Apostol-Bernoulli polynomials $\mathfrak{B}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{N}$ are defined by means of the following generating function:

$$\left(\frac{t}{\lambda e^t - 1}\right)^\alpha \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!}$$

$$\bigl(|t| < 2\pi \text{ when } \lambda = 1; |t| < \bigl|\log(\lambda)\bigr| \text{ when } \lambda \neq 1\bigr). \tag{1.3}$$

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Clearly, the Apostol-Bernoulli polynomials $\mathfrak{B}_n(x;\lambda) := \mathfrak{B}_n^{(1)}(x;\lambda)$ and generalized Bernoulli polynomials $B_n^{\alpha}(x) = \mathfrak{B}_n^{(\alpha)}(x;1)$.

Definition 1.2 (Luo [6]) The generalized Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$ of order $\alpha \in \mathbb{C}$ are defined by means of the following generating function:

$$\left(\frac{2}{\lambda e^t + 1}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{(\alpha)}(x;\lambda)\frac{t^n}{n!} \quad \bigl(|t| < \bigl|\log(-\lambda)\bigr|\bigr). \tag{1.4}$$

Clearly, the Apostol-Euler polynomials $\mathfrak{E}_n(x;\lambda) := \mathfrak{E}_n^{(1)}(x;\lambda)$ and generalized Euler polynomials $E_n^{\alpha}(x) = \mathfrak{E}_n^{(\alpha)}(x;1)$.

Recently, Kurt [7] gave the following generalization of the Bernoulli polynomials of order α.

Definition 1.3 The generalized Bernoulli polynomials $B_n^{[m-1,\alpha]}(x)$, $m \in \mathbb{N}$, are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t^m}{e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} B_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}. \tag{1.5}$$

Clearly, if we take $m = 1$ in (1.5), then the definition (1.5) becomes the definition (1.1).

More recently, Tremblay, Gaboury and Fugère [8] further gave the following generalization of Kurt's definition (1.5) in the following form.

Definition 1.4 For arbitrary real or complex parameters λ and the natural numbers $m, \alpha \in \mathbb{N}$, the generalized Bernoulli polynomials $\mathfrak{B}_n^{[m-1,\alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{t^m}{\lambda e^t - \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}. \tag{1.6}$$

Clearly, if we take $m = 1$ in (1.6), then the definition (1.6) becomes the definition (1.3).

In view of (1.6) in Definition 1.4 and (1.5) in Definition 1.3, we give the following analogous definitions, (1.7) of which is a natural generalization for the generalized Euler polynomials $E_n^{(\alpha)}(x)$.

Definition 1.5 For complex numbers $\alpha \in \mathbb{C}$, natural numbers $m \in \mathbb{N}$, the generalized Euler polynomials $E_n^{[m-1,\alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{2^m}{e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} E_n^{[m-1,\alpha]}(x)\frac{t^n}{n!}. \tag{1.7}$$

Obviously, setting $m = 1$ in (1.7), we have $\mathfrak{E}_n^{[0,\alpha]}(x;1) = E_n^{\alpha}(x)$.

Definition 1.6 For arbitrary real or complex parameters λ, α and natural numbers $m \in \mathbb{N}$, the generalized Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x)$ are defined, in a suitable neighborhood of $t = 0$, by means of the generating function

$$\left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1} \frac{t^l}{l!}}\right)^{\alpha} \cdot e^{xt} = \sum_{n=0}^{\infty} \mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}. \tag{1.8}$$

It is easy to see that setting $m = 1$ in (1.8), we have $\mathfrak{E}_n^{[0,\alpha]}(x;\lambda) = \mathfrak{E}_n^{(\alpha)}(x;\lambda)$. From (1.8) we readily get

$$\mathfrak{E}_0^{[m-1,\alpha]}(x;\lambda) = \left(\frac{2^m}{\lambda+1}\right)^{\alpha}. \tag{1.9}$$

In the present paper, we give some properties of the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and obtain some relationships between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and other polynomials and numbers, for example, the Stirling numbers of the second kind, Jacobi polynomials, Laguerre polynomials, Hermite polynomials and generalized Bernoulli polynomials.

