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# A fourth-order finite difference method based on spline in tension approximation for the solution of one-space dimensional second-order quasi-linear hyperbolic equations

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**Abstract**

In this paper, we propose a new three-level implicit nine-point compact finite difference formulation of order two in time and four in space directions, based on spline in tension approximation in $x$-direction and central finite difference approximation in $t$-direction for the numerical solution of one-space dimensional second-order quasi-linear hyperbolic equations with first-order space derivative term. We describe the mathematical formulation procedure in detail and also discuss how our formulation is able to handle a wave equation in polar coordinates. The proposed method, when applied to a general form of the telegrapher equation, is also shown to be unconditionally stable. Numerical examples are used to illustrate the usefulness of the proposed method.

**MSC:** 65M06; 65M12

**Keywords:** second-order quasilinear hyperbolic equation; spline in tension; wave equation in polar coordinates; stability analysis; maximum absolute errors

## 1 Introduction

We consider the one-space dimensional second-order quasi-linear hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} = A(x,t,u)\frac{\partial^2 u}{\partial x^2} + g(x,t,u,u_x,u_t), \quad 0 < x < 1, t > 0 \tag{1.1}$$

with the following initial conditions:

$$u(x,0) = a(x), \qquad u_t(x,0) = b(x), \quad 0 \le x \le 1 \tag{1.2}$$

and the boundary conditions

$$u(0,t) = p_0(t), \qquad u(1,t) = p_1(t), \quad t \ge 0. \tag{1.3}$$

We assume that the conditions (1.2) and (1.3) are given with sufficient smoothness to maintain the order of accuracy in the numerical method under consideration.

The study of a second-order quasilinear hyperbolic equation is of keen interest in the fields like acoustics, electromagnetics, fluid dynamics, mathematical physics, engineering *etc.* Many efforts are going on to develop efficient and high accuracy methods for such types of partial differential equations. The term 'spline' in the spline function arises from the prefabricated wood or plastic curve board, which is called spline and is used by a draftman to plot smooth curves through connecting the known points. In early 1969, the cubic spline method was proposed to be applied on the differential equation to get their numerical solutions, and it was in 1974 that Raggett *et al.* [1] and Fleck, Jr. [2] successfully implemented cubic spline techniques to 1D wave equations. But, still so far in literature, a very limited number of spline methods are there for second-order quasilinear hyperbolic equations. During last three decades, there has been much effort to develop stable numerical methods based on spline approximations for the solution of the differential equations. Fyfe [3], Jain *et al.* [4, 5], Al-Said [6, 7], Khan *et al.* [8] and Kadalbajoo *et al.* [9] have studied the solution of two point boundary value problems by using the cubic spline method. Kadalbajoo *et al.* [10–12], Marušić [13] and Khan *et al.* [14] have developed the tension spline method for the numerical solution of singular perturbed boundary value problems. Mohanty *et al.* [15, 16] on both uniform as well as non-uniform mesh have used tension spline to find the numerical solution of singular perturbed boundary value problems. The computational techniques based on cubic spline are discussed in detail by Khan *et al.* [17] and Kumar *et al.* [18] for the differential equations. Apart from initial work by Raggett *et al.* [1] and Fleck, Jr. [2] on wave equations using cubic splines, recently, Rashidinia *et al.* [19], Ding *et al.* [20] and Mohanty *et al.* [21–23] have studied the cubic spline and compact finite difference method for the numerical solution of hyperbolic problems. More recently, Ding *et al.* [24, 25] and Rashidinia *et al.* [26, 27] have formulated the solution of second-order telegraph equations and a nonlinear Klein-Gordon, sine-Gordon equation by using parametric spline, respectively. In the present paper, we follow the idea of Jain *et al.* [5, 28] but use non-polynomial tension spline approximation to develop an method of order four for the solution of a wave equation in polar co-ordinates with a significant first-order derivative term. We have shown that our method is in general of order four but, for the sake of computation of the result, we have used the consistency of the first-order continuity condition.

In this paper, using nine grid points, we discuss a new three-level implicit spline in the tension finite difference method of accuracy two in time and four in space for the solution of a one-space dimensional second-order quasilinear hyperbolic equation. In this method, we require only three evaluations of a function $g$. In Section 2, we discuss spline in tension and its properties. In Section 3, we propose a new three-level Numerov-type finite difference method based on spline in tension approximation. In Section 4, we derive the proposed method. In Section 5 and Section 6, we discuss the application of our method to some important physical problems like wave equation in polar coordinates and telegraph equation in a general form and their stability analysis. Difficulties were experienced in the past for the high-order spline solution of a wave equation in polar coordinates. The solution usually deteriorates in the vicinity of the singularity. We modify our technique in such a way that the solution retains its order and accuracy everywhere in the solution region. In Section 7, we establish, under appropriate conditions, the fourth-order convergence of the method. In Section 8, we discuss the higher-order approximation at first time level in order to compute the proposed numerical method of same accuracy and compare the numerical results with the results of the corresponding second-order accuracy spline in tension method. Concluding remarks are given in Section 9.

## 2 Spline in tension approximation

Let $S^j(x)$ be the non-polynomial spline in tension of the function $u(x,t)$ at the grid point $(x_l, t_j)$ and be given by

$$S^j(x) = a_l^j + b_l^j(x - x_l) + c_l^j\left(e^{\omega(x-x_l)} - e^{-\omega(x-x_l)}\right) + d_l^j\left(e^{\omega(x-x_l)} + e^{-\omega(x-x_l)}\right),$$
$$l = 0, 1, 2, \ldots, N+1, \tag{2.1}$$

where $x_l \le x \le x_{l+1}$ and $a_l^j$, $b_l^j$, $c_l^j$, $d_l^j$ are unknowns, and $\omega$ is an arbitrary parameter to be determined. $S^j(x)$ is a class of $C^2[0,1]$, which interpolates $u(x,t)$ at the grid point $(x_l, t_j)$ at $j$th time level.

The derivatives of a non-polynomial spline in tension function $S^j(x)$ are given by

$$S^{j\prime}(x) = b_l^j + \omega c_l^j\left(e^{\omega(x-x_l)} + e^{-\omega(x-x_l)}\right) + \omega d_l^j\left(e^{\omega(x-x_l)} - e^{-\omega(x-x_l)}\right),$$
$$l = 1, 2, \ldots, N+1; j = 1, 2, \ldots, J, \tag{2.2}$$
$$S^{j\prime\prime}(x) = -\omega^2\left(c_l^j\left(e^{\omega(x-x_l)} - e^{-\omega(x-x_l)}\right) + d_l^j\left(e^{\omega(x-x_l)} + e^{-\omega(x-x_l)}\right)\right),$$
$$l = 1, 2, \ldots, N+1; j = 1, 2, \ldots, J. \tag{2.3}$$

We denote

$$M_l^j = S_j''(x_l), \quad l = 0, 1, 2, \ldots, N+1; j = 1, 2, \ldots, J. \tag{2.4}$$

To derive the expression for the coefficients of (2.2) in terms of $U_l^j$, $U_{l+1}^j$, $M_l^j$ and $M_{l+1}^j$, we use

$$S_j(x_l) = U_l^j, \qquad S_j(x_{l+1}) = U_{l+1}^j, \qquad M_l^j = S_j''(x_l), \qquad M_{l+1}^j = S_j''(x_{l+1}).$$

From algebraic manipulation, we get

$$a_l^j = U_l^j - \frac{M_l^j}{\omega^2}, \qquad b_l^j = \frac{U_l^j - U_{l+1}^j}{h} + \frac{M_l^j - M_{l+1}^j}{\omega\theta},$$
$$c_l^j = \frac{2M_{l+1}^j - (e^\theta + e^{-\theta})M_l^j}{2\omega^2(e^\theta - e^{-\theta})}, \qquad d_l^j = \frac{M_l^j}{2\omega^2},$$

where $\theta = \omega h$ and $l = 0, 1, 2, \ldots, N-1$.

Using the continuity of the first derivative at $(x_l, t_j)$, that is, $S^{j\prime}(x_l-) = S^{j\prime}(x_l+)$, we obtain the following relation for $l = 1, 2, \ldots, N-1$:

$$\frac{U_{l+1}^j - 2U_l^j + U_{l-1}^j}{h^2} = \alpha M_{l+1}^j + 2\beta M_l^j + \alpha M_{l-1}^j, \tag{2.5}$$

where

$$\alpha = \frac{1}{\theta^2}\left(1 - \frac{2\theta}{(e^\theta - e^{-\theta})}\right), \qquad \beta = \frac{1}{\theta^2}\left(\frac{\theta(e^\theta + e^{-\theta})}{(e^\theta - e^{-\theta})} - 1\right) \quad \text{and} \quad \theta = \omega h.$$

When $\omega \to 0$, that is, $\theta \to 0$, then $(\alpha, \beta) \to (\frac{1}{6}, \frac{1}{3})$, and the relation (2.5) reduces to the ordinary cubic spline relation

$$U_{l+1}^j - 2U_l^j + U_{l-1}^j = \frac{h^2}{6}\left(M_{l+1}^j + 2M_l^j + M_{l-1}^j\right).$$

From (2.5), we obtain the consistency condition $\alpha + 2\beta + \alpha = 1$, which is equivalent to the equation $\tanh \frac{\theta}{2} = \frac{\theta}{2}$. This equation has an infinite number of roots. Solving graphically, we obtain the smallest nonzero positive value $\theta = 0.001$.

