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Chain components with C^1-stably orbital shadowing

Manseob Lee*

*Correspondence:
lmsds@mokwon.ac.kr
Department of Mathematics,
Mokwon University, Daejeon,
302-729, Korea

Abstract

Let $f: M \to M$ be a diffeomorphism on a C^∞ n-dimensional manifold. Let $C_f(p)$ be the chain component of f associated to a hyperbolic periodic point p. In this paper, we show that (i) if f has the C^1-stably orbitally shadowing property on the chain recurrent set $\mathcal{R}(f)$, then f satisfies both Axiom A and no-cycle condition, and (ii) if f has the C^1-stably orbitally shadowing property on $C_f(p)$, then $C_f(p)$ is hyperbolic.

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1 Introduction

Let M be a closed C^∞ n-dimensional manifold, and let Diff(M) be the space of diffeomorphisms of M endowed with the C^1-topology. Denote by d the distance on M induced from a Riemannian metric $\|\cdot\|$ on the tangent bundle TM. Let $f \in \mathrm{Diff}(M)$. For $\delta > 0$, a sequence of points $\{x_i\}_{i=a}^{b}$ ($-\infty \le a < b \le \infty$) in M is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for all $a \le i \le b-1$. Let $\Lambda \subset M$ be a closed f-invariant set. We say that f has the *shadowing property* on Λ (or Λ is *orbitally shadowable*) if for every $\epsilon > 0$, there is $\delta > 0$ such that for any δ-pseudo-orbit $\{x_i\}_{i=a}^{b} \subset \Lambda$ of f ($-\infty \le a < b \le \infty$), there is a point $y \in M$ such that $d(f^i(y), x_i) < \epsilon$ for all $a \le i \le b-1$. It is easy to see that f has the shadowing property on Λ if and only if f^n has the shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$. The notion of pseudo-orbits often appears in several methods of the modern theory of a dynamical system. Moreover, the shadowing property plays an important role in the investigation of stability theory and ergodic theory. Actually, in [1, 2], the authors showed that every f satisfying both Axiom A and the strong transversality condition has the shadowing property. Since such a system is structurally stable, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that every $g \in \mathcal{U}(f)$ has the shadowing property because f satisfies both Axiom A and the strong transversality condition. And in [3], Sakai proved that if there is a C^1-neighborhood $\mathcal{U}(f)$ of f, for any $g \in \mathcal{U}(f)$, g has the shadowing property, then f satisfies both Axiom A and the strong transversality condition.

For each $x \in M$, let $\mathcal{O}_f(x)$ be the orbit of f through x; that is, $\mathcal{O}_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$. We say that f has the *orbital shadowing property* on Λ (or Λ is *orbitally shadowable*) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, we can find a point $y \in M$ such that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \quad \text{and} \quad \xi \subset B_\epsilon\big(\mathcal{O}_f(y)\big),$$

where $B_\epsilon(A)$ denotes the ϵ-neighborhood of a set $A \subset M$. f is said to have the *weak shadowing property* on Λ (or Λ is *weakly shadowable*) if for any $\epsilon > 0$, there exists $\delta > 0$ such that for any δ-pseudo-orbit $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \Lambda$, there is a point $y \in M$ such that $\xi \subset B_\epsilon(\mathcal{O}_f(y))$.

The orbital shadowing property is a weak version of the shadowing property: the difference is that we do not require a point x_i of a pseudo-orbit ξ and the point $f^i(y)$ of an exact orbit $\mathcal{O}_f(y)$ to be close 'at any time moment'; instead, the sets of the points of ξ and $\mathcal{O}_f(y)$ are required to be close. The weak showing property is a slightly weak version of the orbital shadowing property. The difference is that a set of points of a 'sufficiently precise' pseudo-orbit ξ is required to be contained in a small neighborhood of some exact orbit $\mathcal{O}_f(y)$. We say that Λ is *locally maximal* if there is a compact neighborhood U of Λ such that

$$\bigcap_{n\in\mathbb{Z}} f^n(U) = \Lambda.$$

It is easy to see that f has the orbital shadowing property on Λ if and only if f^n has the orbital shadowing property on Λ for $n \in \mathbb{Z} \setminus \{0\}$.

Now we introduce the notion of the C^1-stably orbitally shadowing property on Λ. We say that f has the *C^1-stably orbitally shadowing property* on Λ if there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood U of Λ such that (i) $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ (locally maximal), (ii) for any $g \in \mathcal{U}(f)$, g has the orbital shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$, where $\Lambda_g(U)$ is the *continuation* of Λ. We say that f has the C^1-stably orbitally shadowing property if $\Lambda = M$ in the above definition. It is known that if any structurally stable diffeomorphism f has a C^1-neighborhood $\mathcal{U}(f)$ of f, then for any $g \in \mathcal{U}(f)$, g has the shadowing property, hence g has the orbital shadowing property. In [4], the authors showed that if f has the C^1-stably orbitally shadowing property, then f satisfies both Axiom A and the strong transversality condition. Thus we can restate the above facts as follows.

Theorem 1.1 [4] *Let $f : M \to M$ be a diffeomorphism. f has the C^1-stably orbitally shadowing property if and only if f is structurally stable.*

For any $x, y \in M$, we write $x \rightsquigarrow y$ if for any $\delta > 0$, there is a δ-pseudo orbit $\{x_i\}_{i=a_\delta}^{b_\delta}$ $(a_\delta < b_\delta)$ of f such that $x_{a_\delta} = x$ and $x_{b_\delta} = y$. The set of points $\{x \in M : x \rightsquigarrow x\}$ is called the *chain recurrent set* of f and is denoted by $\mathcal{R}(f)$. It is well known that $\mathcal{R}(f)$ is a closed and f-invariant set. If we denote the set of periodic points of f by $P(f)$, then $P(f) \subset \Omega(f) \subset \mathcal{R}(f)$. Here $\Omega(f)$ is the non-wandering set of f. We write $x \leftrightsquigarrow y$ if $x \rightsquigarrow y$ and $y \rightsquigarrow x$. The relation $\leftrightsquigarrow$ induces on $\mathcal{R}(f)$ an equivalence relation, whose classes are called *chain components* of f. Note that by [5], the map $f \mapsto \mathcal{R}(f)$ is upper semi-continuous. From the fact, we will show the following result.