2 Some basic properties for the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$

In this section, we state some basic properties for the generalized Apostol-Euler polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ defined by (1.8).

Proposition 2.1 *The generalized Apostol-Euler polynomials* $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *satisfy the following relations*:

$$\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y;\lambda), \tag{2.1}$$

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda)x^{n-k}. \tag{2.2}$$

Proof By (1.8), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha+\beta]}(x+y;\lambda)\frac{t^n}{n!} &= \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\alpha+\beta} e^{(x+y)t} \\
&= \left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\alpha} e^{xt}\left(\frac{2^m}{\lambda e^t + \sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\beta} e^{yt} \\
&= \sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\beta]}(y;\lambda)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;\lambda)\mathfrak{E}_{n-k}^{[m-1,\beta]}(y;\lambda)\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we arrive at (2.1) immediately. In the same way, we may get (2.2), so it is omitted. □

Proposition 2.2 *The generalized Apostol-Euler polynomials* $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *satisfy the following relation*:

$$\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)=2\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;\lambda)\mathfrak{E}_{n-k}^{(-1)}(0;\lambda). \tag{2.3}$$

Proof By (1.4) and (1.8), we have

$$\begin{aligned}&\sum_{n=0}^{\infty}\bigl[\lambda\mathfrak{E}_n^{[m-1,\alpha]}(x+1;\lambda)+\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)\bigr]\frac{t^n}{n!}\\&\quad=\lambda\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x+1;\lambda)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}=\left(\frac{2^m}{\lambda e^t+\sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\alpha}e^{xt}\bigl(\lambda e^t+1\bigr)\\&\quad=2\left(\frac{2^m}{\lambda e^t+\sum_{l=0}^{m-1}\frac{t^l}{l!}}\right)^{\alpha}e^{xt}\left(\frac{2}{\lambda e^t+1}\right)^{(-1)}=2\sum_{n=0}^{\infty}\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)\frac{t^n}{n!}\sum_{n=0}^{\infty}\mathfrak{E}_n^{(-1)}(0;\lambda)\frac{t^n}{n!}\\&\quad=2\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\mathfrak{E}_k^{[m-1,\alpha]}(x;\lambda)\mathfrak{E}_{n-k}^{(-1)}(0;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on the both sides of the above equation, we obtain the identity (2.3) at once. □

Remark 2.3 Setting $m=1$ in (2.3), we obtain the following familiar relations for the generalized Apostol-Euler polynomials (see [6]):

$$\lambda\mathfrak{E}_n^{(\alpha)}(x+1;\lambda)+\mathfrak{E}_n^{(\alpha)}(x;\lambda)=2\mathfrak{E}_n^{(\alpha-1)}(x;\lambda). \tag{2.4}$$

3 Some generalizations of the analogues of the Luo-Srivastava addition theorem

In this section, we give a generalization of the Luo-Srivastava addition theorem and an analogue.

Theorem 3.1 *The relationship*

$$\begin{aligned}&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda)\\&\quad=\sum_{j=0}^{n}\frac{1}{n+1}\binom{n+1}{j}\left[\lambda\sum_{k=0}^{n-j+1}\binom{n-j+1}{k}\mathfrak{E}_k^{[m-1,\alpha-1]}(y;\lambda)-\mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda)\right]\mathfrak{B}_j(x;\lambda)\\&\qquad+\frac{\lambda-1}{n+1}\left(\frac{2^m}{\lambda+1}\right)^{\alpha}\mathfrak{B}_{n+1}(x;\lambda)\end{aligned} \tag{3.1}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Apostol-Bernoulli polynomials* $\mathfrak{B}_n(x;\lambda)$ *defined by* (1.3) (*with* $\alpha=1$).