Now,

$$m_l^j = S^{j\prime}(x_l) = U_{xl}^j = \frac{U_{l+1}^j - U_l^j}{h} - h\left[\alpha M_{l+1}^j + \beta M_l^j\right], \quad x_l \le x \le x_{l+1}, \tag{2.6}$$

and replacing '$h$' by '$-h$', we get

$$m_l^j = S^{j\prime}(x_l) = U_{xl}^j = \frac{U_l^j - U_{l-1}^j}{h} + h\left[\beta M_l^j + \alpha M_{l-1}^j\right], \quad x_{l-1} \le x \le x_l. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain

$$m_l^j = S^{j\prime}(x_l) = U_{xl}^j = \frac{U_{l+1}^j - U_{l-1}^j}{2h} - \frac{\alpha h}{2}\left[M_{l+1}^j - M_{l-1}^j\right]. \tag{2.8}$$

Further, from (2.6), we have

$$m_{l+1}^j = S^{j\prime}(x_{l+1}) = U_{xl+1}^j = \frac{U_{l+1}^j - U_l^j}{h} + h\left[\beta M_{l+1}^j + \alpha M_l^j\right] \tag{2.9}$$

and from (2.7), we have

$$m_{l-1}^j = S^{j\prime}(x_{l-1}) = U_{xl-1}^j = \frac{U_l^j - U_{l-1}^j}{h} - h\left[\beta M_{l-l}^j + \alpha M_l^j\right]. \tag{2.10}$$

Note that (2.4), (2.8), (2.9) and (2.10) are important properties of the non-polynomial cubic spline in tension function $S^j(x)$.

## 3 The finite difference method based on spline in tension approximation

For the sake of the simplicity, first we consider the one-space dimensional nonlinear hyperbolic partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = A(x,t)\frac{\partial^2 u}{\partial x^2} + g(x,t,u,u_x,u_t), \quad 0 < x < 1, t > 0 \tag{3.1}$$

with the given initial conditions (1.2) and boundary conditions (1.3).

The solution domain $[0,1] \times [t > 0]$ is divided into $(N+1) \times J$ mesh with the spatial step size $h = 1/(N+1)$ in $x$-direction and the time step size $k > 0$ in $t$-direction, respectively, where $N$ and $J$ are positive integers. The mesh ratio parameter is given by $\lambda = (k/h) > 0$. Grid points are defined by $(x_l, t_j) = (lh, jk)$, $l = 0, 1, 2, \ldots, N+1$ and $j = 0, 1, 2, \ldots, J$. The notations $u_l^j$ and $U_l^j$ are used for the discrete approximation and the exact value of $u(x,t)$ at the grid point $(x_l, t_j)$, respectively. Similarly, at the grid point $(x_l, t_j)$, we define $A_l^j = A(x_l, t_j)$, $A_{xl}^j = A_x(x_l, t_j), \ldots$, *etc.*

We consider the following approximations:

$$\overline{U}_{tl}^{j} = \left(U_l^{j+1} - U_l^{j-1}\right)/(2k), \tag{3.2a}$$

$$\overline{U}_{tl\pm1}^{j} = \left(U_{l\pm1}^{j+1} - U_{l\pm1}^{j-1}\right)/(2k), \tag{3.2b}$$

$$\overline{U}_{ttl}^{j} = \left(U_l^{j+1} - 2U_l^{j} + U_l^{j-1}\right)/\left(k^2\right), \tag{3.2c}$$

$$\overline{U}_{ttl\pm1}^{j} = \left(U_{l\pm1}^{j+1} - 2U_{l\pm1}^{j} + U_{l\pm1}^{j-1}\right)/\left(k^2\right), \tag{3.2d}$$

$$\overline{U}_{xl}^{j} = \left(U_{l+1}^{j} - U_{l-1}^{j}\right)/(2h), \tag{3.3a}$$

$$\overline{U}_{xl\pm1}^{j} = \left(\pm 3U_{l+1}^{j} \mp 4U_l^{j} \pm U_{l-1}^{j}\right)/(2h), \tag{3.3b}$$

$$\overline{G}_l^{j} = g\left(x_l, t_j, U_l^{j}, \overline{U}_{xl}^{j}, \overline{U}_{tl}^{j}\right), \tag{3.4a}$$

$$\overline{G}_{l\pm1}^{j} = g\left(x_{l\pm1}, t_j, U_{l\pm1}^{j}, \overline{U}_{xl\pm1}^{j}, \overline{U}_{tl\pm1}^{j}\right). \tag{3.4b}$$

Since the derivative values of $S^j(x)$ defined by (2.4), (2.8), (2.9) and (2.10) are not known at each grid point $(x_l, t_j)$, we use the following approximations for the derivatives of $S^j(x)$. Let

$$\overline{M}_l^{j} = \frac{1}{A_l^{j}}\left(\overline{U}_{ttl}^{j} - \overline{G}_l^{j}\right), \tag{3.5a}$$

$$\overline{M}_{l+1}^{j} = \frac{1}{A_{l+1}^{j}}\left(\overline{U}_{ttl+1}^{j} - \overline{G}_{l+1}^{j}\right), \tag{3.5b}$$

$$\overline{M}_{l-1}^{j} = \frac{1}{A_{l-1}^{j}}\left(\overline{U}_{ttl-1}^{j} - \overline{G}_{l-1}^{j}\right), \tag{3.5c}$$

$$\hat{m}_l^{j} = \frac{U_{l+1}^{j} - U_{l-1}^{j}}{2h} - \frac{\alpha h}{2}\left[\overline{M}_{l+1}^{j} - \overline{M}_{l-1}^{j}\right], \tag{3.6a}$$

$$\hat{m}_{l+1}^{j} = \frac{U_{l+1}^{j} - U_l^{j}}{h} + h\left[\beta\overline{M}_{l+1}^{j} + \alpha\overline{M}_l^{j}\right], \tag{3.6b}$$

$$\hat{m}_{l-1}^{j} = \frac{U_l^{j} - U_{l-1}^{j}}{h} - h\left[\beta\overline{M}_{l-1}^{j} + \alpha\overline{M}_l^{j}\right]. \tag{3.6c}$$

Now we define the following approximations:

$$\hat{G}_l^{j} = g\left(x_l, t_j, U_l^{j}, \hat{m}_l^{j}, \overline{U}_{tl}^{j}\right), \tag{3.7a}$$

$$\hat{G}_{l+1}^{j} = g\left(x_{l+1}, t_j, U_{l+1}^{j}, \hat{m}_{l+1}^{j}, \overline{U}_{tl+1}^{j}\right), \tag{3.7b}$$

$$\hat{G}_{l-1}^{j} = g\left(x_{l-1}, t_j, U_{l-1}^{j}, \hat{m}_{l-1}^{j}, \overline{U}_{tl-1}^{j}\right), \tag{3.7c}$$

in which we use spline in tension function $U_l^j = S^j(x_l)$, the approximation of its first-order space derivative defined by (3.6a)-(3.6c) in $x$-direction and central difference approximations of time derivative defined by (3.2a)-(3.2d) in $t$-direction.

Then, at each grid point $(x_l, t_j)$, differential equation (3.1) is discretized by

$$\begin{aligned}
&L_1\left(U_{l+1}^{j} - 2U_l^{j} + U_{l-1}^{j}\right) \\
&\quad = \frac{k^2}{2}\left[R_1\overline{U}_{ttl+1}^{j} + R_2\overline{U}_{ttl-1}^{j} + 10\overline{U}_{ttl}^{j}\right] \\
&\qquad - \frac{k^2}{2}\left[R_1\hat{G}_{l+1}^{j} + R_2\hat{G}_{l-1}^{j} + 10\hat{G}_l^{j}\right] + \hat{T}_l^{j}, \quad l = 1, 2, \ldots, N; j = 0, 1, 2, \ldots,
\end{aligned} \tag{3.8}$$

where

$$L_1 = 6\lambda^2\left[A_l^j - \frac{h^2}{6}\left(\frac{A_{xl}^j}{A_l^j}\right)A_{xl}^j + \frac{h^2}{12}A_{xxl}^j\right],$$

$$R_1 = 1 - \frac{hA_{xl}^j}{A_l^j}, \qquad R_2 = 1 + \frac{hA_{xl}^j}{A_l^j}$$

and the local truncation error

$$\hat{T}_l^j = O\left(k^4 + k^4h^2 + k^2h^4\right). \tag{3.9}$$

## 4 Derivation of the method

For the derivation of a Numorov-type method (3.8) based on spline in tension approximations for the numerical solution of differential equation (3.1), we follow the ideas of Jain *et al.* [5, 28]. We use spline in tension approximations in $x$-direction and second-order finite difference approximation in $t$-direction.