Theorem 1.2 *Let $\mathcal{R}(f)$ be the chain recurrent set of f. f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$ if and only if f satisfies Axiom* A *and the no-cycle condition.*

Let f be an Axiom A diffeomorphism. Then, it is well known that $\Omega(f) = \mathcal{R}(f)$ if and only if f satisfies the no-cycle condition. Hence, f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$ and is characterized as the Ω-stability of the system by Theorem 1.2.

We say that Λ is *hyperbolic* for f if the tangent bundle $T_\Lambda M$ has a Df-invariant splitting $E^s \oplus E^u$ and there exist constants $C > 0$ and $0 < \lambda < 1$ such that

$$\left\| D_x f^n|_{E^s_x} \right\| \le C\lambda^n \quad \text{and} \quad \left\| D_x f^{-n}|_{E^u_x} \right\| \le C\lambda^n$$

for all $x \in \Lambda$ and $n \ge 0$. If $\Lambda = M$, then f is *Anosov*.

If $p \in P(f)$ is a hyperbolic saddle with the period $\pi(p) > 0$, then there is no eigenvalues of $D_p f^{\pi(p)}$ with modulus equal to 1, at least one of them is greater than 1, at least one of them is smaller than 1. Note that there is a C^1-neighborhood $\mathcal{U}(f)$ and a neighborhood U of p such that for all $g \in \mathcal{U}(f)$, there is a unique hyperbolic periodic point $p_g \in U$ of g with the same period as p and $\mathrm{index}(p_g) = \mathrm{index}(p)$. Here $\mathrm{index}(p) = \dim E^s_p$, and the point p_g is called the *continuation* of p. The stable manifold $W^s(p)$ and the unstable manifold $W^u(p)$ are defined as follows. It is well known that if p is a hyperbolic periodic point of f with period k, then the sets

$$W^s(p) = \{x \in M : f^{kn}(x) \to p \text{ as } n \to \infty\} \quad \text{and}$$
$$W^u(p) = \{x \in M : f^{-kn}(x) \to p \text{ as } n \to \infty\}$$

are C^1-injectively immersed submanifolds of M.

Denote by $C_f(p)$ the chain component of f containing p. If p is a sink or source periodic point, then $C_f(p)$ is a periodic orbit itself. Therefore, in this paper, we may assume that all periodic points are the saddle type. Let $q \in P(f)$. We say that p and q are *homoclinic related*, and write $p \sim q$ if $W^s(p) \pitchfork W^u(q) \neq \emptyset$, and $W^u(p) \pitchfork W^s(q) \neq \emptyset$. It is clear that if $p \sim q$ then $\mathrm{index}(p) = \mathrm{index}(q)$.

Denote by $H_f(p)$ the homoclinic points associated with p, that is, $H_f(p) = \{x \in M : x \in W^s(p) \cap W^u(q)\}$, and let $H_f^T(p)$ be the transverse homoclinic points associated with p, that is, $H_f^T(p) = \{x \in M : x \in W^s(p) \pitchfork W^u(p)\}$. Obviously, $H_f^T(p)$ and $H_f(p)$ are closed f-invariant sets, and it is clear that $H_f^T(p) \subset H_f(p)$. Note that by Smale's transverse homoclinic point theorem, $H_f^T(p)$ coincides with the closure of the set of all $q \in P(f)$ such that $p \sim q$. It is known that $H_f^T(p)$ is a transitive set, and if $H_f^T(p)$ is not hyperbolic, then it may contain periodic points having different indices. Note that $H_f^T(p) \subset H_f(p) \subset C_f(p)$ (see [6, examples and counter examples]). For the chain component $C_f(p)$, Lee, Moriyasu and Sakai [7] showed that f has the C^1-stably shadowing property on $C_f(p)$ if and only if the chain component $C_f(p)$ is a hyperbolic homoclinic class of p. From the fact, we consider the orbital shadowing property and the chain component. The following result is the main theorem in this paper.

Theorem 1.3 *Let p be a hyperbolic periodic point of f, and let $C_f(p)$ be the chain component of f containing p. f has the C^1-stably orbitally shadowing property on $C_f(p)$ if and only if $C_f(p)$ is the hyperbolic homoclinic class of p.*

If $C_f(p)$ is hyperbolic, then it is locally maximal, that is, there is a compact neighborhood U of $C_f(p)$ such that $\bigcap_{n\in\mathbb{Z}} f^n(U) = C_f(p)$. By the local stability of a hyperbolic set, there is a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic. Then we know that g has the shadowing property on Λ_g, and so g has the orbital shadowing property on $\Lambda_g(U)$. Thus we get the 'if' part. Thus, we need to show that f has the C^1-stably orbitally shadowing property on $C_f(p)$, then $C_f(p) = H_f^T(p)$ is hyperbolic.

2 Proof of Theorem 1.2

Let M be as before, and for $\epsilon > 0$, we denote by $B_\epsilon(A)$ the closed ϵ-ball $\{x \in M : d(x, A) \leq \epsilon\}$ of a subset A of M.

Proposition 2.1 *Let $f \in \mathrm{Diff}(M)$. Suppose that f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$. Then f satisfies Axiom* A *and the no-cycle condition.*

Denote by $\mathcal{H}(M)$ the set of homeomorphisms of M. For the proof of the following lemma, see [5].

Lemma 2.2 *Let $f \in \mathcal{H}(M)$, and let $\mathcal{R}(f)$ be the chain recurrent set of f. For any $\epsilon > 0$, there is $\delta > 0$ such that if $\rho_0(f, g) < \delta$ ($g \in \mathcal{H}(M)$), then $\mathcal{R}(g) \subset B_\epsilon(\mathcal{R}(f))$.*

The following Franks' lemma will play essential roles in our proofs.

