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On the iterated exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$

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Abstract

In this paper, the authors investigate the iterated exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$ ($j = 0, 1, 2, \ldots$), where f is a solution of some second-order linear differential equation, $\varphi(z) \not\equiv 0$ is an entire function satisfying $\sigma_{p+1}(\varphi) < \sigma_{p+1}(f)$ or $i(\varphi) < i(f)$ ($p \in \mathbb{N}$). We obtain some results which improve and generalize some previous results in (Chen in Acta Math. Sci. Ser. A 20(3):425-432, 2000; Chen and Shon in Chin. Ann. Math. Ser. A 27(4):431-442, 2006; Tu *et al.* in Electron. J. Qual. Theory Differ. Equ. 23:1-17, 2011) and provide us with a method to investigate the iterated exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$ ($j = 0, 1, 2, \ldots$).

MSC: 34A20; 30D35

Keywords: linear differential equations; iterated order; iterated type; iterated exponent of convergence of zero-sequence

1 Introduction

In this paper, we assume that readers are familiar with the fundamental results and standard notation of Nevanlinna's theory of meromorphic functions (see [1, 2]). First, we introduce some notations. Let us define inductively, for $r \in [0, \infty)$, $\exp_1 r = e^r$ and $\exp_{j+1} r = \exp(\exp_j r)$, $j \in N$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{j+1} r = \log(\log_j r)$, $j \in N$; we also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the linear measure and the logarithmic measure of a set $E \subset [1, +\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E dt/t$ respectively. Let $f(z)$, $a(z)$ be meromorphic functions in the complex plane satisfying $T(r,a) = o\{T(r,f)\}$ except possibly for a set of r having finite logarithmic measure, then we call that $a(z)$ is a small function of $f(z)$. We use p to denote a positive integer throughout this paper, not necessarily the same at each occurrence. In order to describe the infinite order of fast growing entire functions precisely, we recall some definitions of entire functions of finite iterated order (*e.g.*, see [3–8]).

Definition 1.1 The p-iterated order of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r}. \tag{1.1}$$

Remark 1.1 If $f(z)$ is an entire function, the p-iterated order of $f(z)$ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log r}. \tag{1.2}$$

It is easy to see that $\sigma_p(f) = \infty$ if $\sigma_{p+1}(f) > 0$. If $p = 2$, the hyper-order of $f(z)$ is defined by (see [9])

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_3 M(r,f)}{\log r}. \tag{1.3}$$

Definition 1.2 The p-iterated type of an entire function $f(z)$ with $0 < \sigma_p(f) = \sigma < \infty$ is defined by

$$\tau_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p M(r,f)}{r^\sigma}. \tag{1.4}$$

Definition 1.3 The finiteness degree of the iterated order of an entire function $f(z)$ is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ polynomial,} \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\} & \text{for } f \text{ transcendental for which some} \\ & j \in \mathbb{N} \text{ with } \sigma_j(f) < \infty \text{ exists,} \\ \infty & \text{for } f \text{ with } \sigma_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.5}$$

Definition 1.4 Suppose that $\varphi(z)$ is an entire function satisfying $\sigma_p(\varphi) < \sigma_p(f)$ or $i(\varphi) < i(f)$, then the p-iterated order exponent of convergence of zero-sequence of $f(z) - \varphi(z)$ is defined by

$$\lambda_p(f - \varphi) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f-\varphi})}{\log r}. \tag{1.6}$$

Especially, if $\varphi(z) = z$, the p-iterated order exponent of convergence of fixed points of $f(z)$ is defined to be

$$\lambda_p(f - z) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f-z})}{\log r}. \tag{1.7}$$

If $\varphi(z) = 0$, the p-iterated exponent of convergence of zero-sequence of $f(z)$ is defined to be

$$\lambda_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log r}. \tag{1.8}$$

Definition 1.5 The p-iterated exponent of convergence of distinct zero-sequence of $f(z) - \varphi(z)$ and the p-iterated exponent of convergence of distinct fixed points of $f(z)$ are respectively defined to be

$$\begin{aligned} \overline{\lambda}_p(f - \varphi) &= \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f-\varphi})}{\log r}, \\ \overline{\lambda}_p(f - z) &= \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r, \frac{1}{f-z})}{\log r}. \end{aligned} \tag{1.9}$$

Definition 1.6 If $\varphi(z)$ is an entire function satisfying $\sigma_p(\varphi) < \sigma_p(f)$ or $i(\varphi) < i(f)$, then the finiteness degree of the iterated exponent of convergence of zero-sequence of $f(z) - \varphi(z)$ is defined by

$$i_\lambda(f-\varphi) = \begin{cases} 0 & \text{for } N(r, \frac{1}{f-\varphi}) = O(\log r), \\ \min\{j \in \mathbb{N} : \lambda_j(f-\varphi) < \infty\} & \text{for some } j \in \mathbb{N} \\ & \text{with } \lambda_j(f-\varphi) < \infty \text{ exists}, \\ \infty & \text{if } \lambda_j(f-\varphi) = \infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.10}$$

Remark 1.2 From Definitions 1.5 and 1.6, we can similarly give the definitions of $\overline{\lambda}_p(f)$, $i_\lambda(f-z)$, $i_{\overline{\lambda}}(f-\varphi)$ and $i_{\overline{\lambda}}(f)$.

2 Main result

In [10], Chen firstly investigated the fixed points of the solutions of equations (2.1) and (2.2) with a polynomial coefficient and a transcendental entire coefficient of finite order and obtained the following Theorems A and B. Two years later in [11], Chen investigated the zeros of $f^{(j)}(z) - \varphi(z)$ $(j = 0, 1, 2)$ and obtained the following Theorems C and D, where $f(z)$ is a solution of equation (2.3) or (2.4), $\varphi(z)$ is an entire function satisfying $\sigma(\varphi) < \infty$. In [12], Tu, Xu and Zhang investigated the hyper-exponent of convergence of zeros of $f^{(j)}(z) - \varphi(z)$ $(j \in \mathbb{N})$ and obtained the following Theorem E, where $f(z)$ is a solution of (2.5), $\varphi(z)$ is an entire function satisfying $\sigma_2(\varphi) < \sigma(B)$. One year later, Xu, Tu and Zheng improved Theorem E to Theorem F in [13] from (2.5) to (2.6). In the following, we list Theorems A-F which have been mentioned above.