At the grid point $(x_l, t_j)$, let us denote

$$U_{ab} = \left(\frac{\partial^{a+b}U}{\partial x^a\,\partial t^b}\right)_l^j, \qquad A_{ab} = \left(\frac{\partial^{a+b}A}{\partial x^a\,\partial t^b}\right)_l^j, \qquad \psi_l^j = \left(\frac{\partial g}{\partial U_x}\right)_l^j. \tag{4.1}$$

At the grid point $(x_l, t_j)$, we may write differential equation (3.1) as

$$U_{ttl}^j - A_l^j U_{xxl}^j = g\left(x_l, t_j, U_l^j, U_{xl}^j, U_{tl}^j\right) \equiv G_l^j \quad \text{(say)}. \tag{4.2}$$

Using the Taylor expansion, we obtain

$$\begin{aligned}
&L_1\left(U_{l+1}^j - 2U_l^j + U_{l-1}^j\right)\\
&\quad= \frac{k^2}{2}\left[R_1\overline{U}_{ttl+1}^j + R_2\overline{U}_{ttl-1}^j + 10\overline{U}_{ttl}^j\right]\\
&\qquad - \frac{k^2}{2}\left[R_1G_{l+1}^j + R_2G_{l-1}^j + 10G_l^j\right] + O\left(k^4 + k^4h^2 + k^2h^4\right),\\
&\qquad l = 1, 2, \ldots, N; j = 0, 1, 2, \ldots.
\end{aligned} \tag{4.3}$$

Simplifying (3.2a)-(3.3b), we get

$$\overline{U}_{tl}^j = U_{tl}^j + O\left(k^2\right), \tag{4.4a}$$

$$\overline{U}_{tl\pm1}^j = U_{tl\pm1}^j + O\left(k^2 \pm k^2h\right), \tag{4.4b}$$

$$\overline{U}_{ttl}^j = U_{ttl}^j + O\left(k^2\right), \tag{4.4c}$$

$$\overline{U}_{ttl\pm1}^j = U_{ttl\pm1}^j + O\left(k^2 \pm k^2h\right), \tag{4.4d}$$

$$\overline{U}_{xl}^j = U_{xl}^j + \frac{h^2}{6}U_{30} + O\left(h^4\right), \tag{4.5a}$$

$$\overline{U}_{xl\pm1}^j = U_{xl\pm1}^j - \frac{h^2}{3}U_{30} \pm O\left(h^3\right). \tag{4.5b}$$

With the help of the approximations (4.4a) and (4.5b), from (3.4a), we obtain

$$\begin{aligned}\overline{G}_l^j &= g\left(x_l, t_j, U_l^j, U_{xl}^j + \frac{h^2}{6}U_{30} + O(h^4), U_{tl}^j + O(k^2)\right)\\ &= g(x_l, t_j, U_l^j, U_{xl}^j, U_{tl}^j) + \frac{h^2}{6}U_{30}\psi_l^j + O(k^2 + h^4)\\ &= G_l^j + \frac{h^2}{6}U_{30}\psi_l^j + O(k^2 + h^4).\end{aligned} \tag{4.6a}$$

Similarly,

$$\overline{G}_{l+1}^j = G_{l+1}^j - \frac{h^2}{3}U_{30}\psi_l^j + O(k^2 + h^4), \tag{4.6b}$$

$$\overline{G}_{l-1}^j = G_{l-1}^j - \frac{h^2}{3}U_{30}\psi_l^j + O(k^2 + h^4). \tag{4.6c}$$

Now, using the approximations (4.4a)-(4.5b), (4.6a)-(4.6c) and simplifying (3.5a), we get

$$\hat{m}_l^j = m_l^j + O(k^2 + h^4). \tag{4.7a}$$

Similarly,

$$\hat{m}_{l+1}^j = m_{l+1}^j + O(k^2 + h^4), \tag{4.7b}$$

$$\hat{m}_{l-1}^j = m_{l-1}^j + O(k^2 + h^4). \tag{4.7c}$$

Now, with the help of the approximations (4.4a) and (4.7a), from (3.7a), we obtain

$$\begin{aligned}\hat{G}_l^j &= g(x_l, t_j, U_l^j, m_l^j + O(k^2 + h^4), U_{tl}^j + O(k^2))\\ &= g(x_l, t_j, U_l^j, m_l^j, U_{tl}^j) + O(k^2 + h^4)\\ &= G_l^j + O(k^2 + h^4).\end{aligned} \tag{4.8a}$$

Similarly,

$$\hat{G}_{l+1}^j = G_{l+1}^j + O(k^2 + h^4), \tag{4.8b}$$

$$\hat{G}_{l-1}^j = G_{l-1}^j + O(k^2 + h^4). \tag{4.8c}$$

Using the approximations (4.4a)-(4.4d) and (4.8a)-(4.8c), from (3.8) and (4.3), we obtain the local truncation error $\hat{T}_l^j = O(k^4 + k^4h^2 + k^2h^4)$.

Now, we consider the numerical method of $O(k^4 + k^4h^2 + k^2h^4)$ for the solution of 1D second-order quasi-linear hyperbolic equation (1.1).

In order to understand the concept to develop the method for quasi-linear case, we consider the following differential equation:

$$u'' = f(x), \quad 0 < x < 1. \tag{4.9}$$

A fourth-order method for differential equation (4.9) is given by

$$U_{l-1} - 2U_l + U_{l+1} = \frac{h^2}{12}\left[f_l + h^2 f_{xxl}\right] + O\left(h^6\right), \tag{4.10}$$

where $f_l = f(x_l), f_{xxl} = \frac{d^2 f_l}{dx^2}$.

Whenever differential equation (4.9) is of the form $u'' = f(x,u)$, then the evaluation of $f_{xx}$ is difficult and the formula (4.10) needs to be modified. Substituting $h^2 f_{xxl} = (f_{l+1} - 2f_l + f_{l-1}) + O(h^4)$ in (4.10), we obtain the modified version of (4.10) due to Numerov as

$$U_{l-1} - 2U_l + U_{l+1} = \frac{h^2}{12}[f_{l+1} + f_{l-1} + 10f_l] + O\left(h^6\right), \tag{4.11}$$

where $f_l = f(x_l, U_l)$. Note that (4.11) is consistent with the differential equation $u'' = f(x,u)$.

Now, we use the above concept to derive the numerical method for quasi-linear equation (1.1). Whenever the coefficient $A$ is a function of $x$, $t$ and $u$, *i.e.*, $A = A(x,t,u)$, the difference scheme (3.8) needs to be modified. For this purpose, we use the following central differences:

$$A_{xl}^j = \frac{A_{l+1}^j - A_{l-1}^j}{2h} + O\left(h^2\right), \tag{4.12a}$$

$$A_{xxl}^j = \frac{A_{l+1}^j - 2A_l^j + A_{l-1}^j}{h^2} + O\left(h^2\right), \tag{4.12b}$$

where

$$A_l^j = A\left(x_l, t_j, U_l^j\right),$$
$$A_{l\pm1}^j = A\left(x_{l\pm1}, t_j, U_{l\pm1}^j\right).$$

With the help of the approximations (4.12a)-(4.12b), it is easy to verify that

$$6\lambda^2\left[A_l^j - \frac{h^2}{6A_l^j}\left(\frac{A_{l+1}^j - A_{l-1}^j}{2h}\right)^2 + \frac{h^2}{12}\left(\frac{A_{l+1}^j - 2A_l^j + A_{l-1}^j}{h^2}\right)\right] = L_1 + O\left(k^4 + k^4h^2 + k^2h^4\right),$$

$$1 - \frac{h}{A_l^j}\left(\frac{A_{l+1}^j - A_{l-1}^j}{2h}\right) = R_1 + O\left(h^3\right),$$

$$1 + \frac{h}{A_l^j}\left(\frac{A_{l+1}^j - A_{l-1}^j}{2h}\right) = R_2 + O\left(h^3\right).$$

Thus, substituting the central differences (4.12a)-(4.12b) into (3.8), we obtain the required numerical method of $O(k^2 + k^2h^2 + h^4)$ for the solution of second-order quasi-linear hyperbolic equation (1.1) and hence the local truncation error retains its order, that is, $\hat{T}_l^j = O(k^4 + k^4h^2 + k^2h^4)$.