Lemma 2.3 [8] *Let $\mathcal{U}(f)$ be any given C^1-neighborhood of f. Then there exist $\epsilon > 0$ and a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that for given $g \in \mathcal{U}_0(f)$, a finite set $\{x_1, x_2, \ldots, x_N\}$, a neighborhood U of $\{x_1, x_2, \ldots, x_N\}$ and linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \epsilon$ for all $1 \leq i \leq N$, there exists $\widehat{g} \in \mathcal{U}(f)$ such that $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \ldots, x_N\} \cup (M \setminus U)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.*

Denote by $\mathfrak{F}(M)$ the set of $f \in \mathrm{Diff}(M)$ such that there is a C^1-neighborhood $\mathcal{U}(f)$ of f with the property that every $p \in P(g)$ ($g \in \mathcal{U}(f)$) is hyperbolic. It is proved by Hayashi [9] that $f \in \mathfrak{F}(M)$ if and only if f satisfies both Axiom A and the no-cycle condition. Let $\Lambda \subset M$ be an invariant submanifold of f. We say that Λ is *normally hyperbolic* if there is a splitting $T_\Lambda M = T\Lambda \oplus N^s \oplus N^u$ such that

(a) the splitting depends continuously on $x \in \Lambda$,

(b) $D_x f(N_x^\sigma) = N_{f(x)}^\sigma$ ($\sigma = s, u$) for all $x \in \Lambda$,

(c) there are constants $C > 0$ and $\lambda \in (0, 1)$ such that for every triple of unit vectors $v \in T_x\Lambda$, $v^s \in N_x^s$ and $v^u \in N_x^u$ ($x \in \Lambda$), we have

$$\frac{\|D_x f^n(v^s)\|}{\|D_x f^n(v)\|} \leq C\lambda^n \quad \text{and} \quad \frac{\|D_x f^n(v^u)\|}{\|D_x f^n(v)\|} \geq C^{-1}\lambda^{-n}$$

for all $n \geq 0$.

Proof of Proposition 2.1 First we suppose that f satisfies both Axiom A and the no-cycle condition. Then $\mathcal{R}(f) = \Omega(f) = \overline{P(f)}$ is hyperbolic, and so $\mathcal{R}(f)$ is locally maximal. By the stability of locally maximal hyperbolic sets, we can choose a compact neighborhood U of $\mathcal{R}(f)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that $\mathcal{R}(f) = \bigcap_{n\in\mathbb{Z}} f^n(U)$, and for any $g \in \mathcal{U}(f)$, $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic for g. Since $\mathcal{R}(f)$ is hyperbolic for f, f has the shadowing property on $\mathcal{R}(f)$, and so f has the orbital shadowing property on $\mathcal{R}(f)$. Thus, we know that f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$. Finally, we show that if f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$, then f satisfies both Axiom A and the no-cycle condition. From the above facts, to complete the proof of the theorem, it is enough to show that if f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$, then $f \in \mathfrak{F}(M)$.

Suppose that f has the C^1-stably orbitally shadowing property on $\mathcal{R}(f)$. Then there are a compact neighborhood U of $\mathcal{R}(f)$ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the orbital shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$, where $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is the continuation of Λ. Choose $\epsilon > 0$ satisfying $B_\epsilon(\mathcal{R}(f)) \subset U$. By Lemma 2.2, there is $\delta > 0$ such that if $\rho_1(f,g) < \delta$ for $g \in \mathcal{U}(f)$, then

$$\mathcal{R}(g) \subset B_\epsilon\big(\mathcal{R}(f)\big) \subset U, \tag{1}$$

where ρ_1 is the usual C^1-metric on Diff(M). Put $\mathcal{U}_0(f) = \{g \in \mathcal{U}(f) : \rho_1(f,g) < \delta\}$. Then for each $g \in \mathcal{U}_0(f)$, $\mathcal{R}(g) \subset U$ and so $\mathcal{R}(g) \subset g^n(U)$ for all $n \in \mathbb{Z}$. This means that $\mathcal{R}(g) \subset \bigcap_{n\in\mathbb{Z}} g^n(U) = \Lambda_g(U)$ for $g \in \mathcal{U}_0(f)$. Since g has the orbital shadowing property on $\Lambda_g(U)$, g has the orbital shadowing property on $\mathcal{R}(g)$.

Let $\epsilon > 0$, and let $\tilde{U}(f) \subset \mathcal{U}_0(f)$ be a C^1-neighborhood of f which is given by Lemma 2.3 with respect to $\mathcal{U}_0(f)$. Let $p \in P(f)$, and let k be the period of p. Let

$$T_pM = E^c(p) \oplus E^s(p) \oplus E^u(p),$$

where $E^\sigma(p)$, $\sigma = c,s,u$, are D_pf^k-invariant subspaces corresponding to eigenvalues λ of D_pf^k for $|\lambda| = 1$, $|\lambda| < 1$ and $|\lambda| > 1$, respectively. It is sufficient to show that each $p \in P(f)$ is hyperbolic. Suppose this is not true. Then there exist eigenvalues $\lambda_1,\dots,\lambda_c$ of D_pf^k with $|\lambda_j| = 1$ or $\lambda_j = e^{i\theta_j}$, $j = 1,2,\dots,c$. Let $\mathcal{U}_{\epsilon_0}(f) \subset \tilde{U}(f)$ be the C^1 ϵ_0-ball of f. Set $C = \sup_{x\in M}\{\|D_xf\|\}$. For $0 < \epsilon_1 < \epsilon_0$, we can obtain a linear automorphism $\mathcal{O} : T_pM \to T_pM$ such that

(a.i) $\|\mathcal{O} - id\| < \frac{\epsilon_1}{C}$,

(a.ii) $\mathcal{O}$ keeps E^σ invariant, where $\sigma = c,s,u$,

(a.iii) all eigenvalues of $\mathcal{O} \circ D_pf^k$, say $\tilde{\lambda}_j$, $j = 1,2,\dots,c$, are roots of unity.