Theorem A [10] *Let $P(z)$ be a polynomial with degree n (≥ 1). Then every non-trivial solution of*

$$f'' + P(z)f = 0 \tag{2.1}$$

has infinitely many fixed points and satisfies $\overline{\lambda}(f-z) = \lambda(f-z) = \sigma(f) = \frac{n+2}{2}$.

Theorem B [10] *Let $A(z)$ be a transcendental entire function with $\sigma(A) = \sigma < \infty$. Then every non-trivial solution of*

$$f'' + A(z)f = 0 \tag{2.2}$$

has infinitely many fixed points and satisfies $\overline{\lambda}_2(f-z) = \lambda_2(f-z) = \sigma_2(f) = \sigma$.

Theorem C [11] *Let $A_j(z)$ $(\not\equiv 0)$ $(j = 1, 2)$ be entire functions with $\sigma(A_j) < 1$. a, b are complex numbers and satisfy $ab \neq 0$ and $\arg a \neq \arg b$ or $a = cb$ $(0 < c < 1)$. If $\varphi(z) \not\equiv 0$ is an entire function of finite order, then every non-trivial solution f of*

$$f'' + A_1(z)e^{az}f' + A_2(z)e^{bz}f = 0 \tag{2.3}$$

satisfies $\overline{\lambda}(f-\varphi) = \overline{\lambda}(f'-\varphi) = \overline{\lambda}(f''-\varphi) = \infty$.

Theorem D [11] *Let $A_1(z) \not\equiv 0$, $\varphi(z) \not\equiv 0$, $Q(z)$ be entire functions with $\sigma(A_1) < 1$ and $1 < \sigma(Q) < \infty$. Then every non-trivial solution f of*

$$f'' + A_1(z)e^{az}f' + Q(z)f = 0 \tag{2.4}$$

satisfies $\overline{\lambda}(f - \varphi) = \overline{\lambda}(f' - \varphi) = \overline{\lambda}(f'' - \varphi) = \infty$, where $a \neq 0$ is a complex number.

Theorem E [12] *Let $A(z)$ and $B(z)$ be entire functions with finite order. If $\sigma(A) < \sigma(B) < \infty$ or $0 < \sigma(A) = \sigma(B) < \infty$ and $0 \le \tau(A) < \tau(B) < \infty$, then for every solution $f \not\equiv 0$ of*

$$f'' + A(z)f' + B(z)f = 0 \tag{2.5}$$

and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(B)$, we have
(i) $\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = \overline{\lambda}_2(f''' - \varphi) = \sigma_2(f) = \sigma(B)$;
(ii) $\overline{\lambda}_2(f^{(j)} - \varphi) = \sigma_2(f) = \sigma(B)$ $(j > 3, j \in \mathbb{N})$.

Theorem F [13] *Let $A_j(z)$ $(j = 0, 1, \ldots, k-1)$ be entire functions of finite order and satisfy one of the following conditions*:
(i) $\max\{\sigma(A_j), j = 1, 2, \ldots, k-1\} < \sigma(A_0) < \infty$;
(ii) $0 < \sigma(A_{k-1}) = \cdots = \sigma(A_1) = \sigma(A_0) < \infty$ *and* $\max\{\tau(A_j), j = 1, 2, \ldots, k-1\} < \tau(A_0) < \infty$.
Then for every solution $f \not\equiv 0$ of

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = 0 \tag{2.6}$$

and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_2(\varphi) < \sigma(A_0)$, we have

$$\overline{\lambda}_2\bigl(f^{(j)} - \varphi\bigr) = \sigma_2(f) = \sigma(A_0), \quad j \in \mathbb{N}.$$

The main purpose of this paper is to improve Theorem E from entire coefficients of finite order in (2.5) to entire coefficients of finite iterated order. And we obtain the following results.

Theorem 2.1 *Let $A(z)$ and $B(z)$ be entire functions of finite iterated order satisfying $\sigma_p(A) < \sigma_p(B) < \infty$ or $0 < \sigma_p(A) = \sigma_p(B) < \infty$ and $0 \le \tau_p(A) < \tau_p(B) \le \infty$. Then for every solution $f \not\equiv 0$ of (2.5) and for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_{p+1}(\varphi) < \sigma_p(B)$, we have*
(i) $\overline{\lambda}_{p+1}(f - \varphi) = \overline{\lambda}_{p+1}(f' - \varphi) = \overline{\lambda}_{p+1}(f'' - \varphi) = \overline{\lambda}_{p+1}(f''' - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$;
(ii) $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$, $j > 3$, $j \in \mathbb{N}$.

Theorem 2.2 *Let $A(z)$, $B(z)$ be entire functions satisfying $i(A) < i(B) = p$. Then for every solution $f \not\equiv 0$ of (2.5) and for any entire functions $\varphi(z) \not\equiv 0$ with $i(\varphi) \le p$, we have*
(i) $i_{\overline{\lambda}}(f^{(j)} - \varphi) = i_{\lambda}(f^{(j)} - \varphi) = i(f^{(j)} - \varphi) = p + 1$ $(j = 0, 1, 2, \ldots)$;
(ii) $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f^{(j)} - \varphi) = \sigma_p(B)$ $(j = 0, 1, 2, \ldots)$.

Theorem 2.3 *Under the hypotheses of Theorem 2.1, let $L(f) = a_k f^{(k)} + a_{k-1}f^{(k-1)} + \cdots + a_0 f$, where a_j $(j = 0, 1, \ldots, k)$ are entire functions which are not all equal to zero and satisfy $\sigma_p(a_j) < \sigma_p(B)$. Then for any solution $f \not\equiv 0$ of (2.5), we have $\sigma_{p+1}(L(f)) = \sigma_{p+1}(f) = \sigma_p(B)$.*

Corollary 2.1 *Under the hypotheses of Theorem* 2.1, *if* $\varphi(z) = z$, *we have*

(i) $\overline{\lambda}_{p+1}(f - z) = \overline{\lambda}_{p+1}(f' - z) = \overline{\lambda}_{p+1}(f'' - z) = \overline{\lambda}_{p+1}(f''' - z) = \sigma_{p+1}(f) = \sigma_p(B)$;

(ii) $\overline{\lambda}_{p+1}(f^{(j)} - z) = \sigma_{p+1}(f) = \sigma_p(B)$, $j > 3$, $j \in \mathbb{N}$.

Corollary 2.2 *Under the hypotheses of Theorem* 2.2, *if* $\varphi(z) = z$, *we have*

(i) $i_{\overline{\lambda}}(f^{(j)} - z) = i_{\lambda}(f^{(j)} - z) = i(f^{(j)} - z) = p + 1$ $(j = 0, 1, 2, \ldots)$;

(ii) $\overline{\lambda}_{p+1}(f^{(j)} - z) = \lambda_{p+1}(f^{(j)} - z) = \sigma_{p+1}(f^{(j)} - z) = \sigma_p(B)$ $(j = 0, 1, 2, \ldots)$.