Note that the initial and Dirichlet boundary conditions are given by (1.2) and (1.3), respectively. Incorporating the initial and boundary conditions, we can write the method (3.8) in a tri-diagonal matrix form. If differential equation (1.1) is linear, we can solve the linear system using the Gauss-elimination (tri-diagonal solver) method; in the nonlinear case, we can use the Newton-Raphson iterative method to solve the nonlinear system (see Kelly [29], Hageman and Young [30]).

## 5 Application to wave equation with singular coefficients and stability analysis

We consider the one-space dimensional wave equation in polar co-ordinates

$$u_{tt} = u_{rr} + \frac{\gamma}{r}u_r - \frac{\gamma}{r^2}u + f(r,t), \quad 0 < r < 1, t > 0. \tag{5.1}$$

For $\gamma = 1$ and 2, the above equation represents the one-space dimensional wave equation in cylindrical and spherical co-ordinates, respectively. The initial and the Dirichlet boundary conditions are prescribed by

$$u(r,0) = \phi(r), \qquad u_t(r,0) = \varphi(r), \quad 0 \le r \le 1, \tag{5.2}$$

$$u(0,t) = q_0(t), \qquad u(1,t) = q_1(t), \quad t \ge 0. \tag{5.3}$$

Assume that $f(r,t) \in C^2(0,1) \times [t > 0]$ and the conditions (5.2) and (5.3) are given with sufficient smoothness to maintain the order of accuracy in the numerical method under consideration.

Replacing the variable $x$ by $r$, applying the spline in tension method (3.8) to (5.1) and neglecting the local truncation error, we obtain

$$\begin{aligned} &6\lambda^2\left[u_{l+1}^j - 2u_l^j + u_{l-1}^j\right] \\ &\quad = \frac{k^2}{2}\left[\overline{u}_{ttl+1}^j + \overline{u}_{ttl-1}^j + 10\overline{u}_{ttl}^j\right] - \frac{\gamma k^2}{2}\left[\frac{1}{r_{l+1}}\hat{u}_{rl+1}^j + \frac{1}{r_{l-1}}\hat{u}_{rl-1}^j + \frac{10}{r_l}\hat{u}_{rl}^j\right] \\ &\qquad + \frac{\gamma k^2}{2}\left[\frac{1}{r_{l+1}^2}\hat{u}_{l+1}^j + \frac{1}{r_{l-1}^2}\hat{u}_{l-1}^j + \frac{10}{r_l^2}\hat{u}_l^j\right] \\ &\qquad - \frac{k^2}{2}\left[f_{l+1}^j + f_{l-1}^j + 10f_l^j\right], \quad l = 1(1)N, j = 1,2,\dots,J, \end{aligned} \tag{5.4}$$

where the approximations associated with (5.4) are defined in Section 3.

Note that the scheme (5.4) is of $O(k^2 + k^2h^2 + h^4)$ accuracy for the solution of wave equation (5.1). Since $r_0 = 0$, the scheme (5.4) fails to compute at $l = 1$ due to zero division. In order to get a stable spline in tension scheme of $O(k^2 + k^2h^2 + h^4)$ accuracy, we need the following approximations:

$$\frac{1}{r_{l\pm1}} = \frac{1}{r_l} \mp \frac{h}{r_l^2} + \frac{h^2}{r_l^3} \mp O(h^3), \tag{5.5a}$$

$$\frac{1}{(r_{l\pm1})^2} = \frac{1}{r_l^2} \mp \frac{2h}{r_l^3} + \frac{3h^2}{r_l^4} \mp O(h^3), \tag{5.5b}$$

$$f_{l\pm1}^j = f_{00} \pm hf_{10} + \frac{h^2}{2}f_{20} \pm O(h^3), \tag{5.5c}$$

where

$$f_l^j = f(r_l,t_j) = f_{00}, \qquad f_{rl}^j = f_r(r_l,t_j) = f_{10}, \qquad f_{rrl}^j = f_{rr}(r_l,t_j) = f_{20}, \dots, \ etc.$$

Now, with the help of the approximations defined in Section 3 and (5.5a)-(5.5c), neglecting high-order terms, we can re-write the scheme (5.4) in three-level operator compact implicit form

$$\begin{aligned}&\left[T_0+\frac{1}{12}\left(\delta_r^2+T_1(2\mu_r\delta_r)\right)\right]\delta_t^2u_l^j\\&\quad=\lambda^2\left[T_2\delta_r^2+T_3(2\mu_r\delta_r)+2T_4\right]u_l^j+\sum f;\quad l=1(1)N, j=1(1)J,\end{aligned}\tag{5.6}$$

where

$$\begin{aligned}&T_0=1+\frac{\gamma}{12l^2},\qquad T_1=\frac{1}{2}\frac{\gamma}{l},\qquad T_2=1+\frac{\gamma(\gamma-2)}{12l^2},\\&T_3=T_1+\frac{\gamma(6-\gamma)}{24l^3},\qquad T_4=-\frac{\gamma}{2l^2}+\frac{\gamma(\gamma-6)}{24l^4}\end{aligned}$$

and

$$\sum f=\frac{k^2}{12}\left[\left(12+\frac{\gamma}{l^2}\right)f_{00}+\frac{\gamma h}{l}f_{10}+h^2f_{20}\right],$$

and $\mu_r u_l=\frac{1}{2}(u_{l+\frac{1}{2}}+u_{l-\frac{1}{2}})$ and $\delta_r u_l=(u_{l+\frac{1}{2}}-u_{l-\frac{1}{2}})$ are averaging and central difference operators with respect to $r$-direction, *etc.* This implies $(2\mu_r\delta_r)u_l^j=u_{l+1}^j-u_{l-1}^j$, $\delta_r^2u_l^j=u_{l+1}^j-2u_l^j+u_{l-1}^j$, $\delta_t^2u_l^j=u_l^{j+1}-2u_l^j+u_l^{j-1}$, *etc.* The spline in tension finite difference scheme (5.6) has a local truncation error of $O(k^2+h^4)$ and is free from the terms $\frac{1}{l\pm1}$ and hence it can be solved for $l=1(1)N$, $j=1(1)J$ in the region $0<r<1$, $t>0$.

For the stability of the method (5.6), we follow the technique used by Mohanty [31]. We may re-write (5.6) as

$$\begin{aligned}&\left[T_0+\frac{1}{12}\left(T_2\delta_r^2+T_3(2\mu_r\delta_r)\right)\right]\delta_t^2u_l^j\\&\quad=\lambda^2\left[T_2\delta_r^2+T_3(2\mu_r\delta_r)+2T_4\right]u_l^j+\sum f.\end{aligned}\tag{5.7}$$

The additional terms are of high orders and do not affect the accuracy of the scheme. The exact value $U_l^j=u(r_l,t_j)$ satisfies

$$\begin{aligned}&\left[T_0+\frac{1}{12}\left(T_2\delta_r^2+T_3(2\mu_r\delta_r)\right)\right]\delta_t^2U_l^j\\&\quad=\lambda^2\left[T_2\delta_r^2+T_3(2\mu_r\delta_r)+2T_4\right]U_l^j+\sum f+O\left(k^4+k^2h^4\right).\end{aligned}\tag{5.8}$$

We assume that there exists an error $\varepsilon_l^j=U_l^j-u_l^j$ at the grid point $(x_l,t_j)$. Subtracting (5.7) from (5.8), we obtain the error equation

$$\begin{aligned}&\left[T_0+\frac{1}{12}\left(T_2\delta_r^2+T_3(2\mu_r\delta_r)\right)\right]\delta_t^2\varepsilon_l^j\\&\quad=\lambda^2\left[T_2\delta_r^2+T_3(2\mu_r\delta_r)+2T_4\right]\varepsilon_l^j+O\left(k^4+k^2h^4\right).\end{aligned}\tag{5.9}$$

For the stability of the modified scheme (5.7), we assume that $\varepsilon_l^j=A^le^{i\phi j}e^{i\theta l}$ (where $\xi=e^{i\phi}$ such that $|\xi|=1$) at the grid point $(x_l,t_j)$, where $\xi$ is in general complex, $\theta$ is an arbitrary real number and $A$ is a non-zero real parameter to be determined. Substituting $\varepsilon_l^j = A^l e^{i\phi j} e^{i\theta l}$ in the homogeneous part of error equation (5.9), we obtain the amplification factor

$$\begin{aligned} -4\sin^2\left(\frac{\phi}{2}\right) = {} & \bigl(\lambda^2\bigl[T_2\bigl\{\bigl(A + A^{-1}\bigr)\cos\theta - 2 + i\bigl(A - A^{-1}\bigr)\sin\theta\bigr\} \\ & + T_3\bigl\{\bigl(A - A^{-1}\bigr)\cos\theta + i\bigl(A + A^{-1}\bigr)\sin\theta\bigr\} + 2T_4\bigr]\bigr) \\ & \Big/\left(T_0 + \frac{1}{12}\bigl[T_2\bigl\{\bigl(A + A^{-1}\bigr)\cos\theta - 2 + i\bigl(A - A^{-1}\bigr)\sin\theta\bigr\}\right. \\ & \left. + T_3\bigl\{\bigl(A - A^{-1}\bigr)\cos\theta + i\bigl(A + A^{-1}\bigr)\sin\theta\bigr\}\bigr]\right). \end{aligned} \tag{5.10}$$