Let F be a finite set $\{p, f(p),\dots,f^{k-1}(p)\}$. Define

$$G_j = \begin{cases} D_{f^j(p)}f, & j = 0,1,\dots,k-2, \\ \mathcal{O} \circ D_{f^{k-1}(p)}f, & j = k-1. \end{cases}$$

Observe that $\|G_{k-1} - D_{f^{k-1}(p)}f\| \le \|\mathcal{O} - id\|\|D_{f^{k-1}(p)}f\| < \epsilon_0$. Thus $\|G_j - D_{f^j(p)}f\| < \epsilon_0$ for all $j = 0,1,\dots,k-1$. By Lemma 2.3, we can find a diffeomorphism $g \in \mathcal{U}_{\epsilon_0}(f)$ and $\delta_0 > 0$ such that

(b.i) $B_{4\delta_0}(f^j(p)) \subset U$,

(b.ii) $B_{4\delta_0}(f^i(p)) \cap B_{4\delta_0}(p) = \emptyset$, $0 \le i,j \le k-1$, $i \ne j$,

(b.iii) $g = f$ on $F \cup (M - \bigcup_{j=0}^{k-1} B_{4\delta_0}(f^j(p)))$, and

(b.iv) $g = \exp_{f^{j+1}(p)} \circ G_j \circ \exp^{-1}_{f^j(p)}$ on $B_{\delta_0}(f^j(p))$, $0 \le j \le k-1$.

Define

$$G = \mathcal{O} \circ D_pf^k = \prod_{j=0}^{k-1} G_j.$$

Then by (a.iii) we can find $m > 0$ such that $G^m|_{E^c(p)} = id|_{E^c(p)}$. Choose a small δ_1 satisfying $0 < 4\delta_1 < \delta_0$ such that

$$G^{mk}\big(T_pM(4\delta_1)\big) \subset T_pM(\delta_0),$$

where $T_pM(\delta_1) = \{v \in T_pM \mid \|v\| \le \delta_1\}$. Then by (b.iv) we have

$$(g^k)^m = g^{mk} = \exp_p \circ G^m \circ \exp_p^{-1}$$

on $\exp_p(T_pM(4\delta_1))$. Since $\exp_p(T_pM(4\delta_1)) \subset B_{\delta_0}(f^j(p))$, we get

$$\exp_p(T_pM(4\delta_1)) \subset \bigcap_{n\in\mathbb{Z}} g^n(B_{\delta_0}(f^j(p))) \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U).$$

We write

$$T_pM(\delta_1) = E^c(p,\delta_1) \oplus E^s(p,\delta_1) \oplus E^u(p,\delta_1),$$

where $E^\sigma(p,\delta_1) = E^\sigma(p) \cap T_pM(\delta_1)$, $\sigma = c, s, u$. Then $\exp_p(E^c(p,4\delta_1))$ is $(g^k)^m$-invariant. If the eigenvalue is real, then $\dim(\exp_p(E^c(p,4\delta_1))) = 1$ and $\exp_p(E^c(p,4\delta_1))$ is an arc $\mathfrak{I}_\mathfrak{p}$ centered at p; and if the eigenvalue is complex, then $\dim(\exp_p(E^c(p,4\delta_1))) = 2$ and $\exp_p(E^c(p,4\delta_1))$ is a disk $\mathfrak{D}_\mathfrak{p}$ centered at p. We know that $\mathfrak{I}_\mathfrak{p} \subset \mathcal{R}(g)$ and $\mathfrak{D}_\mathfrak{p} \subset \mathcal{R}(g)$. By (1), we get

$$\mathfrak{I}_\mathfrak{p} \subset \mathcal{R}(g) \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$$

and

$$\mathfrak{D}_\mathfrak{p} \subset \mathcal{R}(g) \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U).$$

Since g has the orbital shadowing property on $\mathcal{R}(g)$, g must have the orbital shadowing property on $\mathfrak{I}_\mathfrak{p}$ and $\mathfrak{D}_\mathfrak{p}$. Since $\mathfrak{I}_\mathfrak{p}$ and $\mathfrak{D}_\mathfrak{p}$ are normally hyperbolic, we can see that a shadowing point is in $\mathfrak{I}_\mathfrak{p}$ and $\mathfrak{D}_\mathfrak{p}$. Observe that $(g^k)^m = id$ on $\exp_p(E^c(p,4\delta_1))$. By our construction, $(g^k)^m$ is the identity on the arc $\mathfrak{I}_\mathfrak{p}$ as well as on the disk $\mathfrak{D}_\mathfrak{p}$. Since the identity map does not have the orbital shadowing property. Indeed, let $\xi = \{x_i\}_{i\in\mathbb{Z}} \subset \mathfrak{I}_\mathfrak{p}$ be a δ-pseudo orbit. Then, by the orbital shadowing property and normally hyperbolicity, there is a point $y \in \mathfrak{I}_\mathfrak{p}$ such that $\mathcal{O}_g(y) \subset B_\epsilon(\mathcal{O}_g(y))$ and $\xi \subset B_\epsilon(\mathcal{O}_g(y))$. But since g is the identity map on $\mathfrak{I}_\mathfrak{p}$ for all $n \in \mathbb{Z}$, $g^n(y) = y$. Thus we have $\xi \not\subset B_\epsilon(\mathcal{O}_g(y))$. This is a contradiction. This completes the proof of Proposition 2.1. □

3 Proof of Theorem 1.3

Let $f \in \mathrm{Diff}(M)$, and let $p \in P(f)$ be a hyperbolic saddle with period $\pi(p) > 0$. Then there are the local stable manifold $W^s_\epsilon(p)$ and the unstable manifold $W^u_\epsilon(p)$ of p for some $\epsilon = \epsilon(p) > 0$. It is easily seen that if $d(f^n(x), f^n(p)) \le \epsilon$, for all $n \ge 0$, then $x \in W^s_\epsilon(p)$, and if $d(f^n(x), f^n(p)) \le \epsilon$ for all $n \le 0$, then $x \in W^u_\epsilon(p)$. Note that the local stable manifold $W^s_\epsilon(p) \subset W^s(p)$ (resp. the local unstable manifold $W^u_\epsilon(p) \subset W^u(p)$).