Remark 2.1 Theorem 2.1 is an extension and improvement of Theorem E. As for Theorem C, if $a = cb$ $(0 < c < 1)$, it is easy to see that $\sigma(A_1e^{az}) = \sigma(A_2e^{bz}) = 1$ and $\tau(A_1e^{az}) = a < \tau(A_2e^{bz}) = b$. By Theorem E, for every solution $f \not\equiv 0$ of (2.3) and for any entire function $\varphi(z) \not\equiv 0$ with $\sigma_2(\varphi) < 1$, we have $\overline{\lambda}_2(f - \varphi) = \overline{\lambda}_2(f' - \varphi) = \overline{\lambda}_2(f'' - \varphi) = 1$, therefore Theorem E is also a partial extension of Theorem C. Theorem B is a special case of Corollary 2.1 for $p = 1$.

Remark 2.2 Nevanlinna's second fundamental theorem is an important tool to investigate the distribution of zeros of meromorphic functions. From Nevanlinna's second fundamental theorem [1, p.47, Theorem 2.5], we have that

$$\bigl(1 + o(1)\bigr)T(r,f) < \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f - \varphi}\right) + S(r,f),$$

where $\varphi(z)$ is a small function of $f(z)$. For example, set $f(z) = e^{a(z)}$, $a(z)$ is a transcendental entire function with $i(a) = p$, then we have $\overline{\lambda}_{p+1}(f - \varphi) = \sigma_{p+1}(f) = \sigma_p(a)$ if $i(\varphi) < p + 1$. Our Theorem 2.1 and Theorem 2.2 also provide us with a method to investigate the iterated exponent of zero sequence of $f^{(j)}(z) - \varphi(z)$ $(j = 0, 1, 2, \ldots)$, where $f(z)$ and $\varphi(z)$ are entire functions satisfying $\sigma_{p+1}(\varphi) < \sigma_{p+1}(f)$ or $i(\varphi) < i(f)$. If we can find equation (2.5) with entire coefficients $A(z)$, $B(z)$ satisfying $\sigma_p(A) < \sigma_p(B)$ or $0 < \sigma_p(A) = \sigma_p(B) < \infty$ and $0 \le \tau_p(A) < \tau_p(B) \le \infty$ such that $f(z)$ is a solution of (2.5), then we have $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_p(B)$ $(j = 0, 1, 2, \ldots)$. By the above example, set $f(z) = e^{a(z)}$, $a(z)$ is transcendental with $i(a) = p$, then $f(z)$ is a solution of $f'' - (a'' + a'^2)f = 0$. Since $\sigma_p(a'' + a'^2) = \sigma_p(a)$ and by Theorem 2.1, we have $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_p(a)$ $(j = 0, 1, 2, \ldots)$ for any entire function $\varphi(z) \not\equiv 0$ satisfying $\sigma_{p+1}(\varphi) < \sigma_p(a)$ or $i(\varphi) < p + 1$.

3 Lemmas

Lemma 3.1 [5, 7] *Let* $f(z)$ *be an entire function with* $\sigma_p(f) = \sigma$, *and* $\nu_f(r)$ *denote the central index of* $f(z)$. *Then*

$$\overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log r} = \sigma. \tag{3.1}$$

Lemma 3.2 *Let* $f(z)$ *be an entire function with* $\sigma_p(f) = \sigma$, *then there exists a set* $E_1 \subset [1, +\infty)$ *with infinite logarithmic measure such that for all* $r \in E_1$, *we have*

$$\lim_{r\to\infty} \frac{\log_p T(r,f)}{\log r} = \sigma, \quad r \in E_1. \tag{3.2}$$

Proof By Definition 1.1 $\overline{\lim}_{r\to\infty}\frac{\log_p T(r,f)}{\log r} = \sigma$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ satisfying $(1+\frac{1}{n})r_n < r_{n+1}$ and

$$\lim_{r_n\to\infty}\frac{\log_p T(r_n,f)}{\log r_n} = \sigma. \tag{3.3}$$

There exists an n_1 such that for $n \geq n_1$ and for any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n,f)}{\log(1+\frac{1}{n})r_n} \leq \frac{\log_p T(r,f)}{\log r}. \tag{3.4}$$

Set $E_1 = \bigcup_{n=n_1}^{\infty}[r_n, (1+\frac{1}{n})r_n]$, by (3.4) and Definition 1.1, then for any $r \in E_1$, we have

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log r} = \lim_{r_n\to\infty}\frac{\log_p T(r_n,f)}{\log r_n} = \sigma,$$

and $m_l E_1 = \sum_{n=n_1}^{\infty}\int_{r_n}^{(1+\frac{1}{n})r_n}\frac{dt}{t} = \sum_{n=n_1}^{\infty}\log(1+\frac{1}{n}) = \infty$. Thus, we complete the proof of this lemma. □

By Lemma 3.1 and the same proof in Lemma 3.2, we have the following lemma.

Lemma 3.3 *Let $f(z)$ be an entire function with $\sigma_p(f) = \sigma$ and $\nu_f(r)$ denote the central index of $f(z)$. Then there exists a set $E_1 \subset [1,+\infty)$ with infinite logarithmic measure such that for all $r \in E_1$, we have*

$$\lim_{r\to\infty}\frac{\log_p \nu_f(r)}{\log r} = \sigma, \quad r \in E_1. \tag{3.5}$$

Lemma 3.4 [5, 7] *Let $A_0, A_1, \ldots, A_{k-1}, F \not\equiv 0$ be meromorphic functions. If f is a meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_0 f = F, \tag{3.6}$$

then we have the following two statements:

(i) *If* $\max\{i(A_j), j = 0,1,\ldots,k-1, i(F)\} < i(f)$, *then* $i_{\overline{\lambda}}(f) = i_{\lambda}(f) = i(f)$;

(ii) *If* $\max\{\sigma_p(A_j), j = 0,1,\ldots,k-1, \sigma_p(F)\} < \sigma_p(f)$, *then* $\overline{\lambda}_p(f) = \lambda_p(f) = \sigma_p(f)$.