Since the left-hand side of (5.10) is a real quantity, hence the imaginary part of the right-hand side of (5.10) must be zero, from which we obtain

$$T_2\bigl(A - A^{-1}\bigr) + T_3\bigl(A + A^{-1}\bigr) = 0$$

or

$$A = \sqrt{\frac{T_2 - T_3}{T_2 + T_3}}, \tag{5.11}$$

where $T_2 \pm T_3 > 0$. Substituting the values of $A$ and $A^{-1}$ in (5.10), we get

$$\sin^2\left(\frac{\phi}{2}\right) = \frac{\lambda^2[T_2 + \sqrt{(T_2^2 - T_3^2)}(2\sin^2(\frac{\theta}{2}) - 1) - T_4]}{2T_0 - \frac{1}{3}[T_2 + \sqrt{(T_2^2 - T_3^2)}(2\sin^2(\frac{\theta}{2}) - 1)]}. \tag{5.12}$$

Since $0 \le \sin^2(\frac{\phi}{2}) \le 1$, $\max\sin^2(\frac{\theta}{2}) = 1$, $\min\sin^2(\frac{\theta}{2}) = 0$, it follows from (5.12) that the spline in tension finite difference scheme (5.4) is stable if

$$0 < \lambda^2 \le \frac{2T_0 - \frac{1}{3}[T_2 + \sqrt{T_2^2 - T_3^2}]}{T_2 - T_4 + \sqrt{T_2^2 - T_3^2}} \tag{5.13}$$

leading to $|\xi| = 1$. It is easy to verify that as $l \to \infty$, $0 < \lambda^2 \le 1$.

## 6 Application to telegraph equation in a general form and stability analysis

Consider the telegraph equation in a general form

$$u_{tt} + (\alpha + \beta)u_t + \alpha\beta u = c^2 u_{xx} + f(x,t), \quad 0 < x < 1, t > 0, \tag{6.1}$$

where $\alpha > 0$, $\beta \ge 0$ are real parameters. For $\beta = 0$, the equation above represents a damped wave equation. The initial and boundary conditions are prescribed by (1.2) and (1.3), respectively. The telegraph equation is a linear differential equation which describes the voltage and current on an electric transmission line with distance and time. In fact, the telegraph equation is more suitable than an ordinary diffusion equation in modeling reaction-diffusion for such branches of science. In equation (6.1), $u(x,t)$ is either voltage or current through the wire at position $x$ and time $t$, and $\alpha = Z/C$, $\beta = \Re/I$ and $c^2 = 1/(IC)$, where $Z$ is the conductance of a resistor, $\Re$ is the resistance of the resistor, $I$ is the inductance of the coil and $C$ is the capacitance of the capacitor.

Now onwards, we denote $a = (\frac{\alpha+\beta}{2})^2k^2$, $b = \alpha\beta k^2$; and $f_l^j = f(x_l, t_j)$.

Applying the scheme (3.8) to differential equation (6.1), we may obtain a numerical approximation of $O(k^2 + h^4)$ as

$$\left(1+\frac{\delta_x^2}{12}\right)\delta_t^2 u_l^j + \sqrt{a}\left(1+\frac{\delta_x^2}{12}\right)(2\mu_t\delta_t)u_l^j + \left(\frac{b}{12}-\lambda^2c^2\right)\delta_x^2 u_l^j + bu_l^j$$
$$= \frac{k^2}{12}\left[f_{l+1}^j + f_{l-1}^j + 10f_l^j\right]. \tag{6.2}$$

The corresponding error equation is

$$\left(1+\frac{\delta_x^2}{12}\right)\delta_t^2 \varepsilon_l^j + \sqrt{a}\left(1+\frac{\delta_x^2}{12}\right)(2\mu_t\delta_t)\varepsilon_l^j + \left(\frac{b}{12}-\lambda^2c^2\right)\delta_x^2 \varepsilon_l^j + b\varepsilon_l^j$$
$$= O\left(k^4 + k^2h^4\right). \tag{6.3}$$

To establish the stability for the scheme (6.2), it is necessary to assume that the solution of the homogeneous part of error equation (6.3) is of the form $\varepsilon_l^j = \xi^j e^{i\theta l}$, where $i = \sqrt{-1}$, $\theta$ is real and we obtain the characteristic equation

$$p\xi^2 + q\xi + r = 0, \tag{6.4}$$

where

$$p = (1+\sqrt{a})\left(1-\frac{1}{3}\sin^2\left(\frac{\theta}{2}\right)\right),$$
$$q = b - 2\left(1-\frac{1}{3}\sin^2\left(\frac{\theta}{2}\right)\right) + 4\left(\lambda^2c^2-\frac{b}{12}\right)\sin^2\left(\frac{\theta}{2}\right),$$
$$r = (1-\sqrt{a})\left(1-\frac{1}{3}\sin^2\left(\frac{\theta}{2}\right)\right).$$

The necessary and sufficient condition for $|\xi| < 1$ is that $p+q+r > 0$, $p-r > 0$, $p-q+r > 0$. The conditions $p+q+r > 0$ and $p-r > 0$ are satisfied for $\alpha > 0$, $\beta \geq 0$ and for all $\theta$ except $(\theta, \beta) = (0, 0)$ or $(2\pi, 0)$. We can treat this case separately.

The condition $p - q + r > 0$ is satisfied if

$$0 < \lambda^2 \leq \frac{4-b}{6c^2} \quad \text{provided} \quad 0 \leq b < 4. \tag{6.5}$$

In order to obtain an unconditionally stable spline in tension finite difference scheme of accuracy of $O(k^2 + h^4)$, we may re-write the scheme (6.2) as

$$\left[1+\eta b^2 - \gamma\lambda^2c^2\delta_x^2 + \frac{\delta_x^2}{12}\right]\delta_t^2 u_l^j + \sqrt{a}\left(1+\frac{\delta_x^2}{12}\right)(2\mu_t\delta_t)u_l^j + \left(\frac{b}{12}-\lambda^2c^2\right)\delta_x^2 u_l^j + bu_l^j$$
$$= \frac{k^2}{12}\left[f_{l+1}^j + f_{l-1}^j + 10f_l^j\right]; \quad l = 1(1)N, j = 1(1)J, \tag{6.6}$$

where $\eta$ and $\gamma$ are free parameters to be determined. The additional terms $\eta b^2\delta_t^2 u_l^j$ and $-\gamma\lambda^2c^2\delta_x^2\delta_t^2 u_l^j$ are of higher order and do not affect the consistency and accuracy of the scheme. Now using the technique discussed in [21] and [22], we find that for $\alpha > 0$, $\beta \geq 0$, $\eta \geq \frac{1}{64}$; and $\gamma \geq \frac{1+3\lambda^2c^2}{12\lambda^2c^2}$, the spline in tension finite difference scheme (6.6) is stable for all choices of $h > 0$ and $k > 0$.

## 7 Convergence analysis

In this section, we establish under appropriate conditions the fourth-order convergence of the proposed method. For simplicity, we consider the nonlinear hyperbolic differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(x,t,u,u_x,u_t), \quad 0 < x < 1, t > 0 \tag{7.1}$$

with the initial and boundary conditions which are prescribed by (1.2) and (1.3), respectively.

We assume that the initial value problem (7.1), (1.2)-(1.3) has a unique smooth solution $u(x,t)$ and the following conditions are satisfied:

(i) $g(x,t,u,u_x,u_t)$ is continuous,

(ii) $g(x,t,u,u_x,u_t)$ satisfies the Lipschitz condition, namely,

$$\begin{aligned}&\left|g(x,t,u+\xi_1,u_x+\xi_2,u_t+\xi_3) - g\left(x,t,u+\xi_1^*,u_x+\xi_2^*,u_t+\xi_3^*\right)\right| \\ &\quad \leq L\left(\left|\xi_1-\xi_1^*\right| + \left|\xi_2-\xi_2^*\right| + \left|\xi_3-\xi_3^*\right|\right),\end{aligned}$$

where $\xi_i$ and $\xi_i^*$ are arbitrary real numbers and $L$ is a Lipschitz constant,

(iii) $a(x)$ and $b(x)$ are continuously differentiable up to order four and two, respectively.