Proof First of all, according to the equation [9, p.634, Eq. (29)], we substitute

$$x^n=\frac{1}{n+1}\left[\lambda\sum_{j=0}^{n+1}\binom{n+1}{j}\mathfrak{B}_j(x;\lambda)-\mathfrak{B}_{n+1}(x;\lambda)\right] \tag{3.2}$$

into the right-hand side of (2.2) and we get

$$
\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) x^{n-k} \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \left[\lambda \sum_{j=0}^{n+1-k} \binom{n+1-k}{j} \mathfrak{B}_j(x;\lambda) - \mathfrak{B}_{n+1-k}(x;\lambda) \right] \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \left[\lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) + (\lambda-1)\mathfrak{B}_{n-k+1}(x;\lambda) \right] \\
&\quad = \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) \\
&\qquad + \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{\lambda-1}{n+1-k} \mathfrak{B}_{n-k+1}(x;\lambda).
\end{aligned} \tag{3.3}
$$

The first sum in (3.3) is equal to

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) \\
&\quad = \sum_{j=0}^{n} \sum_{k=0}^{n-j} \frac{\lambda}{n+1} \binom{n+l}{n-k+1} \binom{n-k+l}{j} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \mathfrak{B}_j(x;\lambda) \\
&\quad = \sum_{j=0}^{n} \frac{\lambda}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x;\lambda) \sum_{k=0}^{n-j} \binom{n+l-j}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \\
&\quad = \sum_{j=0}^{n} \frac{\lambda}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x;\lambda) \left[\mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y+1;\lambda) - \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda) \right].
\end{aligned} \tag{3.4}
$$

Upon inverting the order of summation and using the following elementary combinational identity:

$$
\binom{m}{l}\binom{l}{n} = \binom{m}{n}\binom{m-n}{n-l} \quad (m \geq l \geq n; l, m, n \in \mathbb{N}_0), \tag{3.5}
$$

the second sum in (3.3) is equal to (noting that $\mathfrak{B}_0(x;\lambda) = 0$)

$$
\begin{aligned}
&\sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{\lambda-1}{n+1-k} \mathfrak{B}_{n-k+1}(x;\lambda) \\
&\quad = \sum_{k=0}^{n} \binom{n+1}{k} \frac{\lambda-1}{n+1} \mathfrak{B}_{n-k+1}(x;\lambda) \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \\
&\quad = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{\lambda-1}{n+1} \mathfrak{B}_{n-k+1}(x;\lambda) \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) - \frac{\lambda-1}{n+1} \mathfrak{B}_0(x;\lambda) \mathfrak{E}_{n+1-k}^{[m-1,\alpha]}(y;\lambda)
\end{aligned}
$$

$$= \sum_{j=0}^{n+1} \binom{n+1}{j} \frac{\lambda - 1}{n+1} \mathfrak{B}_j(x;\lambda) \mathfrak{E}_{n-j+1}^{[m-1,\alpha]}(y;\lambda)$$

$$= \sum_{j=0}^{n} \binom{n+1}{j} \frac{\lambda - 1}{n+1} \mathfrak{B}_j(x;\lambda) \mathfrak{E}_{n-j+1}^{[m-1,\alpha]}(y;\lambda) + \frac{\lambda - 1}{n+1} \mathfrak{B}_{n+1}(x;\lambda) \mathfrak{E}_{0}^{[m-1,\alpha]}(y;\lambda). \tag{3.6}$$

Combining (3.4) and (3.6), and noting that (2.2) with $y = 1$ and (1.9), we obtain that