Lemma 3.5 [14] *Let $f(z)$ be an entire function of finite iterated order with $i(f) = p$. Then there exist entire functions $\beta(z)$ and $D(z)$ such that*

$$f(z) = \beta(z)e^{D(z)},$$
$$\sigma_p(f) = \max\{\sigma_p(\beta), \sigma_p(e^{D(z)})\}$$

and

$$\sigma_p(\beta) = \overline{\lim_{r\to\infty}}\frac{\log_p N(r,\frac{1}{f})}{\log r}.$$

Moreover, for any $\varepsilon > 0$, we have

$$\log\left|\beta(z)\right| \geq -\exp_{p-1}\left\{r^{\sigma_p(\beta)+\varepsilon}\right\}, \quad r \notin E_2, \tag{3.7}$$

where $E_2 \subset [1,+\infty)$ is a set of r of finite linear measure.

Lemma 3.6 *Let $f(z)$ be an entire function of finite iterated order with $\sigma_p(f) = \sigma < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E_3 \subset [1,\infty)$ that has finite linear measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$, we have*

$$\exp\left\{-\exp_{p-1} r^{\sigma+\varepsilon}\right\} \leq \left|f(z)\right| \leq \exp_p\left\{r^{\sigma+\varepsilon}\right\}. \tag{3.8}$$

Proof Let $f(z)$ be an entire function of finite iterated order with $\sigma_p(f) = \sigma$. By Definition 1.1, it is easy to obtain that $|f(z)| < \exp_p\{r^{\sigma+\varepsilon}\}$ holds for all sufficiently larger $|z| = r$. By Lemma 3.5, there exist entire functions $\beta(z)$ and $D(z)$ such that

$$f(z) = \beta(z)e^{D(z)}, \qquad \sigma_p(f) = \max\left\{\sigma_p(\beta), \sigma_p\left(e^{D(z)}\right)\right\}.$$

For any $\varepsilon > 0$, we have

$$\left|\beta(z)\right| \geq \exp\left\{-\exp_{p-1}\left\{r^{\sigma_p(\beta)+\varepsilon}\right\}\right\} \geq \exp\left\{-\exp_{p-1}\left\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\right\}\right\}, \quad r \notin E_3, \tag{3.9}$$

hold outside a set $E_3 \subset [1,+\infty)$ of finite linear measure. Since $\sigma_{p-1}(D(z)) = \sigma_p(e^{D(z)}) \leq \sigma_p(f)$, by Definition 1.1, we have that $|D(z)| \leq \exp_{p-1}\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}$ holds for all sufficiently large r. From $|e^{D(z)}| \geq e^{-|D(z)|} \geq \exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}\}$ and (3.9), we have

$$\begin{aligned}\left|f(z)\right| &\geq \left|\beta(z)\right|e^{-|D(z)|} \geq \exp\left\{-2\exp_{p-1}\left\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\right\}\right\} \\ &\geq \exp\left\{-\exp_{p-1}\left\{r^{\sigma_p(f)+\varepsilon}\right\}\right\}, \quad r \notin E_3,\end{aligned} \tag{3.10}$$

where E_3 is a set of r of finite linear measure. By Definition 1.1 and (3.10), we obtain the conclusion of Lemma 3.6. □

Remark 3.1 Lemma 3.6 gives the modulus estimation of an entire function with finite iterated order and extends the conclusion of [15, p.84, Lemma 4].

Lemma 3.7 *Let $f(z)$ be an entire function of finite iterated order with $\sigma_p(f) = \sigma > 0$ $(p \geq 2)$, and let $L(f) = a_2f'' + a_1f' + a_0f$, where a_0, a_1, a_2 are entire functions of finite iterated order which are not all equal to zero and satisfy $b = \max\{\sigma_{p-1}(a_j), j = 0,1,2\} < \alpha$, then $\sigma_p(L(f)) = \sigma_p(f) = \sigma$.*

Proof $L(f)$ can be written as

$$L(f) = f\left(a_2\frac{f''}{f} + a_1\frac{f'}{f} + a_0\right). \tag{3.11}$$

By the Wiman-Valiron lemma (see [2, 16]), for all z satisfying $|z| = r$ and $|f(z)| = M(r,f)$, we have

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^k\left(1 + o(1)\right), \quad k \in N, r \notin E_4, \tag{3.12}$$

where E_4 is a set of finite logarithmic measure. From (1.4.5) in [16, p.26], for any given $\varepsilon > 0$, we have

$$\nu_f(r) < \left[\log \mu_f(r)\right]^{1+\varepsilon} \tag{3.13}$$

holds outside a set E_5 with finite logarithmic measure, where $\mu_f(r)$ is the maximum term of f. By the Cauchy inequality, we have $\mu_f(r) \le M(r,f)$. Substituting it into (3.13), we have

$$\nu_f(r) < \left[\log M(r,f)\right]^{1+\varepsilon}, \quad r \notin E_5. \tag{3.14}$$

By Lemma 3.1, there exists a set E_1 having infinite logarithmic measure such that for all $|z| = r \in E_1$, we have

$$\lim_{r\to\infty} \frac{\log_p \nu_f(r)}{\log r} = \sigma, \quad r \in E_1. \tag{3.15}$$

By (3.15) and Lemma 3.6, for all $r \in E_1 - E_3$ and for any ε $(0 < 2\varepsilon < \sigma - b)$, we have

$$\begin{aligned} \exp\{-\exp_{p-2} r^{b+\varepsilon}\} < \left|a_j(z)\right| < \exp_{p-1}\{r^{b+\varepsilon}\} < \exp_{p-1}\{r^{\sigma-\varepsilon}\} \\ < \nu_f(r) < \exp_{p-1}\{r^{\sigma+\varepsilon}\} \quad (j = 0,1,2). \end{aligned} \tag{3.16}$$

Substituting (3.16) into (3.14), we have

$$\exp_p\{r^{\sigma-2\varepsilon}\} < M(r,f) \quad \big(r \in E_1 - (E_3 \cup E_5)\big). \tag{3.17}$$

By (3.11), we have

$$\left|L(f)\right| = |f|\left|a_2\frac{f''}{f} + a_1\frac{f'}{f} + a_0\right| \ge |f|\left[\left|a_2\frac{f''}{f} + a_1\frac{f'}{f}\right| - |a_0|\right]. \tag{3.18}$$