The method (3.8) becomes

$$\begin{aligned}&12\lambda^2\left(U_{l+1}^j - 2U_l^j + U_{l-1}^j\right) - k^2\left[\overline{U}_{ttl+1}^j + \overline{U}_{ttl-1}^j + 10\overline{U}_{ttl}^j\right] \\ &\quad + k^2\left[\hat{G}_{l+1}^j + \hat{G}_{l-1}^j + 10\hat{G}_l^j\right] = T_l^j; \quad l = 1(1)N, j = 0,1,2,\ldots,\end{aligned} \tag{7.2}$$

where $\overline{G}_l^j = g(x_l,t_j,U_l^j,\overline{U}_{xl}^j,\overline{U}_{tl}^j)$, $\hat{G}_l^j = g(x_l,t_j,U_l^j,\hat{U}_{xl}^j,\overline{U}_{tl}^j)$, *etc.*, … and $T_l^j = O(k^4 + k^4h^2 + k^2h^4)$.

Let $\mathbf{U}^j = [U_1^j, U_2^j, \ldots, U_N^j]^T$ ($T$ denotes transpose) and $\mathbf{u}^j = [u_1^j, u_2^j, \ldots, u_N^j]^T$ be the exact and approximate solution vectors of the solution $u(x,t)$ at the grid point $(x_l,t_j)$, respectively, and $\mathbf{T} = [T_1^j, T_2^j, \ldots, T_N^j]^T$ be the local truncation error vector.

Let

$$\boldsymbol{\phi}(\mathbf{U}) \equiv \boldsymbol{\phi}\left(\mathbf{U}^{j+1},\mathbf{U}^j,\mathbf{U}^{j-1}\right) = k^2\left[\hat{G}_{l+1}^j + \hat{G}_{l-1}^j + 10\hat{G}_l^j\right]$$

and

$$\boldsymbol{\phi}(\mathbf{u}) \equiv \boldsymbol{\phi}\left(\mathbf{u}^{j+1},\mathbf{u}^j,\mathbf{u}^{j-1}\right) = k^2\left[\hat{g}_{l+1}^j + \hat{g}_{l-1}^j + 10\hat{g}_l^j\right],$$

where $\overline{g}_l^j = g(x_l,t_j,u_l^j,\overline{u}_{xl}^j,\overline{u}_{tl}^j)$, $\hat{g}_l^j = g(x_l,t_j,u_l^j,\hat{u}_{xl}^j,\overline{u}_{tl}^j)$, *etc.*, … .

Then, the spline in tension method described by (7.2) can be expressed in a matrix form as follows:

$$\mathbf{D}\mathbf{U}^{j+1} + 2\mathbf{C}\mathbf{U}^{j} + \mathbf{D}\mathbf{U}^{j-1} + \boldsymbol{\phi}(\mathbf{U}) = \mathbf{T}, \tag{7.3}$$

where $\mathbf{D} = [-1, -10, -1]^T$ and $\mathbf{C} = [1 + 6\lambda^2, 10 - 12\lambda^2, 1 + 6\lambda^2]^T$ are tri-diagonal matrices of order $N$.

The method consists of obtaining an approximation $\mathbf{u}^{j+1}$ for $\mathbf{U}^{j+1}$ by solving the tri-diagonal system

$$\mathbf{D}\mathbf{u}^{j+1} + 2\mathbf{C}\mathbf{u}^{j} + \mathbf{D}\mathbf{u}^{j-1} + \boldsymbol{\phi}(\mathbf{u}) = \mathbf{0}. \tag{7.4}$$

Let $\varepsilon_l^j = u_l^j - U_l^j$ and $\mathbf{E}^j = \mathbf{u}^j - \mathbf{U}^j = [\varepsilon_1^j, \varepsilon_2^j, \ldots, \varepsilon_N^j]^T$.

We may write $\overline{u}_{tl}^j - \overline{U}_{tl}^j = (\varepsilon_l^{j+1} - \varepsilon_l^{j-1})/(2k)$, $\overline{u}_{xl}^j - \overline{U}_{xl}^j = (\varepsilon_{l+1}^j - \varepsilon_{l-1}^j)/(2h), \ldots,$ *etc.*

$$\overline{g}_{l\pm1}^j - \overline{G}_{l\pm1}^j = \varepsilon_{l\pm1}^j W_{l\pm1}^j + \frac{1}{2h}\left(\pm 3\varepsilon_{l\pm1}^j \mp 4\varepsilon_l^j \pm \varepsilon_{l\mp1}^j\right)H_{l\pm1}^j + \frac{1}{2k}\left(\varepsilon_{l\pm1}^{j+1} - \varepsilon_{l\pm1}^{j-1}\right)I_{l\pm1}^j, \tag{7.5a}$$

$$\overline{g}_{l}^j - \overline{G}_{l}^j = \varepsilon_l^j W_l^j + \frac{1}{2h}\left(\varepsilon_{l+1}^j - \varepsilon_{l-1}^j\right)H_l^j + \frac{1}{2k}\left(\varepsilon_l^{j+1} - \varepsilon_l^{j-1}\right)I_l^j, \tag{7.5b}$$

$$\begin{aligned}
\hat{g}_{l\pm1}^j - \hat{G}_{l\pm1}^j = {} & \varepsilon_{l\pm1}^j W_{l\pm1}^j \pm \frac{1}{h}\left(\varepsilon_{l\pm1}^j - \varepsilon_l^j\right)H_{l\pm1}^j \pm \frac{\alpha h}{k^2}\left(\varepsilon_l^{j+1} - 2\varepsilon_l^j + \varepsilon_l^{j-1}\right)H_{l\pm1}^j \\
& \pm \frac{\beta h}{k^2}\left(\varepsilon_{l\pm1}^{j+1} - 2\varepsilon_{l\pm1}^j + \varepsilon_{l\pm1}^{j-1}\right)H_{l\pm1}^j \\
& \mp \alpha h\left[\varepsilon_l^j W_l^j + \frac{1}{2h}\left(\varepsilon_{l+1}^j - \varepsilon_{l-1}^j\right)H_l^j + \frac{1}{2k}\left(\varepsilon_l^{j+1} - \varepsilon_l^{j-1}\right)I_l^j\right]H_{l\pm1}^j \\
& \mp \beta h\left[\varepsilon_{l\pm1}^j W_{l\pm1}^j + \frac{1}{2h}\left(\pm 3\varepsilon_{l\pm1}^j \mp 4\varepsilon_l^j \pm \varepsilon_{l\mp1}^j\right)H_{l\pm1}^j\right. \\
& \left. + \frac{1}{2k}\left(\varepsilon_{l\pm1}^{j+1} - \varepsilon_{l\pm1}^{j-1}\right)I_{l\pm1}^j\right]H_{l\pm1}^j + \frac{1}{2k}\left(\varepsilon_{l\pm1}^{j+1} - \varepsilon_{l\pm1}^{j-1}\right)I_{l\pm1}^j,
\end{aligned} \tag{7.5c}$$

$$\begin{aligned}
\hat{g}_{l}^j - \hat{G}_{l}^j = {} & \varepsilon_l^j W_l^j + \frac{1}{2h}\left(\varepsilon_{l+1}^j - \varepsilon_{l-1}^j\right)H_l^j - \frac{\alpha h}{2k^2}\left[\left(\varepsilon_{l+1}^{j+1} - 2\varepsilon_{l+1}^j + \varepsilon_{l+1}^{j-1}\right) - \left(\varepsilon_{l-1}^{j+1} - 2\varepsilon_{l-1}^j + \varepsilon_{l-1}^{j-1}\right)\right]H_l^j \\
& + \frac{\alpha h}{2}\left[\varepsilon_{l+1}^j W_{l+1}^j + \frac{1}{2h}\left(3\varepsilon_{l+1}^j - 4\varepsilon_l^j + \varepsilon_{l-1}^j\right)H_{l+1}^j + \frac{1}{2k}\left(\varepsilon_{l+1}^{j+1} - \varepsilon_{l+1}^{j-1}\right)I_{l+1}^j\right]H_l^j \\
& - \frac{\alpha h}{2}\left[\varepsilon_{l-1}^j W_{l-1}^j + \frac{1}{2h}\left(-3\varepsilon_{l-1}^j + 4\varepsilon_l^j - \varepsilon_{l+1}^j\right)H_{l-1}^j + \frac{1}{2k}\left(\varepsilon_{l-1}^{j+1} - \varepsilon_{l-1}^{j-1}\right)I_{l-1}^j\right]H_l^j \\
& + \frac{1}{2k}\left(\varepsilon_l^{j+1} - \varepsilon_l^{j-1}\right)I_l^j
\end{aligned} \tag{7.5d}$$

for suitable $W$, $H$ and $I$. Further, we may write

$$W_{l\pm1}^j = W_l^j \pm W_{xl}^j + O(h^2), \qquad H_{l\pm1}^j = H_l^j \pm H_{xl}^j + O(h^2) \quad \text{and} \quad I_{l\pm1}^j = I_l^j \pm I_{xl}^j + O(h^2).$$

With the help of (7.5c) and (7.5d), we obtain

$$\boldsymbol{\phi}(\mathbf{u}) - \boldsymbol{\phi}(\mathbf{U}) = \mathbf{P}\mathbf{E}^{j+1} + 2\mathbf{Q}\mathbf{E}^{j} + \mathbf{R}\mathbf{E}^{j-1}, \tag{7.6}$$

where $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are the coefficient matrices of error vectors $\mathbf{E}^{j+1}$, $\mathbf{E}^j$ and $\mathbf{E}^{j-1}$, respectively.