$$\begin{aligned}
&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{1}{n+1-k} \lambda \sum_{j=0}^{n-k} \binom{n-k+l}{j} \mathfrak{B}_j(x;\lambda) \\
&\quad + \sum_{k=0}^{n} \binom{n}{k} \mathfrak{E}_k^{[m-1,\alpha]}(y;\lambda) \frac{\lambda - 1}{n+1-k} \mathfrak{B}_{n-k+1}(x;\lambda) \\
&= \sum_{j=0}^{n} \frac{\lambda}{n+1} \binom{n+1}{j} \big[\mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y+1;\lambda) - \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda)\big] \mathfrak{B}_j(x;\lambda) \\
&\quad + \sum_{j=0}^{n} \frac{\lambda - 1}{n+1} \binom{n+1}{j} \mathfrak{B}_j(x;\lambda) \mathfrak{E}_{n-j+1}^{[m-1,\alpha]}(y;\lambda) + \frac{\lambda - 1}{n+1} \mathfrak{B}_{n+1}(x;\lambda) \mathfrak{E}_0^{[m-1,\alpha]}(y;\lambda) \\
&= \sum_{j=0}^{n} \frac{1}{n+1} \binom{n+1}{j} \\
&\quad \times \big[\lambda \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y+1;\lambda) - \lambda \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda) + (\lambda - 1) \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda)\big] \mathfrak{B}_j(x;\lambda) \\
&\quad + \frac{\lambda - 1}{n+1} \mathfrak{E}_0^{[m-1,\alpha]}(y;\lambda) \mathfrak{B}_{n+1}(x;\lambda) \\
&= \sum_{j=0}^{n} \frac{1}{n+1} \binom{n+1}{j} \left[\lambda \sum_{k=0}^{n-j+1} \binom{n-j+1}{k} \mathfrak{E}_k^{[m-1,\alpha-1]}(y;\lambda) - \mathfrak{E}_{n+1-j}^{[m-1,\alpha]}(y;\lambda)\right] \mathfrak{B}_j(x;\lambda) \\
&\quad + \frac{\lambda - 1}{n+1} \left(\frac{2^m}{\lambda + 1}\right)^{\alpha} \mathfrak{B}_{n+1}(x;\lambda).
\end{aligned}$$

This completes the proof. □

Remark 3.2 Letting $m = 1$ in (3.1) and noting (2.4), we get the Luo-Srivastava addition theorem (see [10, p.5711, Theorem B]):

$$\begin{aligned}
&\mathfrak{E}_n^{(\alpha)}(x+y;\lambda) \\
&= \sum_{k=0}^{n} \binom{n}{k} \frac{2}{k+1} \big(\mathfrak{E}_{k+1}^{(\alpha-1)}(y;\lambda) - \mathfrak{E}_{k+1}^{(\alpha)}(y;\lambda)\big) \mathfrak{B}_{n-k}(x;\lambda) \\
&\quad + \frac{\lambda - 1}{n+1} \left(\frac{2}{\lambda + 1}\right)^{\alpha} \mathfrak{B}_{n+1}(x;\lambda).
\end{aligned} \tag{3.7}$$

4 Some relationships between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and other polynomials and numbers

In this section, by applying the same method as in the proof of (3.1) of Theorem 3.1, we give and display some relationships between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ and other polynomials and numbers, for example, the Genocchi polynomials, Stirling numbers of the second kind, Laguerre polynomials, Jacobi polynomials, Hermite polynomials, generalized Bernoulli polynomials $B_n^{[m-1]}(x)$ and generalized Bernoulli polynomials $B_n^{(\alpha)}(x)$.

Theorem 4.1 *The relationship*

$$\begin{aligned}&\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda)\\&\quad=\frac{1}{2}\sum_{k=0}^{n}\frac{1}{k+1}\left[\binom{n}{k}\mathfrak{E}_{n-k}^{[m-1,\alpha]}(y;\lambda)+\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda)\right]G_{k+1}(x)\end{aligned}\tag{4.1}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Genocchi polynomials* $G_n(x;\lambda)$ *defined by* [11, p.291, Eq. (1.1)].

Proof By substituting (see [11])

$$x^n=\frac{1}{2(n+1)}\left[\sum_{k=0}^{n}\binom{n+l}{k+1}G_{k+1}(x)+G_{n+1}(x)\right]\tag{4.2}$$

into the right-hand side of (2.2), we can get (4.1). □

Theorem 4.2 *The relationship*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda)=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=k}^{n}\binom{n}{j}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda)S(j,k)\tag{4.3}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Stirling numbers* $S(n,k)$ *of the second kind defined by* [12, p.58, Eq. (15)].