Lemma 3.1 *Let p be a hyperbolic periodic point of f, and let $C_f(p)$ be the chain component of f containing p. If f has the orbital shadowing property on $C_f(p)$, then $C_f(p) = H_f(p)$.*

Proof We know that $H_f(p) \subset C_f(p)$. We now show that $C_f(p) \subset H_f(p)$. Let $x \in C_f(p)$. Then $x \rightsquigarrow p$ and $p \rightsquigarrow x$. Thus, for any $\eta > 0$, there is a periodic η-pseudo orbit $\{x_i\}_{i=-n_1}^{n_2}$ of f such that $x_{-n_1} = p$, $x_0 = x$ and $x_{n_2} = p$ for some $n_1 = n_1(\eta)$ and $n_2 = n_2(\eta) > 0$. By [10, Proposition 1.6], $\{x_i\}_{i=-n_1}^{n_2} \subset C_f(p)$. To simplify, we may assume that $f(p) = p$. Then we extend the pseudo orbit as follows:

(i) $x_i = f^{n_1+i}(p)$ for all $i \le -n_1$ and
(ii) $x_i = f^{i-n_2}(p)$ for all $i \ge n_2$.

Thus we see that $x_i \to p$ as $i \to \pm\infty$. Then we get an η-pseudo orbit:

$$\begin{aligned}\xi &= \{\dots, f^{-n_1}(p) = p, x_{-n_1+1}, \dots, x_{-1}, x, x_1, \dots, x_{n_2-1}, f^{n_2}(p) = p, \dots\}\\ &= \{\dots, x_{-n_1-1}, x_{-n_1}, x_{-n_1+1}, \dots, x_{-1}, x, x_1, \dots, x_{n_2-1}, x_{n_2}, x_{n_2+1}, \dots\}.\end{aligned}$$

Since p is a hyperbolic periodic point of f, we can take an $\epsilon(p) > 0$ such that $d(f^i(x), f^i(p)) = d(f^i(x), p) \le \epsilon(p)$ for all $i \ge 0$ implies $x \in W^s_{\epsilon(p)}(p)$ and $d(f^{-i}(x), f^{-i}(p)) = d(f^{-i}(x), p) \le \epsilon(p)$ for all $i \ge 0$ implies $x \in W^u_{\epsilon(p)}(p)$. Take $\epsilon = \min\{\epsilon_0, d(x,p)/4\}$, and let $0 < \delta = \delta(\epsilon) < \epsilon$ be the number in the definition of the orbital shadowing property for f. From the above, we set $\eta = \delta$. Then we see that $\xi \subset C_f(p)$. Since f has the orbital shadowing property on $C_f(p)$, there is $y \in M$ such that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \quad \text{and} \quad \xi \subset B_\epsilon(\mathcal{O}_f(y)).$$

There are $l_1 > 0$ and $l_2 > 0$ such that $d(f^{-n_1}(p), f^{-l_1}(y)) = d(p, f^{-l_1}(y)) < \epsilon$, and $d(f^{n_2}(p), f^{l_2}(y)) = d(p, f^{l_2}(y)) < \epsilon$.

Then $d(p, f^{-l_1-i}(y)) < \epsilon$ for all $i \ge 0$, and $d(p, f^{l_2+i}(y)) < \epsilon$ for all $i \ge 0$. Hence $f^{-l_1}(y) \in W^u_{\epsilon(p)}(p)$ and $f^{l_2}(y) \in W^s_{\epsilon(p)}(p)$. Therefore, we see that $y \in W^s(p) \cap W^u(p)$. Thus we can find $j \in \mathbb{Z}$ such that $f^j(y) \in W^s(p) \cap W^u(p) \cap B_\epsilon(x) \subset H_f(p)$. □

Let Λ be a closed f-invariant set. Note that if f has the orbital shadowing property on a locally maximal Λ, then the shadowing point can be taken from Λ.

Lemma 3.2 *Suppose that f has the C^1-stably orbitally shadowing property on Λ. Then for every $p \in \Lambda \cap P(f)$ is hyperbolic.*

Proof Suppose that f has the C^1-stably orbitally shadowing property on Λ. Then there exist a compact neighborhood U of Λ and a C^1-neighborhood $\mathcal{U}(f)$ of f such that for any $g \in \mathcal{U}(f)$, g has the orbital shadowing property on $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. Take a C^1-neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f as in Lemma 2.3. Then arguing similarly as in the proof of Theorem 1.2, $\mathcal{U}_0(f)$ is seen to be a desired neighborhood of f. In fact, suppose $p \in \Lambda_g$ is a non-hyperbolic periodic point for some $g \in \mathcal{U}_0(f)$. Observe that we can choose smaller $\mathcal{U}_0(f)$ if necessary so that $p \in \operatorname{int} U$. Then, by making use of Lemma 2.3, we can construct a diffeomorphism $h \in \mathcal{U}_0(f)$ C^1-close to g which has the invariant hyperbolic small arc $\mathfrak{I}_q$ and disk $\mathfrak{D}_q$, centered at q, contained in $\Lambda_h(U)$, where $\Lambda_h(U) = \bigcap_{n\in\mathbb{Z}} h^n(U)$. Note that $h^k_{|\mathfrak{I}} = id$ for some $k > 0$, where either $\mathfrak{I} = \mathfrak{I}_q$ or $\mathfrak{I} = \mathfrak{D}_q$. Since the identity map does not have the orbital shadowing property and h^k has the orbital shadowing property on $\mathfrak{I}_q$ as well as on $\mathfrak{D}_q$, we have a contradiction. This completes the proof. □