Subtracting (7.4) from (7.3), we have

$$(\mathbf{D}+\mathbf{P})\mathbf{E}^{j+1}+2(\mathbf{C}+\mathbf{Q})\mathbf{E}^{j}+(\mathbf{D}+\mathbf{R})\mathbf{E}^{j-1}=\mathbf{T}. \tag{7.7}$$

Assume that the exact solution values of $u(x,t)$ are known exactly at the initial and first time levels so that $\mathbf{E}^{j}=\mathbf{E}^{j-1}=0$. Then from (7.7), we obtain the error equation

$$(\mathbf{D}+\mathbf{P})\mathbf{E}^{j+1}=\mathbf{T}. \tag{7.8}$$

Let $P_{l,m}$ be the $(l,m)$th element of matrix $\mathbf{P}$, then it is easy to verify that

$$-1+\mathbf{P}_{l,l\pm1}<0 \quad \text{for } l=1(1)N-1,2(1)N$$

and hence $\mathbf{D}+\mathbf{P}$ is irreducible (see Varga [32]).

Let $S_m$ be the sum of the elements of the $m$th row of $\mathbf{D}+\mathbf{P}$ and $H_*=\min[(5\alpha-\beta)H_l^j\cdot I_{x_l}^j-(2\alpha+\beta I_l^j)H_{x_l}^j]$, then for sufficiently small $h$ and $k$, we obtain

$$S_m>\frac{kh^2H_*}{2},\quad m=1 \text{ and } N, \tag{7.9a}$$

$$S_m\ge kh^2H_*,\quad m=2(1)N-1 \tag{7.9b}$$

and hence, $\mathbf{D}+\mathbf{P}$ is also monotone.

Then, $(\mathbf{D}+\mathbf{P})^{-1}$ exists and $(\mathbf{D}+\mathbf{P})^{-1}\ge0$ (see Varga [32]), *i.e.*, $(\mathbf{D}+\mathbf{P})^{-1}_{l,m}\ge0$.

Since

$$\sum_{m=1}^{N}(\mathbf{D}+\mathbf{P})^{-1}_{l,m}\cdot S_m=1,\quad l=1(1)N,$$

hence

$$(\mathbf{D}+\mathbf{P})^{-1}_{l,m}\le\frac{1}{S_m}\le\frac{2}{kh^2H_*},\quad l=1(1)N;m=1 \text{ and } N \tag{7.10a}$$

and

$$\sum_{m=2}^{N}(\mathbf{D}+\mathbf{P})^{-1}_{l,m}\le\frac{1}{\min S_m}\le\frac{1}{kh^2H_*},\quad l=1(1)N. \tag{7.10b}$$

From (7.8), we have

$$\left\|\mathbf{E}^{j+1}\right\|\le\left\|(\mathbf{D}+\mathbf{P})^{-1}\right\|\|\mathbf{T}\|. \tag{7.11}$$

Now,

$$\left|\varepsilon_l^{j+1}\right|\le(\mathbf{D}+\mathbf{P})^{-1}_{l,1}|\mathbf{T}_1|+\sum_{m=2}^{N-2}(\mathbf{D}+\mathbf{P})^{-1}_{l,m}\cdot|\mathbf{T}_m|+(\mathbf{D}+\mathbf{P})^{-1}_{l,N}|\mathbf{T}_N|,\quad l=1(1)N. \tag{7.12}$$

Let $\|E^{j+1}\|=\max\{|\varepsilon_l^{j+1}|:l=1(1)N\}$.

With the help of (3.9), (7.10a) and (7.10b), for a fixed value of $\sigma = \frac{k}{h^2}$, from (7.12), we obtain for sufficiently small $h$ and $k$

$$\|\mathbf{E}^{j+1}\| = O(h^4).$$

This establishes the fourth-order convergence of the method.

## 8 Numerical illustrations

In this section, we solve some benchmark problems using the method described by equation (3.8) and compare our results with those obtained by the numerical method of $O(k^2 + h^2)$ accuracy based on spline in tension approximations. The exact solutions are provided in each case. The linear difference equation was solved using a tri-diagonal solver, whereas nonlinear difference equations were solved using the Newton-Raphson method. While using the Newton-Raphson method, the iterations were stopped when absolute error tolerance $\leq 10^{-10}$ was achieved. In order to demonstrate the fourth-order convergence of the proposed method, throughout the computation, we chose the fixed value of the parameter $\sigma = \frac{k}{h^2} = 3.2$. All computations were carried out using double precision arithmetic.

Note that, the proposed spline in tension finite difference method (3.8) for second-order quasilinear hyperbolic equations is a three-level scheme. The value of $u$ at $t = 0$ is known from the initial condition. To start any computation, it is necessary to know the numerical value of $u$ of required accuracy at $t = k$. In this section, we discuss an explicit scheme of $O(k^2)$ for $u$ at first time level, *i.e.*, at $t = k$ in order to solve differential equation (1.1) using the method (3.8), which is applicable to problems in Cartesian and polar coordinates.

Since the values of $u$ and $u_t$ are known explicitly at $t = 0$, this implies all their successive tangential derivatives are known at $t = 0$, *i.e.*, the values of $u, u_x, u_{xx}, \ldots, u_t, u_{tx}, \ldots$, *etc.* are known at $t = 0$.

An approximation for $u$ of $O(k^2)$ at $t = k$ may be written as

$$u_l^1 = u_l^0 + k u_{tl}^0 + \frac{k^2}{2}(u_{tt})_l^0 + O(k^3). \tag{8.1}$$

From equation (1.1), we have

$$(u_{tt})_l^0 = \left[A(x,t,u)u_{xx} + g(x,t,u,u_x,u_t)\right]_l^0. \tag{8.2}$$

Thus, using the initial values and their successive tangential derivative values, from (8.2) we can obtain the value of $(u_{tt})_l^0$, and then ultimately, from (8.1) we can compute the value of $u$ at first time level, *i.e.*, at $t = k$. Replacing the variable $x$ by $r$ in (8.1), we can also obtain an approximation of $O(k^2)$ for $u$ at $t = k$ in polar coordinates.

**Example 1** (Wave equation in polar coordinates)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\gamma}{r}\frac{\partial u}{\partial r} - \frac{\gamma}{r^2}u + f(r,t), \quad 0 < r < 1, t > 0. \tag{8.3a}$$

The initial and boundary conditions are given by

$$u(r,0) = 0, \qquad u_t(r,0) = r^2, \quad 0 \leq r \leq 1, \tag{8.3b}$$

**Table 1 Example 1: The maximum absolute errors**

| $h$ | $O(k^2 + h^4)$-method | | $O(k^4 + h^4)$-method discussed in [33] | |
|---|---|---|---|---|
| | $\gamma = 1, t = 5$ | $\gamma = 2, t = 5$ | $\gamma = 1, t = 5$ | $\gamma = 2, t = 5$ |
| $\frac{1}{8}$ | 0.4559(–04) | 0.3865(–04) | 0.6486(–04) | 0.5784(–04) |
| $\frac{1}{16}$ | 0.2850(–05) | 0.2416(–05) | 0.4291(–05) | 0.3808(–05) |
| $\frac{1}{32}$ | 0.1781(–06) | 0.1510(–06) | 0.2774(–06) | 0.2424(–06) |
| $\frac{1}{64}$ | 0.1113(–07) | 0.9442(–08) | 0.1612(–07) | 0.1566(–07) |

**Table 2 Example 2: The maximum absolute errors**

| $h$ | $\alpha = 6, \beta = 4, \eta = 0.5, \gamma = 1$ | $\alpha = \pi, \beta = \pi, \eta = 1, \gamma = 1$ | $\alpha = 3\pi, \beta = \pi, \eta = 10, \gamma = 20$ |
|---|---|---|---|
| $\frac{1}{8}$ | 0.2305(–04)<br>*0.7179(–04) | 0.2422(–04)<br>*0.2551(–04) | 0.2103(–04)<br>*0.9534(–03) |
| $\frac{1}{16}$ | 0.1743(–05)<br>*0.5295(–05) | 0.1704(–05)<br>*0.2030(–05) | 0.9626(–05)<br>*0.6674(–04) |
| $\frac{1}{32}$ | 0.1101(–06)<br>*0.4329(–06) | 0.1076(–06)<br>*0.1888(–06) | 0.6386(–06)<br>*0.4462(–05) |
| $\frac{1}{64}$ | 0.6889(–08)<br>*0.5214(–07) | 0.6730(–08)<br>*0.2704(–07) | 0.4006(–07)<br>*0.3416(–06) |

*Result obtained by Mohanty [34].

$$u(0,t) = 0, \qquad u(1,t) = \sinh t, \quad t \geq 0. \tag{8.3c}$$

We solve equation (8.3a) using the method (5.6) in the region bounded by $0 < r < 1$, $t > 0$. The exact solution is given by $u(r,t) = r^2 \sinh t$. The maximum absolute errors (MAE) are tabulated in Table 1 at $t = 5.0$ and for $\gamma = 1$, $\gamma = 2$.