Proof By substituting (see [12, p.58, Eq. (14)])

$$x^n=\sum_{k=0}^{n}\binom{x}{k}k!S(n,k)\tag{4.4}$$

into the right-hand side of (2.2), we can get (4.3). □

Theorem 4.3 *The relationship*

$$\mathfrak{E}_n^{[m-1,\mu]}(x+y;\lambda)=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{n}{j}\binom{j+\alpha}{j-k}\mathfrak{E}_{n-j}^{[m-1,\mu]}(y;\lambda)L_k^{(\alpha)}(x)\tag{4.5}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Laguerre polynomials defined by* [12, p.55, Eq. (71)].

Proof By substituting (see [13, p.207, Eq. (2)])

$$x^n=n!\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}L_k^{(\alpha)}(x)\tag{4.6}$$

into the right-hand side of (2.2), we can obtain (4.5). □

Theorem 4.4 *The relationship*

$$\begin{aligned}&\mathfrak{E}_n^{[m-1,\mu]}(x+y;\lambda)\\&\quad=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!\binom{j+\alpha}{j-k}\binom{n}{j}\frac{\alpha+\beta+2k+1}{(\alpha+\beta+k+1)_{j+1}}\mathfrak{E}_{n-j}^{[m-1,\mu]}(y;\lambda)P_k^{(\alpha,\beta)}(1-2x)\end{aligned}\tag{4.7}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Jacobi polynomials defined by* [12, p.49, Eq. (35)].

Proof By substituting (see [13, p.262, Eq. (2)])

$$x^n=n!\sum_{k=0}^{n}(-1)^k\binom{n+\alpha}{n-k}\frac{\alpha+\beta+2k+1}{(\alpha+\beta+k+1)_{n+1}}P_k^{(\alpha,\beta)}(1-2x)\tag{4.8}$$

into the right-hand side of (2.2), we can get (4.7). □

Theorem 4.5 *The relationship*

$$\mathfrak{E}_n^{[m-1,\mu]}(x+y;\lambda)=\sum_{k=0}^{[n/2]}\sum_{j=2k}^{n}2^{(-j)}\binom{n}{j}\binom{j}{2k}\frac{(2k)!}{k!}\mathfrak{E}_{n-j}^{[m-1,\mu]}(y;\lambda)H_{j-2k}(x)\tag{4.9}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the Hermite polynomials defined by* [12, p.55, Eq. (70)].

Proof By substituting (see [13, p.194, Eq. (4)])

$$(2x)^n=\sum_{k=0}^{[n/2]}\binom{n}{2k}\frac{(2k)!}{k!}H_{n-2k}(x)\tag{4.10}$$

into the right-hand side of (2.2), we can get (4.9). □

Theorem 4.6 *The relationship*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda)=\sum_{k=0}^{n}\sum_{j=k}^{n}\frac{k!}{(j+m)!}\binom{n}{j}\binom{j}{k}\mathfrak{E}_{n-j}^{[m-1,\alpha]}(y;\lambda)B_{j-k}^{[m-1]}(x)\tag{4.11}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the generalized Bernoulli polynomials* $B_n^{[m-1]}(x)$ *defined by* (1.5) (*with* $\alpha=1$).

Proof According to the equation [14, p.158, (2.6)], we substitute

$$x^n=\sum_{k=0}^{n}\binom{n}{k}\frac{k!}{(k+m)!}B_{n-k}^{[m-1]}(x)\quad(m\in\mathbb{N})\tag{4.12}$$

into the right-hand side of(2.2) and we can obtain (4.11). □

Theorem 4.7 *The relationship*

$$\mathfrak{E}_n^{[m-1,\alpha]}(x+y;\lambda) = \sum_{k=0}^{n}\sum_{l=k}^{n}\binom{n}{l}\binom{l}{k}\binom{k+j}{j}^{-1}\mathfrak{E}_{n-l}^{[m-1,\alpha]}(y;\lambda)S(k+j,j)B_{l-k}^{(j)}(x) \tag{4.13}$$

holds between the polynomials $\mathfrak{E}_n^{[m-1,\alpha]}(x;\lambda)$ *and the generalized Bernoulli polynomials defined by* (1.1).