Substituting (3.12), (3.16), (3.17) into (3.18), for all z satisfying $|f(z)| = M(r,f)$ and $|z| = r \in E_1 - (E_3 \cup E_4 \cup E_5)$, we have

$$\begin{aligned} \left|L(f)\right| &\ge |f|\left[\left|\frac{\nu_f(r)}{z}\left(a_2\frac{\nu_f(r)}{z} + a_1\right)\right| - |a_0|\right] \\ &\ge |f|\left[\left|\frac{\nu_f(r)}{z}\right|\left|\left|a_2\frac{\nu_f(r)}{z}\right| - |a_1|\right| - |a_0|\right] \\ &\ge \exp_p\{r^{\sigma-2\varepsilon}\}\left[\exp_{p-1}\{r^{\sigma-\varepsilon}\} - \exp_{p-1}\{r^{b+\varepsilon}\}\right]. \end{aligned} \tag{3.19}$$

By (3.19), we can obtain that $\sigma_p(L(f)) \ge \sigma_p(f)$. On the other hand, it is easy to get $\sigma_p(L(f)) \le \sigma_p(f)$. Hence $\sigma_p(L(f)) = \sigma_p(f)$. □

Remark 3.2 The assumption $\sigma_{p-1}(a_j) < \sigma_p(f)$ in Lemma 3.7 is necessary. For example, if $a(z)$ is an entire function satisfying $\sigma_{p-1}(a) > 0$ $(p \ge 2)$, set $f(z) = e^{a(z)}$, $L(f) = f'' - a'f - a''f$, then we have $\sigma_p(f) = \sigma_{p-1}(a) > 0$ and $L(f) \equiv 0$, *i.e.*, $\sigma_p(L(f)) = 0 < \sigma_p(f)$.

By a similar proof to that in Lemma 3.7, we can easily get the following lemma.

Lemma 3.8 *Let $f(z)$ be an entire function with $\sigma_p(f) = \sigma > 0$ ($p \geq 2$) and $L(f) = a_k f^{(k)} + a_{k-1} f^{(k-1)} + \cdots + a_0 f$, where $a_0, a_1, \ldots, a_k$ are entire functions which are not all equal to zero satisfying $b = \max\{\sigma_{p-1}(a_j), j = 0, 1, \ldots, k\} < \sigma$. Then $\sigma_p(L(f)) = \sigma_p(f) = \sigma$.*

Remark 3.3 By a similar proof to that of Lemma 2.2 in [3] or Lemma 6 in [8], we can easily get the following lemma which is a better result than that in [3] or [8] by allowing $\tau_p(f) = \infty$.

Lemma 3.9 *Let $f(z)$ be an entire function satisfying $0 < \sigma_p(f) = \sigma < \infty$, $0 < \tau_p(f) = \tau \leq \infty$, then for any given $\beta < \tau$, there exists a set $E_6 \subset [1, +\infty)$ that has infinite logarithmic measure such that for all $r \in E_6$, we have*

$$\log_p M(r, f) > \beta r^{\sigma}. \tag{3.20}$$

Lemma 3.10 [17] *Let $f(z)$ be a transcendental meromorphic function and $\alpha > 1$ be a given constant, for any given $\varepsilon > 0$, there exists a set $E_7 \subset [1, \infty)$ that has finite logarithmic measure and a constant $B > 0$ that depends only on α and (m, n) ($m, n \in \{0, \ldots, k\}$) with $m < n$ such that for all z satisfying $|z| = r \notin [0, 1] \cup E_7$, we have*

$$\left| \frac{f^{(n)}(z)}{f^{(m)}(z)} \right| \leq B \left(\frac{T(\alpha r, f)}{r} \left(\log^{\alpha} r\right) \log T(\alpha r, f) \right)^{n-m}. \tag{3.21}$$

Lemma 3.11 [6] *Let $A_j(z)$ ($j = 0, 1, \ldots, k-1$) be entire functions with finite iterated order satisfying $\max\{\sigma_p(A_j), j \neq 0\} < \sigma_p(A_0)$, then every solution $f \not\equiv 0$ of (2.6) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

Lemma 3.12 [3, 8] *Let $A_j(z)$ ($j = 0, 1, \ldots, k-1$) be entire functions with finite iterated order satisfying $\max\{\sigma_p(A_j), j \neq 0\} \leq \sigma_p(A_0)$ ($0 < \sigma_p(A_0) < \infty$) and $\max\{\tau_p(A_j), \sigma_p(A_j) = \sigma_p(A_0), j \neq 0\} < \tau_p(A_0) < \infty$. Then every solution $f \not\equiv 0$ of (2.6) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0)$.*

Remark 3.4 The conclusion of Lemma 3.12 also holds if $\tau_p(A_0) = \infty$.

Lemma 3.13 *Let $A(z)$, $B(z)$ be entire functions of finite iterated order satisfying $\sigma_p(A) < \sigma_p(B)$. Let $G(z)$, $H(z)$ be meromorphic functions with $\sigma_p(H) \leq \sigma_p(A)$, $\sigma_p(G) \leq \sigma_p(B)$, if $f(z)$ is an entire solution of equation*

$$f'' + \left(A + \frac{G'}{G}\right) f' + \left(B + H' + \frac{HG'}{G}\right) f = 0, \tag{3.22}$$

then $\sigma_{p+1}(f) \geq \sigma_p(B)$.

Proof From (3.22), we have

$$m\left(r, B + H' + \frac{HG'}{G}\right) \leq m\left(r, \frac{f''}{f}\right) + m\left(r, \frac{f'}{f}\right) + m\left(r, A + \frac{G'}{G}\right). \tag{3.23}$$

By the lemma of logarithmic derivative and (3.23), we have

$$m(r, B) \leq O\left\{\log rT(r, f)\right\} + m(r, A) + 2T(r, H) + O\left\{\log rT(r, G)\right\}, \quad r \notin E_0, \tag{3.24}$$

where E_0 is a set having finite linear measure. By Lemma 3.2, there exists a set E_1 having infinite logarithmic measure such that for all $|z| = r \in E_1 - E_0$, we have

$$\exp_{p-1}\{r^{\sigma_p(B)-\varepsilon}\} \leq O\{\log rT(r,f)\} + 4\exp_{p-1}\{r^{\sigma_p(A)+\varepsilon}\}, \tag{3.25}$$

where $0 < 2\varepsilon < \sigma(B) - \sigma(A)$. By (3.25), we have $\sigma_{p+1}(f) \geq \sigma_p(B)$. □