**Example 2** (Telegraphic equation in general form)

$$\frac{\partial^2 u}{\partial t^2} + (\alpha + \beta)\frac{\partial u}{\partial t} + \alpha\beta u = c^2 \frac{\partial^2 u}{\partial x^2} + f(x,t), \quad 0 < x < 1, t > 0. \tag{8.4a}$$

The initial and boundary conditions are given by

$$u(x,0) = \sinh x, \qquad u_t(x,0) = -\sinh x, \quad 0 \leq x \leq 1, \tag{8.4b}$$

$$u(0,t) = 0, \qquad u(1,t) = e^{-t}\sinh 1, \quad t \geq 0, \tag{8.4c}$$

where $\alpha = 6$, $\beta = 4$; $\alpha = \pi$, $\beta = \pi$, and $\alpha = 3\pi$, $\beta = \pi$. We solve equation (8.4a) using the method (6.6). The exact solution is given by $u = e^{-t}\sinh x$. The MAE are tabulated in Table 2.

**Example 3** (Nonlinear hyperbolic equation)

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \gamma u\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}\right) + f(x,t), \quad 0 < x < 1, t > 0. \tag{8.5a}$$

The initial and boundary conditions are given by

$$u(x,0) = 0, \qquad u_t(x,0) = x^2, \quad 0 \leq x \leq 1, \tag{8.5b}$$

**Table 3 Example 3: The maximum absolute errors**

| $h$ | $O(k^2+h^4)$-method | | | $O(k^2+h^4)$-method | | |
|---|---|---|---|---|---|---|
| | $\gamma=1$ | $\gamma=5$ | $\gamma=10$ | $\gamma=1$ | $\gamma=5$ | $\gamma=10$ |
| $\frac{1}{8}$ | 0.1015(–03) | 0.1831(–02) | 0.4422(–01) | 0.3267(–03) | 0.2746(–02) | 0.4598(–01) |
| $\frac{1}{16}$ | 0.6438(–05) | 0.1080(–03) | 0.3198(–02) | 0.6032(–04) | 0.3570(–03) | 0.4750(–02) |
| $\frac{1}{32}$ | 0.4032(–06) | 0.6806(–05) | 0.1840(–03) | 0.1398(–04) | 0.6526(–04) | 0.8192(–03) |
| $\frac{1}{64}$ | 0.2194(–07) | 0.4252(–06) | 0.1126(–04) | 0.3438(–05) | 0.1515(–04) | 0.1713(–03) |

**Table 4 Example 4: The maximum absolute errors**

| $h$ | $O(k^2+h^4)$-method | | | $O(k^2+h^4)$-method | | |
|---|---|---|---|---|---|---|
| | $\gamma=1$ | $\gamma=5$ | $\gamma=10$ | $\gamma=1$ | $\gamma=5$ | $\gamma=10$ |
| $\frac{1}{8}$ | 0.5927(-05) | 0.6006(-04) | 0.5058(-03) | 0.3267(-03) | 0.2746(-02) | 0.4598(-01) |
| $\frac{1}{16}$ | 0.3601(-06) | 0.3476(-05) | 0.3100(-04) | 0.6032(-04) | 0.3570(-03) | 0.4750(-02) |
| $\frac{1}{32}$ | 0.2225(-07) | 0.2150(-06) | 0.1952(-05) | 0.1398(-04) | 0.6526(-04) | 0.8192(-03) |
| $\frac{1}{64}$ | 0.1389(-08) | 0.1339(-07) | 0.1222(-06) | 0.3438(-05) | 0.1515(-04) | 0.1713(-03) |

$$u(0,t)=0, \qquad u(1,t)=\sinh t, \quad t\geq 0. \tag{8.5c}$$

We solve equation (8.5a) using the method (3.8) in the region bounded by $0<x<1$, $t>0$. The exact solution is given by $u(x,t)=x^2\sinh t$. The MAE are tabulated in Table 3 for $\gamma=1,5$ and 10 at $t=2.0$.

**Example 4** (Quasi-linear hyperbolic equation)

$$\frac{\partial^2 u}{\partial t^2}=\left(1+x^2+u^2\right)\frac{\partial^2 u}{\partial x^2}+\gamma u\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}\right)+f(x,t), \quad 0<x<1, t>0. \tag{8.6a}$$

The initial and the boundary conditions are given by

$$u(x,0)=\sinh x, \qquad u_t(x,0)=-\sinh x, \quad 0\leq x\leq 1, \tag{8.6b}$$

$$u(0,t)=0, \qquad u(1,t)=e^{-t}\sinh 1, \quad t\geq 0. \tag{8.6c}$$

The exact solution is given by $u=e^{-t}\sinh x$. The MAE are tabulated in Table 4 for $\gamma=1,5$ and 10 at $t=2$.

The order of convergence may be obtained by using the formula

$$\frac{\log(e_{h_1})-\log(e_{h_2})}{\log(h_1)-\log(h_2)}, \tag{8.7}$$

where $e_{h_1}$ and $e_{h_2}$ are maximum absolute errors for two uniform mesh widths $h_1$ and $h_2$, respectively. For computation of order of convergence of the proposed method, we have considered $h_1=\frac{1}{32}$ and $h_2=\frac{1}{64}$ for all cases, and results are reported in Table 5.

## 9 Concluding remarks

Available numerical methods based on spline in tension approximations for the numerical solution of second-order quasilinear hyperbolic equations are of $O(k^2+h^2)$ accuracy only and require 9-grid points. In this article, using the same number of grid points and

**Table 5 Order of convergence: $h_1 = \frac{1}{32}$, $h_2 = \frac{1}{64}$**

| Example | Parameters | Order of the method |
|---|---|---|
| 01 | $\gamma = 1$ at $t = 5$ | 4.002 |
| | $\gamma = 2$ at $t = 5$ | 3.999 |
| 02 | $\alpha = 6$, $\beta = 4$, $\eta = 0.5$, $\gamma = 1$ at $t = 2$ | 3.998 |
| | $\alpha = \pi$, $\beta = \pi$, $\eta = 1$, $\gamma = 1$ at $t = 2$ | 3.998 |
| | $\alpha = 3\pi$, $\beta = \pi$, $\eta = 10$, $\gamma = 20$ at $t = 2$ | 3.994 |
| 03 | $\gamma = 1$ at $t = 1$ | 4.199 |
| | $\gamma = 5$ at $t = 2$ | 4.000 |
| | $\gamma = 10$ at $t = 2$ | 4.030 |
| 04 | $\gamma = 1$ at $t = 2$ | 4.001 |
| | $\gamma = 2$ at $t = 2$ | 4.005 |
| | $\gamma = 3$ at $t = 2$ | 3.997 |

three evaluations of the function $g$ (as compared to five and nine evaluations of the function $g$ discussed in [33] and [35]), we have derived a new stable spline in tension finite difference method of $O(k^2 + h^4)$ accuracy for the solution of second-order quasi-linear hyperbolic equation (1.1). For a fixed parameter $\sigma = \frac{k}{h^2}$, the proposed method behaves like a fourth-order method, which is exhibited by the computed results. The proposed numerical method for the wave equation in polar coordinates is conditionally stable, whereas for the damped wave equation and the telegraphic equation, the method is shown to be unconditionally stable. From Table 5, we found that the order of the method is nearly equal to four.

**Competing interests**

The authors declare that they have no competing interest.

**Authors' contributions**

RKM derived the difference method based on spline in tension approximation and discussed the convergence analysis of the method. VG has discussed the application of proposed method to wave equation in polar coordinates and telegraphic equation, and stability analysis. VG also carried out all computational work. All authors read and approved the final manuscript.

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