Proof By substituting (see [15, p.1329, (2.15)])

$$x^n = \sum_{l=0}^{n}\binom{n}{l}\binom{l+j}{j}^{-1}S(l+j,j)B_{n-l}^{(j)}(x) \quad (j \in \mathbb{N}_0) \tag{4.14}$$

into the right-hand side of (2.2), we can get (4.13). □

Remark 4.8 If we set $m = 1$ in (4.1), (4.3), (4.5), (4.7), (4.9) and (4.11), then we obtain the corresponding results of the Apostol-Euler polynomials $\mathfrak{E}_n^{(\alpha)}(x;\lambda)$.

If we set $m = 1$, $\lambda = 1$ in (4.1), (4.3), (4.5), (4.7), (4.9) and (4.11), then we obtain the corresponding results of the generalized Euler polynomials $E_n^{(\alpha)}(x)$.

5 A remark for the Apostol-type polynomials

In [16, pp.939-940], Srivastava, Kurt and Simsek gave the following remarks about the Apostol-type polynomials:

'… It should be reiterated in passing that the investigations of the corresponding generalizations of the Apostol-Euler polynomials, which are associated with any admissible (real or complex) order f, are not at all affected by the observations made here.

For the sake of the interested readers, we list below the following additional sequels, some relevant parts of which are believed to be similarly affected by the works of Luo and Srivastava (see [5, 7–10, 14, 15, 17–34]).

In each of the main results in most of the aforecited works, which involve the generalized Apostol-Bernoulli polynomials and/or the generalized Apostol-Genocchi polynomials, only the nonnegative integer orders of these polynomials are considered and used correctly. Finally, it should be mentioned here that a suitable research-cum-expository article which would deal in detail, both analytically and rigorously, with each and every aspect of this situation is under preparation.'

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally in writing this paper, and read and approved the final manuscript.