Lemma 3.14 *Let $A(z)$, $B(z)$ be entire functions satisfying $0 < \sigma_p(A) = \sigma_p(B) = \sigma_2 < \infty$ and $\tau_p(A) < \tau_p(B) \leq \infty$ and let $G(z)$, $H(z)$ be meromorphic functions satisfying $\sigma_p(H) \leq \sigma_2$, $\sigma_p(G) \leq \sigma_2$ and $|H^{(j)}(z)| \leq \exp_p\{(\tau(A) + \varepsilon)r^{\sigma_2}\}$ $(j = 0,1)$ outside of a set E_8 of finite logarithmic measure, where $0 < 2\varepsilon < \tau_p(B) - \tau_p(A)$. If $f(z)$ is an entire solution of* (3.22), *then $\sigma_{p+1}(f) \geq \sigma_2$.*

Proof Without loss of generality, we suppose that $\tau_p(A) < \tau_p(B) < \infty$. From (3.22), we have

$$|B(z)| \leq \left|\frac{f''}{f}\right| + \left[|A| + \left|\frac{G'}{G}\right|\right]\left|\frac{f'}{f}\right| + |H'| + |H|\left|\frac{G'}{G}\right|. \tag{3.26}$$

By Lemma 3.9, for any given β ($\tau_p(A) + 2\varepsilon < \beta < \tau_p(B)$), there exists a set E_6 having infinite logarithmic measure such that for all $|z| = r \in E_6$, we have

$$M(r,B) > \exp_p\{\beta r^{\sigma_2}\}. \tag{3.27}$$

By Lemma 3.10, there exists a set E_7 having finite logarithmic measure such that for all $|z| = r \notin E_7$, we have

$$\begin{aligned} &\left|\frac{f''}{f}\right| \leq B[T(2r,f)]^2, \qquad \left|\frac{f'}{f}\right| \leq B[T(2r,f)], \\ &\left|\frac{G'}{G}\right| \leq B[T(2r,G)] < \exp_{p-1}\{r^{\sigma_2+\varepsilon}\}, \end{aligned} \tag{3.28}$$

where $M > 0$ is a constant. By the hypotheses, for all $|z| = r \notin E_8$, we have

$$|H'| < \exp_p\{(\tau(A) + \varepsilon)r^{\sigma_2}\}, \qquad |H| < \exp_p\{(\tau(A) + \varepsilon)r^{\sigma_2}\}. \tag{3.29}$$

By (3.26)-(3.29), for all z satisfying $|B(z)| = M(r,B)$ and $|z| = r \in E_6 - (E_7 \cup E_8)$, we have

$$\exp_p\{\beta r^{\sigma_2}\} \leq 4\exp_p\{(\tau_p(A) + 2\varepsilon)r^{\sigma_2}\}[T(2r,f)]^2. \tag{3.30}$$

By (3.30), we have $\sigma_{p+1}(f) \geq \sigma_2$. □

By the above proof, we can easily obtain that Lemma 3.14 also holds if $\tau_p(B) = \infty$.

4 Proof of Theorem 2.1

Now we divide the proof of Theorem 2.1 into two cases: case (i) $\sigma_p(A) < \sigma_p(B)$ and case (ii) $\tau_p(A) < \tau_p(B)$ and $\sigma_p(A) = \sigma_p(B) > 0$.

Case (i): (1) We prove that $\overline{\lambda}_{p+1}(f - \varphi) = \sigma_{p+1}(f)$. Assume that $f \not\equiv 0$ is a solution of (2.5), then $\sigma_{p+1}(f) = \sigma_p(B)$ by Lemma 3.11. Set $g = f - \varphi$, since $\sigma_{p+1}(\varphi) < \sigma_p(B)$, then $\sigma_{p+1}(g) =$

$\sigma_{p+1}(f) = \sigma_p(B)$, $\overline{\lambda}_{p+1}(g) = \overline{\lambda}_{p+1}(f - \varphi)$. Substituting $f = g + \varphi$, $f' = g' + \varphi'$, $f'' = g'' + \varphi''$ into (2.5), we have

$$g'' + Ag' + Bg = -(\varphi'' + A\varphi' + B\varphi). \tag{4.1}$$

If $\varphi'' + A\varphi' + B\varphi \equiv 0$, by Lemma 3.11, we have $\sigma_{p+1}(\varphi) = \sigma_p(B)$, which is a contradiction. Since $\varphi'' + A\varphi' + B\varphi \not\equiv 0$ and $\sigma_{p+1}(\varphi'' + A\varphi' + B\varphi) < \sigma_{p+1}(f) = \sigma_{p+1}(g)$, by Lemma 3.4 and (4.1), we have $\overline{\lambda}_{p+1}(g) = \lambda_{p+1}(g) = \sigma_{p+1}(g) = \sigma_p(B)$, therefore $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$.

(2) We prove that $\overline{\lambda}_{p+1}(f' - \varphi) = \sigma_{p+1}(f)$. Set $g_1 = f' - \varphi$, then $\sigma_{p+1}(g_1) = \sigma_{p+1}(f) = \sigma_p(B)$ and

$$f' = g_1 + \varphi, \qquad f'' = g_1' + \varphi', \qquad f''' = g_1'' + \varphi''. \tag{4.2}$$

By (2.5), we get

$$f = -\frac{1}{B}(f'' + Af'). \tag{4.3}$$

The derivation of (2.5) is

$$f''' + Af'' + (A + B)f' + B'f = 0. \tag{4.4}$$

Substituting (4.2), (4.3) into (4.4), we obtain

$$\begin{aligned} &g_1'' + \left(A - \frac{B'}{B}\right)g_1' + \left(A' + B - \frac{AB'}{B}\right)g_1 \\ &\quad = -\left(\varphi'' + \left(A - \frac{B'}{B}\right)\varphi' + \left(A' + B - \frac{AB'}{B}\right)\varphi\right). \end{aligned} \tag{4.5}$$

Let $F_1 = \varphi'' + (A - \frac{B'}{B})\varphi' + (A' + B - \frac{AB'}{B})\varphi$. We affirm that $F_1 \not\equiv 0$. If $F_1 \equiv 0$, by Lemma 3.13, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(B)$, which is a contradiction; therefore $F_1 \not\equiv 0$. Since $\sigma_{p+1}(F_1) < \sigma_p(B) = \sigma_{p+1}(g_1)$, by Lemma 3.4 and (4.5), we get $\overline{\lambda}_{p+1}(f' - \varphi) = \lambda_{p+1}(f' - \varphi) = \sigma_{p+1}(f)$.