For $f \in \mathrm{Diff}(M)$, we say that a compact f-invariant set Λ admits a *dominated splitting* if the tangent bundle $T_\Lambda M$ has a continuous Df-invariant splitting $E \oplus F$ and there exist

$C > 0$, $0 < \lambda < 1$ such that for all $x \in \Lambda$ and $n \geq 0$, we have

$$\|Df^n|_{E(x)}\| \cdot \|Df^{-n}|_{F(f^n(x))}\| \leq C\lambda^n \ldots.$$

Remark 3.3 If Λ admits a dominated splitting $T_\Lambda M = E \oplus F$ such that for any $x \in \Lambda$, $\dim E(x)$ is constant, then there are a C^1-neighborhood $\mathcal{U}(f)$ of f and a compact neighborhood V of Λ such that for any $g \in \mathcal{U}(f)$, $\bigcap_{n\in\mathbb{Z}} g^n(V)$ admits a dominated splitting $T_{\bigcap_{n\in\mathbb{Z}} g^n(V)}M = E' \oplus F'$ with $\dim E = \dim E'$.

From Lemma 3.2, for any $g \in \mathcal{U}(f)$, the family of periodic sequences of linear isomorphisms of $\mathbb{R}^{\dim M}$ generated by Dg along the hyperbolic periodic points $q \in \Lambda_g(U) \cap P(g)$ is uniformly hyperbolic (see [11]). Indeed, there is $\epsilon > 0$ such that for any $g \in \mathcal{U}(f)$, $q \in \Lambda_g(U) \cap P(g)$, and any sequence of linear maps $L_i : T_{g^i(q)}M \to T_{g^{i+1}(q)}M$ with $\|L_i - D_{g^i(q)}g\| < \epsilon$ for $i = 1, 2, \ldots, \pi(q) - 1$, and $\prod_{i=0}^{\pi(q)-1} L_i$ is hyperbolic. Here $\mathcal{U}(f)$ is the C^1-neighborhood of f. Then we can apply Proposition 2.1 in [11] to obtain the following proposition.

Proposition 3.4 *Suppose that f has the C^1-stably orbitally shadowing property on $C_f(p)$. Then there exist a C^1-neighborhood $\mathcal{U}(f)$ of f, constants $C > 0$, $0 < \lambda < 1$ and $m \in \mathbb{Z}^+$ such that*

(1) *for each $g \in \mathcal{U}(f)$, if q is a periodic point of g in $\Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$ with period $\pi(q,g)$ $(\pi(q,g) \geq m)$, then*

$$\prod_{i=0}^{k-1} \|Dg^m|_{E^s(g^{im}(q))}\| < C\lambda^k \quad \text{and} \quad \prod_{i=0}^{k-1} \|Dg^{-m}|_{E^u(g^{-im}(q))}\| < C\lambda^k,$$

where $k = [\pi(q,g)/m]$.

(2) *$C_f(p)$ admits a dominated splitting $T_{C_f(p)}M = E \oplus F$ with* $\dim E = \operatorname{index}(p)$.

Remark 3.5 By Proposition 3.4(2) and [12, Theorem A], the homoclinic class $H_f(p)$ is the transverse homoclinic class $H_f^T(p)$. Thus if f has the C^1-stably orbital shadowing property on $C_f(p)$, then we see that $H_f^T(p) = H_f(p) = C_f(p)$.

Let j denote the index(p), $0 < j < \dim M$, and let

$$\begin{aligned} P_i(f|_{C_f(p)}) = P_i(f|_{H_f^T(p)}) = P_i(f|_{H_f(p)}) \\ = \{q \in H_f^T(p) \cap P(f) : \operatorname{index}(q) = i\}, \end{aligned}$$

where $0 < i < \dim(M)$. We write $\Lambda_i(f) = \overline{P_i(f|_{H_f(p)})}$. Then we know that these are basic sets and $\Lambda_i(f) = H_f(p)$. In general, a non-hyperbolic homoclinic class $H_f(p)$ contains saddle periodic points with different indices. Since $H_f(p) \subset C_f(p)$, we know that the chain component $C_f(p)$ may contain saddle periodic points with different indices in general. However, if f has the C^1-stably orbitally shadowing property on $C_f(p)$, then such a case cannot happen. Thus we need the following proposition to prove Theorem 1.3.

Proposition 3.6 *Suppose that f has the C^1-stably orbitally shadowing property on $C_f(p)$. Then for any $q \in C_f(p) \cap P(f)$,* $\operatorname{index}(p) = \operatorname{index}(q)$.

To prove Proposition 3.6, we need the following lemma.

Lemma 3.7 *Suppose that f has the orbital shadowing property on $C_f(p)$. Then, for any hyperbolic $q \in C_f(p) \cap P(f)$,*

$$W^s(p) \cap W^u(q) \neq \emptyset \quad \textit{and} \quad W^u(p) \cap W^s(q) \neq \emptyset.$$

Proof Suppose that f has the orbital shadowing property on $C_f(p)$. Let $\epsilon(p) > 0$ and $\epsilon(q) > 0$ be as in the definition of $W^{s,u}_{\epsilon(p)}(p)$ and $W^{s,u}_{\epsilon(q)}(q)$ with respect to p and q. To simplify notation in this proof, we may assume that $f(p) = p$ and $f(q) = q$. Take $\epsilon = \min\{\epsilon(p), \epsilon(q)\}$, and let $0 < \delta = \delta(\epsilon) \leq \epsilon$ be the number of the definition of the orbital shadowing property for f.

For $x \in C_f(p)$, there is a finite δ-pseudo orbit $\{x_i\}_{i=-n}^{n}$ $(n \geq 1)$ such that

(i) $\{x_i\}_{i=-n}^{n} \subset C_f(p)$, and

(ii) $x_{-n} = q$, $x_0 = x$ and $x_n = p$.