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References

1. Erdélyi, A, Magnus, W, Oberhettinger, F, Tricomi, F: Higher Transcendental Functions, vol. 1 (1953)
2. Erdélyi, A, Magnus, W, Oberhettinger, F, Tricomi, F: Higher Transcendental Functions, vol. 2 (1953)
3. Erdélyi, A, Magnus, W, Oberhettinger, F, Tricomi, F: Higher Transcendental Functions, vol. 3 (1953)
4. Luck, Y: The Special Functions and Their Approximations (1969)
5. Luo, Q-M, Srivastava, HM: Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials. J. Math. Anal. Appl. **308**, 290-302 (2005)
6. Luo, Q-M: Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions. Taiwan. J. Math. **10**(4), 917-925 (2006)
7. Kurt, B: A further generalization of the Bernoulli polynomials and on the 2D-Bernoulli polynomials $B_n^2(x, y)$. Appl. Math. **233**, 3005-3017 (2010)
8. Tremblay, R, Gaboury, S, Fugère, B-J: A new class of generalized Apostol-Bernoulli polynomials and some analogues of the Srivastava-Pintér addition theorem. Appl. Math. Lett. **24**, 1888-1893 (2011)
9. Luo, Q-M, Srivastava, HM: Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials. Comput. Math. Appl. **51**, 631-642 (2006)
10. Luo, Q-M, Srivastava, HM: Some generalizations of the Apostol-Genocchi polynomials and the Stirling numbers of the second kind. Appl. Math. Comput. **217**, 5702-5728 (2011)
11. Luo, Q-M: Extension for the Genocchi polynomials and its Fourier expansions and integral representations. Osaka J. Math. **48**, 291-309 (2011)
12. Srivastava, HM, Choi, J: Series Associated with the Zeta and Related Functions. Kluwer Academic, Dordrecht (2001)
13. Rainville, ED: Special Functions. Macmillan Company, New York (1960); Reprinted by Chelsea publishing Company, Bronx (1971)
14. Natalini, P, Bernardini, A: A generalization of the Bernoulli polynomials. J. Appl. Math. **3**, 155-163 (2003)
15. Wang, W, Jia, C, Wang, T: Some results on the Apostol-Bernoulli and Apostol-Euler polynomials. Comput. Math. Appl. **55**, 1322-1332 (2008)
16. Srivastava, HM, Kurt, B, Simsek, Y: Corrigendum to some families of Genocchi type polynomials and their interpolation functions. Integral Transforms Spec. Funct. **23**, 939-940 (2009)
17. Choi, J, Jang, DS, Srivastava, HM: A generalization of the Hurwitz-Lerch Zeta function. Integral Transforms Spec. Funct. **19**, 65-79 (2008)
18. Garg, M, Jain, K, Srivastava, HM: Some relationships between the generalized Apostol-Bernoulli polynomials and Hurwitz-Lerch Zeta functions. Integral Transforms Spec. Funct. **17**, 803-815 (2006)
19. Lu, D-Q, Srivastava, HM: Some series identities involving generalized Apostol type and related polynomials. Comput. Math. Appl. **62**, 3591-3602 (2011)
20. Luo, Q-M: The multiplication formulas for the Apostol type polynomials and the multiple alternating sum. Math. Notes **91**, 46-57 (2012) (Published in Russian in Mat. Zametki **91**, 54-66 (2012))
21. Luo, Q-M: An explicit relationship between the generalized Apostol-Bernoulli and Apostol-Euler polynomials associated with λ-Stirling numbers of the second kind. Houst. J. Math. **36**, 1159-1171 (2010)
22. Luo, Q-M: The multiplication formulas for the Apostol-Bernoulli and Apostol-Euler polynomials of higher order. Integral Transforms Spec. Funct. **20**, 377-391 (2009)
23. Luo, Q-M: Some formulas for Apostol-Euler polynomials associated with Hurwitz zeta function at rational arguments. Appl. Anal. Discrete Math. **3**, 336-346 (2009)
24. Luo, Q-M: q-extensions for the Apostol-Genocchi polynomials. Gen. Math. **17**(2), 113-125 (2009)
25. Özarslan, MA: Unified Apostol-Bernoulli, Euler and Genocchi polynomials. Comput. Math. Appl. **62**, 2452-2462 (2011)
26. Dere, R, Simsek, Y: Applications of umbral algebra to some special polynomials. Adv. Stud. Contemp. Math. **22**, 433-438 (2012)
27. Kurt, B, Simsek, Y: Frobenious-Euler type polynomials related to Hermite-Bernoulli polynomials, analysis and applied math. AIP Conf. Proc. **1389**, 385-388 (2011)
28. Ozden, H, Simsek, Y, Srivastava, HM: A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. Comput. Math. Appl. **60**, 2779-2787 (2010)
29. Srivastava, HM, Choi, J: Zeta and q-Zeta Functions and Associated Series and Integrals. Elsevier, Amsterdam (2012)
30. Luo, Q-M: Fourier expansions and integral representations for the Apostol-Bernoulli and Apostol-Euler polynomials. Math. Comput. **78**, 2193-2208 (2009)
31. Luo, Q-M: An explicit formula for the Euler polynomials of higher order. Appl. Math. Inform. Sci. **3**(1), 53-58 (2009)
32. Luo, Q-M: Some results for the q-Bernoulli and q-Euler polynomials. J. Math. Anal. Appl. **363**, 7-18 (2010)
33. Luo, Q-M: q-analogues of some results for the Apostol-Euler polynomials. Adv. Stud. Contemp. Math. **20**(1), 103-113 (2010)
34. Luo, Q-M, Zhou, Y: Extension of the Genocchi polynomials and its q-analogue. Util. Math. **85**, 281-297 (2011)

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