(3) We prove that $\overline{\lambda}_{p+1}(f'' - \varphi) = \sigma_{p+1}(f)$. Set $g_2 = f'' - \varphi$, then $\sigma_{p+1}(g_2) = \sigma_{p+1}(f) = \sigma_p(B)$ and

$$f'' = g_2 + \varphi, \qquad f''' = g_2' + \varphi', \qquad f^{(4)} = g_2'' + \varphi''. \tag{4.6}$$

Substituting (4.3) into (4.4), we have

$$f''' + \left(A - \frac{B'}{B}\right)f'' + \left(A' + B - \frac{AB'}{B}\right)f' = 0. \tag{4.7}$$

The derivation of (4.7) is

$$\begin{aligned} &f^{(4)} + \left(A - \frac{B'}{B}\right)f''' + \left[\left(A - \frac{B'}{B}\right)' + \left(A' + B - \frac{AB'}{B}\right)\right]f'' \\ &\quad - \frac{(A' + B - \frac{AB'}{B})'}{A' + B - \frac{AB'}{B}}\left[f''' + \left(A - \frac{B'}{B}\right)f''\right] = 0. \end{aligned} \tag{4.8}$$

Set $Q(z) = A' + B - \frac{AB'}{B}$, $S(z) = A - \frac{B'}{B}$, it is easy to see that

$$\overline{\lim_{r\to\infty}} \frac{\log_p m(r,Q)}{\log r} = \sigma_p(B) \quad \text{and} \quad \overline{\lim_{r\to\infty}} \frac{\log_p m(r,S)}{\log r} = \sigma_p(A),$$

then by (4.8), we get

$$f^{(4)} + \left(S - \frac{Q'}{Q}\right)f''' + \left(S' + Q - \frac{SQ'}{Q}\right)f'' = 0. \tag{4.9}$$

Substituting (4.6) into (4.9), we have

$$\begin{aligned} &g_2'' + \left(S - \frac{Q'}{Q}\right)g_2' + \left(S' + Q - \frac{SQ'}{Q}\right)g_2 \\ &\quad = -\left(\varphi'' + \left(S - \frac{Q'}{Q}\right)\varphi' + \left(S' + Q - \frac{SQ'}{Q}\right)\varphi\right). \end{aligned} \tag{4.10}$$

If $F_2(z) = \varphi'' + (S - \frac{Q'}{Q})\varphi' + (S' + Q - \frac{SQ'}{Q})\varphi \equiv 0$, by Lemma 3.13, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(B)$, which is a contradiction; therefore $F_2 \not\equiv 0$. Since $\sigma_{p+1}(F_2) < \sigma_p(B) = \sigma_{p+1}(g_2)$, by Lemma 3.4 and (4.10), we have $\overline{\lambda}_{p+1}(f'' - \varphi) = \lambda_{p+1}(f'' - \varphi) = \sigma_{p+1}(f)$.

(4) We prove that $\overline{\lambda}_{p+1}(f''' - \varphi) = \sigma_{p+1}(f)$. Set $g_3 = f''' - \varphi$, then $\sigma_{p+1}(g_3) = \sigma_{p+1}(f) = \sigma_p(B)$ and

$$g_3' = f^{(4)} - \varphi', \qquad g_3'' = f^{(5)} - \varphi''. \tag{4.11}$$

The derivation of (4.9) is

$$\begin{aligned} &f^{(5)} + \left(S - \frac{Q'}{Q}\right)f^{(4)} + \left(S - \frac{Q'}{Q}\right)' f''' + \left(S' + Q - \frac{SQ'}{Q}\right)f''' \\ &\quad + \left(S' + Q - \frac{SQ'}{Q}\right)' f'' = 0. \end{aligned} \tag{4.12}$$

By (4.9), we have

$$f'' = -\frac{1}{S' + Q - \frac{SQ'}{Q}}\left[f^{(4)} + \left(S - \frac{Q'}{Q}\right)f'''\right]. \tag{4.13}$$

Substituting (4.13) into (4.12), we have

$$\begin{aligned} &f^{(5)} + \left(S - \frac{Q'}{Q}\right)f^{(4)} + \left[\left(S - \frac{Q'}{Q}\right)' + \left(S' + Q - \frac{SQ'}{Q}\right)\right]f''' \\ &\quad - \frac{(S' + Q - \frac{SQ'}{Q})'}{S' + Q - \frac{SQ'}{Q}}\left[f^{(4)} + \left(S - \frac{Q'}{Q}\right)f'''\right] = 0. \end{aligned} \tag{4.14}$$

Let $U(z) = S' + Q - \frac{SQ'}{Q}$, $V(z) = S - \frac{Q'}{Q}$, it is easy to obtain that

$$\overline{\lim_{r\to\infty}} \frac{\log_p m(r,U)}{\log r} = \sigma_p(B) \quad \text{and} \quad \overline{\lim_{r\to\infty}} \frac{\log_p m(r,V)}{\log r} = \sigma_p(A),$$

by (4.14), we have

$$f^{(5)} + \left(V - \frac{U'}{U}\right)f^{(4)} + \left(V' + U - \frac{VU'}{U}\right)f''' = 0. \tag{4.15}$$

Substituting (4.11) into (4.15), we have

$$\begin{aligned} &g_3'' + \left(V - \frac{U'}{U}\right)g_3' + \left(V' + U - \frac{VU'}{U}\right)g_3 \\ &\quad = -\left(\varphi'' + \left(V - \frac{U'}{U}\right)\varphi' + \left(V' + U - \frac{VU'}{U}\right)\varphi\right). \end{aligned} \tag{4.16}$$

Let $F_3(z) = \varphi'' + (V - \frac{U'}{U})\varphi' + (V' + U - \frac{VU'}{U})\varphi$. By Lemma 3.13, we have $F_3(z) \not\equiv 0$. Since $\sigma_{p+1}(F_3) < \sigma_p(B) = \sigma_{p+1}(g_3)$, by Lemma 3.4 and (4.16), we have $\overline{\lambda}_{p+1}(f''' - \varphi) = \lambda_{p+1}(f''' - \varphi) = \sigma_{p+1}(f)$.

(5) We prove that $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f)$ $(j > 3)$. Set $f^{(j)} = g_j + \varphi$ $(j > 3)$, then $f^{(j+1)} = g_k' + \varphi'$, $f^{(j+2)} = g_j'' + \varphi''$ $(j > 3)$ and $\sigma_{p+1}(g_j) = \sigma_{p+1}(f^{(j)}) = \sigma_p(B)$. By successive derivation on (4.14), we can also get the following equation which has a similar form to (4.16):

$$\begin{aligned} &g_j'' + \left(A + \frac{G'}{G}\right)g_j' + \left(B + H' + \frac{HG'}{G}\right)g_j \\ &\quad = -\left(\varphi'' + \left(A + \frac{G'}{G}\right)\varphi' + \left(B + H' + \frac{HG'}{G}\right)\varphi\right), \end{aligned} \tag{4.17}$$

where G, H are meromorphic functions which have the same form as $U(z)$, $V(z)$ and satisfy $\sigma_p(G) \leq \sigma_p(B)$ and $\sigma_p(H) \leq \sigma_p(A)$. By Lemma 3.13, we have $F_j = \varphi'' + (A + \frac{G'}{G})\varphi' + (B + H' + \frac{HG'}{G})\varphi \not\equiv 0$. Since $\sigma_{p+1}(F_j) < \sigma_{p+1}(g_j) = \sigma_p(B)$, by Lemma 3.4, we have $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$ $(j > 3)$.