We extend the finite δ-pseudo orbit as follows: Put

(i) $x_{-n-i} = f^i(q)$ for all $i \geq 0$, and

(ii) $x_{n+i} = f^i(p)$ for all $i \geq 0$.

Then we get a δ-pseudo orbit

$$\xi = \{\ldots, q, q, x_{-n+1}, \ldots, x_0(= x), \ldots, x_{n-1}, p, p, \ldots\}.$$

Since f has the orbital shadowing property on $C_f(p)$, there is a point $y \in M$ such that

$$\mathcal{O}_f(y) \subset B_\epsilon(\xi) \quad \text{and} \quad \xi \subset B_\epsilon(\mathcal{O}_f(y)).$$

Then, we see that $\mathcal{O}_f(y) \cap W^u_\epsilon(q) \cap W^s_\epsilon(p) \neq \emptyset$. Thus $W^u(q) \cap W^s(p) \neq \emptyset$. The other case is similar. □

A diffeomorphism f is said to be *Kupka-Smale* if the periodic points of f are hyperbolic, and if $p, q \in P(f)$, then $W^s(p)$ is transversal to $W^u(q)$. It is well known that the set of Kupka-Smale diffeomorphisms is a C^1-residual set in Diff(M).

Proof of Proposition 3.6 Suppose that f has the C^1-stably orbitally shadowing property on $C_f(p)$. Let $\mathcal{U}(f)$ be a C^1-neighborhood of f and U be a compact neighborhood of $C_f(p)$ as in the definition. Note that $C_f(p)$ is upper semi continuous and $H_f(p)$ is lower semi continuous. By Lemma 3.1, $C_f(p) = H_f(p)$, $C_f(p)$ is semi continuous. By the definition, $C_g(p_g) \subset \Lambda_g(U) = \bigcap_{n\in\mathbb{Z}} g^n(U)$. To derive a contradiction, we may assume that there is a point $q \in C_f(p) \cap P(f)$ such that for any $g \in \mathcal{U}_0(f) \subset \mathcal{U}(f)$, $q \in C_g(p_g) \cap P(g)$ and index$(p) <$ index(q), where p_g is the continuation. By Lemma 3.2, for every $q \in C_f(p) \cap P(f)$ is hyperbolic. Then we know that $\dim W^s(p) + \dim W^u(q) < \dim M$. Take a Kupka-Smale diffeomorphism $g \in \mathcal{U}_0(f)$. Then there are the p_g and q_g that are the continuations of p and q respectively, and $q_g \in C_g(p_g) \cap P(g) \subset \Lambda_g(U) \cap P(g)$. Since $\dim W^s(p_g) = \dim W^s(p)$ and $\dim W^u(q_g) = \dim W^u(q)$, we know that $W^s(p_g) \cap W^u(q_g) = \emptyset$, where $W^s(p_g)$ and $W^u(q_g)$ are the stable and the unstable manifolds of p_g and q_g with respect to g. On the other hand, since $g \in \mathcal{U}_0(f)$, g has the orbital shadowing property on $\Lambda_g(U)$. Thus g has the shadowing property on $C_g(p_g)$. By Lemma 3.7, $W^s(p_g) \cap W^u(q_g) \neq \emptyset$. This is a contradiction. □

Let us recall Mañé's ergodic closing lemma obtained in [11]. Denote by $B_\epsilon(f,x)$ an ϵ-tubular neighborhood of $\mathcal{O}_f(x)$; that is,

$$B_\epsilon(f,x)\{y\in M : d(f^n(x),y) < \epsilon \text{ for some } n\in\mathbb{Z}\}.$$

Let Σ_f be the set of points $x\in M$ such that for any C^1-neighborhood $\mathcal{U}(f)$ and $\epsilon>0$, there are $g\in\mathcal{U}(f)$ and $\gamma\in P(g)$ such that $g=f$ on $M\setminus B_\epsilon(f,x)$ and $d(f^i(x),g^i(\gamma))\le\epsilon$, for $0\le i\le\pi(\gamma)$. The following lemma is in [11].

Lemma 3.8 [11] *For any f-invariant probability measure μ, we have $\mu(\Sigma_f)=1$.*

End of the Proof of Theorem 1.3 Suppose that f has the C^1-stably orbitally shadowing property on $C_f(p)$. Let $\mathcal{U}_0(f)$ be the C^1-neighborhood of f given by Proposition 3.4. To get the conclusion, it is sufficient to show that $\Lambda_i(f)$ is hyperbolic, where $\Lambda_i(f)=\overline{P_i(f|_{C_f(p)})}$, and i is the index of p. Fix any neighborhood $U_i\subset U$ of $\Lambda_i(f)$. Note that by Proposition 3.6, $\Lambda_j(f)=\overline{P_j(f|_{C_f(p)})}=\emptyset$ if $i\neq j$.

Thus we show the following: Let $\mathcal{V}(f)\subset\mathcal{U}_0(f)$ be a small connected C^1-neighborhood of f. If any $g\in\mathcal{V}(f)$ satisfies $q=f$ on $M\setminus U_i$, then $\text{index}(p)=\text{index}(q)$ for any $p,q\in\Lambda_g(U)\cap P(g)$. Indeed, suppose not, then there are $g_1\in\mathcal{V}(f)$ and $q\in\Lambda_g(U)\cap P(g_1)$ such that $g_1=f$ on $M\setminus U_i$ and $\text{index}(p)\neq\text{index}(q)$. Suppose that $g_1^n(q)=q$, $k=\text{index}(q)$, and define $\gamma:\mathcal{V}(f)\to\mathbb{Z}$ by

$$\gamma(g)=\sharp\{y\in\Lambda_g(U)\cap P(g) : g^n(y)=y \text{ and } \text{index}(y)=k\}.$$

By Lemma 3.2, the function γ is continuous, and since $\mathcal{V}(f)$ is connected, it is constant. But the property of g_1 implies $\gamma(g_1)>\gamma(f)$. This is a contradiction.