Case (ii): (1) We prove that $\overline{\lambda}_{p+1}(f - \varphi) = \sigma_{p+1}(f)$. Assume that $f \not\equiv 0$ is a solution of (2.5), by Lemma 3.12, we know that $\sigma_{p+1}(f) = \sigma_p(B) > 0$. Set $g = f - \varphi$, $\varphi \not\equiv 0$ is an entire function with $\sigma_{p+1}(\varphi) < \sigma_p(B)$, then we have $\sigma_{p+1}(g) = \sigma_{p+1}(f) = \sigma_p(B)$, $\overline{\lambda}_{p+1}(g) = \overline{\lambda}_{p+1}(f - \varphi)$. Substituting $f = g + \varphi$, $f' = g' + \varphi'$, $f'' = g'' + \varphi''$ into (2.6), we have (4.1). We affirm that $\varphi'' + A\varphi' + B\varphi \not\equiv 0$. If $\varphi'' + A\varphi' + B\varphi \equiv 0$, by Lemma 3.12, we have $\sigma_{p+1}(\varphi) = \sigma_p(B)$, which is a contradiction. Since $\varphi'' + A\varphi' + B\varphi \not\equiv 0$ and $\sigma_{p+1}(\varphi'' + A\varphi' + B\varphi) < \sigma_{p+1}(f) = \sigma_{p+1}(g)$, by Lemma 3.4 and (4.1), we have $\overline{\lambda}_{p+1}(g) = \lambda_{p+1}(g) = \sigma_{p+1}(g) = \sigma_p(B)$; therefore $\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$.

(2) We prove that $\overline{\lambda}_{p+1}(f' - \varphi) = \sigma_{p+1}(f)$. Set $g_1 = f' - \varphi$, then $\sigma_{p+1}(g_1) = \sigma_{p+1}(f) = \sigma_p(B)$. By the same proof as that of (2) in case (i), we have (4.5). Set $F_1 = \varphi'' + (A - \frac{B'}{B})\varphi' + (A' + B - \frac{AB'}{B})\varphi$, we affirm $F_1 \not\equiv 0$, if $F_1 \equiv 0$, then by Lemma 3.14, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(B)$, which is a contradiction to $\sigma_{p+1}(\varphi) < \sigma_p(B)$; therefore $F_1 \not\equiv 0$. Since $\sigma_{p+1}(F_1) < \sigma_p(B) = \sigma_{p+1}(g_1)$, by Lemma 3.4 and (4.5), we have $\overline{\lambda}_{p+1}(f' - \varphi) = \lambda_{p+1}(f' - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$.

(3) We prove that $\overline{\lambda}_{p+1}(f'' - \varphi) = \sigma_{p+1}(f)$. Set $g_2 = f'' - \varphi$, then $\sigma_{p+1}(g_2) = \sigma_{p+1}(f) = \sigma_p(B)$ and $f'' = g_2 + \varphi$, $f''' = g_2' + \varphi'$, $f^{(4)} = g_2'' + \varphi''$ By the same proof as that of (3) in case (i), we can obtain (4.10). Set $F_2 = \varphi'' + (S - \frac{Q'}{Q})\varphi' + (S' + Q - \frac{SQ'}{Q})\varphi$, where $Q(z) = A' + B - \frac{AB'}{B}$, $S(z) = A - \frac{B'}{B}$. In the following we prove that $F_2 \not\equiv 0$. By Definition 1.2 and Lemma 3.10, for all sufficiently large $|z| = r \notin E_7$ and for any $\varepsilon > 0$, we have

$$\left|S(z)\right| \leq \exp_p\left\{\left(\tau_p(A) + \varepsilon\right)r^{\sigma_p(A)}\right\}, \qquad \left|S'(z)\right| \leq \exp_p\left\{\left(\tau_p(A) + \varepsilon\right)r^{\sigma_p(A)}\right\}. \tag{4.18}$$

By Lemma 3.9 and Lemma 3.10, for all sufficiently large $|z| = r \in E_6 - E_7$ and for any ε $(0 < 2\varepsilon < \tau_p(B) - \tau_p(A))$, we have

$$M(r, Q) \geq \exp_p\{(\tau_p(B) - \varepsilon)r^{\sigma_p(B)}\}. \tag{4.19}$$

By (4.18)-(4.19) and Lemma 3.10, it is easy to obtain

$$\begin{aligned} &\left|S - \frac{Q'}{Q}\right| \leq 2\exp_p\{(\tau_p(A) + \varepsilon)r^{\sigma_p(A)}\}, \\ &\left|S' + Q - \frac{SQ'}{Q}\right| \geq \frac{1}{2}\exp_p\{(\tau_p(B) - \varepsilon)r^{\sigma_p(B)}\}, \quad r \in E_6 - E_7. \end{aligned} \tag{4.20}$$

If $F_2 \equiv 0$, by (4.20) and by a similar proof to that in Lemma 3.14, we have $\sigma_{p+1}(\varphi) \geq \sigma_p(B)$, which is a contradiction. Therefore $F_2 \not\equiv 0$, then by Lemma 3.4 and $\sigma_{p+1}(F_2) < \sigma_{p+1}(g_2) = \sigma_p(B)$, we have $\overline{\lambda}_{p+1}(f'' - \varphi) = \lambda_{p+1}(f'' - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$.

By following the proof of (4)-(5) in case (i) and the proof of (3) in case (ii), we can obtain $\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \sigma_{p+1}(f) = \sigma_p(B)$ $(j \geq 3)$.

5 Proof of Theorems 2.2-2.3

Using a similar proof to that in case (i) of Theorem 2.1 and by Lemma 3.4, we can easily obtain Theorem 2.2. Theorem 2.3 is a direct result of Theorem 2.1 and Lemma 3.8.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

TJ completed the main part of this article, TJ, XZX and XHY corrected the main theorems. All authors read and approved the final manuscript.

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