Finally, to prove Theorem 1.3, we use the proof of Theorem B in [11]. Thus we show that

$$\liminf_{n\to\infty}\|D_xf^n|_{E_x}\|=0 \quad\text{and}\quad \liminf_{n\to\infty}\|D_xf^{-n}|_{F_x}\|=0$$

for all $x\in C_f(p)$, and thus, the splitting is hyperbolic.

More precisely, we will prove the case of $\liminf_{n\to\infty}\|D_xf^n_{|E}\|=0$ (the other case is similar). It is enough to show that for any $x\in C_f(p)$, there exists $n=n(x)>0$ such that

$$\prod_{j=0}^{n-1}\|Df^m|_{E_{f^{mj}(x)}}\|<1.$$

We will derive a contraction. If it is not true, then there is $x\in C_f(p)$ such that

$$\prod_{j=0}^{n-1}\|Df^m|_{E_{f^{mj}(x)}}\|\ge 1$$

for all $n\ge 0$. Thus

$$\frac{1}{n}\sum_{j=0}^{n-1}\log\|Df^m|_{E_{f^{mj}(x)}}\|\ge 0$$

for all $n \geq 0$. Define a probability measure

$$\mu_n := \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(x)}.$$

Then there exists μ_{n_k} ($k \geq 0$) such that $\mu_{n_k} \to \mu_0 \in \mathcal{M}_f(M)$, as $k \to \infty$, where M is a compact metric space. Thus

$$\begin{aligned}\int \log \|Df|_{E_x}\| \, d\mu_0 &= \lim_{k\to\infty} \int \log \|Df|_{E_x}\| \, d\mu_{n_k} \\ &= \lim_{k\to\infty} \int \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df|_{E_{f^j(x)}}\| \geq 0.\end{aligned}$$

By Mañè [11, p.521],

$$\int_{C_f(p)} \log \|Df|_{E_x}\| \, d\mu_0 = \int_{C_f(p)} \frac{1}{n}\sum_{j=0}^{n-1} \log \|D_{f^j(r)}f|_{E_{f^j(x)}}\| \, d\mu_0 \geq 0,$$

where μ_0 is an f-invariant measure. Let

$$B_\epsilon(f,x) = \{y \in M : d(f^n(x), y) < \epsilon \text{ for some } n \in \mathbb{Z}\},$$

and $\Sigma_f = \{x \in M : d(f^n(x), y) < \epsilon$, there exist $g \in \mathcal{U}(f)$ and $y \in P(g)$ such that $g = f$ on $M \setminus B_\epsilon(f,x)$ and $d(f^i(x), f^i(y)) \leq \epsilon$ for $0 \leq i \leq \pi(y)\}$.

Note that if $x \notin P(f)$, $0 \leq \pi(y) = N$ such that $d(f^N(x), f^N(y)) = d(f^N(x), y) \to 0$ as $N \to \infty$, then $d(x,y) \to 0$. So, this is a contradiction.

For any $\mu \in \mathcal{M}_f(M)$, $\mu(\Sigma_f) = 1$. Then, for any $\mu \in \mathcal{M}_f(C_f(p))$,

$$\mu(C_f(p) \cap \Sigma_f) = 1,$$

since $\mu(C_f(p)) = 1$ and $\mu(\Sigma_f) = 1$. Thus, $C_f(p) = C_f(p) \cap \Sigma(f)$ almost everywhere. Therefore,

$$\int_{C_f(p)\cap\Sigma(f)} \lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df|_{E_{f^j(x)}}\| \, d\mu \geq 0.$$

By Birkhoff's theorem and the ergodic closing lemma, we can take $z_0 \in C_f(p) \cap \Sigma(f)$ such that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=0}^{n-1} \log \|Df|_{E_{f^j(z_0)}}\| \geq 0.$$

By Proposition 3.4, this is a contradiction.

Thus, by Proposition 3.4, $z_0 \notin P(f)$.

Let $K > 0$, $m > 0$ and $\lambda \in (0,1)$ be given by Proposition 3.4 and take $\lambda < \lambda_0 < 1$ and $n_0 > 0$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1} \log \left\| Df^m_{|E_{f^{mj}(z_0)}} \right\| \geq \log \lambda_0 \quad \text{if } n \geq n_0.$$

Then, by Mañé's ergodic closing lemma, we can find $g \in \mathcal{V}_0(f)$ $g = f$ on $M \setminus U_j$ and $r_g \in \Lambda_g \cap P(g)$ near by r.

Moreover, we know that $\text{index}(r_g) = \text{index}(p)$ since $g = f$ on $M \setminus U_j$. By applying Lemma 2.3, we can construct $h \in \mathcal{V}_0(f)$ $(\subset \mathcal{V}(f))$ C^1-nearby g such that

$$\lambda_0^k \leq \prod_{i=0}^{k-1} \left\| g_1^{im}(r_{g_1}) h^m_{|E_{g_1^{im}(r_{g_1})}} \right\|$$

(see [11, pp.523-524]). On the other hand, by Proposition 3.4, we see that

$$\prod_{i=0}^{k-1} \left\| D_{g_1^{im}(r_{g_1})} h^m_{|E_{g_1^{im}(r_{g_1})}} \right\| < K\lambda^k.$$

We can choose the period $\pi(r_{g_1})$ $(> n_0)$ of r_{g_1} as large as $\lambda_0^k \geq K\lambda^k$. Here $k = [\pi(r_{g_1})/m]$. This is a contradiction. Thus,

$$\liminf_{n\to\infty} \left\| D_x f^n_{|E_x} \right\| = 0$$

for all $x \in C_f(p)$. Therefore, $C_f(p)$ is hyperbolic. This completes the proof of the 'only if' part of Theorem 1.3. □

Competing interests

The author declares that they have no competing interests.

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