Advances in Difference Equations
a SpringerOpen Journal

RESEARCH **Open Access**

Stability of quadratic functional equations in generalized functions

Young-Su Lee*

*Correspondence:
masuri@sogang.ac.kr
Hana Academy Seoul, Seoul, Korea

Abstract

In this paper, we consider the following generalized quadratic functional equation with n-independent variables in the spaces of generalized functions:

$$\sum_{1\le i<j\le n} \left(f(x_i+x_j)+f(x_i-x_j)\right) = 2(n-1)\sum_{i=1}^{n} f(x_i).$$

Making use of the fundamental solution of the heat equation, we solve the general solutions and the stability problems of the above equation in the spaces of tempered distributions and Fourier hyperfunctions. Moreover, using the Dirac sequence of regularizing functions, we extend these results to the space of distributions.

MSC: 39B82; 46F05

Keywords: quadratic functional equation; stability; generalized function; heat kernel; Gauss transform; distribution

1 Introduction

In 1940, Ulam [1] raised a question concerning the stability of group homomorphisms as follows:

> Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

The case of approximately additive mappings was solved by Hyers [2] under the assumption that G_2 is a Banach space. In 1978, Rassias [3] generalized Hyers' result to the unbounded Cauchy difference. During the last decades, stability problems of various functional equations have been extensively studied and generalized by a number of authors (see [4–6]).

Quadratic functional equations are used to characterize the inner product spaces. Note that a square norm on an inner product space satisfies the parallelogram equality

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all vectors x, y. By virtue of this equality, the following functional equation is induced:

$$f(x+y)+f(x-y)=2f(x)+2f(y). \tag{1.1}$$

It is easy to see that the quadratic function $f(x)=ax^2$ on a real field, where a is an arbitrary constant, is a solution of (1.1). Thus, it is natural that (1.1) is called a quadratic functional equation. It is well known that a function f between real vector spaces satisfies (1.1) if and only if there exists a unique symmetric biadditive function B such that $f(x)=B(x,x)$ (see [4–7]). The biadditive function B is given by

$$B(x,y)=\frac{1}{4}\bigl(f(x+y)-f(x-y)\bigr).$$

The Hyers-Ulam stability for quadratic functional equation (1.1) was proved by Skof [8]. Thereafter, many authors studied the stability problems of (1.1) in various settings (see [9–11]). In particular, the Hyers-Ulam-Rassias stability of (1.1) was proved by Czerwik [12].

Recently, Eungrasamee *et al.* [13] considered the following functional equation:

$$\sum_{1\le i<j\le n}\bigl(f(x_i+x_j)+f(x_i-x_j)\bigr)=2(n-1)\sum_{i=1}^{n}f(x_i), \tag{1.2}$$

where n is a positive integer with $n>1$. They proved that (1.2) is equivalent to (1.1). Also, they proved the Hyers-Ulam-Rassias stability of this equation.

In this paper, we solve the general solutions and the stability problems of (1.2) in the spaces of generalized functions such as $\mathcal{S}'$ of tempered distributions, $\mathcal{F}'$ of Fourier hyperfunctions and $\mathcal{D}'$ of distributions. Using the notions as in [14–19], we reformulate (1.2) and the related inequality in the spaces of generalized functions as follows:

$$\sum_{1\le i<j\le n}(u\circ A_{ij}+u\circ B_{ij})=2(n-1)\sum_{i=1}^{n}u\circ P_i, \tag{1.3}$$

$$\left\|\sum_{1\le i<j\le n}(u\circ A_{ij}+u\circ B_{ij})-2(n-1)\sum_{i=1}^{n}u\circ P_i\right\|\le\epsilon, \tag{1.4}$$

where A_{ij}, B_{ij} and P_i are the functions defined by

$$A_{ij}(x_1,\dots,x_n)=x_i+x_j,\quad 1\le i<j\le n,$$
$$B_{ij}(x_1,\dots,x_n)=x_i-x_j,\quad 1\le i<j\le n,$$
$$P_i(x_1,\dots,x_n)=x_i,\quad 1\le i\le n.$$

Here, $\circ$ denotes the pullback of generalized functions and the inequality $\|v\|\le\epsilon$ in (1.4) means that $|\langle v,\varphi\rangle|\le\epsilon\|\varphi\|_{L^1}$ for all test functions φ. We refer to [20] for pullbacks and to [14–17] for more details on the spaces of generalized functions. Recently, Chung [14, 15, 17] solved the general solutions and the stability problems of (1.1) in the spaces of generalized functions. As a matter of fact, our approaches are based on the methods as in [14–17].

This paper is organized as follows. In Section 2, we solve the general solutions and the stability problems of (1.2) in the spaces of $\mathcal{S}'$ and $\mathcal{F}'$. We prove that every solution u in $\mathcal{F}'$ (or $\mathcal{S}'$) of equation (1.3) has the form

$$u = ax^2,$$

where $a \in \mathbb{C}$. Also, we prove that every solution u in $\mathcal{S}'$ (or $\mathcal{F}'$) of inequality (1.4) can be written uniquely in the form

$$u = ax^2 + \mu(x),$$

where $a \in \mathbb{C}$ and μ is a bounded measurable function such that $\|\mu\|_{L^\infty} \le \frac{n^2-n-1}{n^2-n}\epsilon$. Subsequently, in Section 3, making use of the Dirac sequence of regularizing functions, we extend these results to the space $\mathcal{D}'$.

2 Stability in $\mathcal{F}'$

We first introduce the spaces of tempered distributions and Fourier hyperfunctions. Here, $\mathbb{N}_0$ denotes the set of nonnegative integers.

Definition 2.1 [20, 21] We denote by $\mathcal{S}(\mathbb{R})$ the set of all infinitely differentiable functions φ in $\mathbb{R}$ satisfying

$$\|\varphi\|_{\alpha,\beta} = \sup_x \left|x^\alpha \partial^\beta \varphi(x)\right| < \infty \tag{2.1}$$

for all $\alpha, \beta \in \mathbb{N}_0$. A linear functional u on $\mathcal{S}(\mathbb{R})$ is said to be a tempered distribution if there exists a constant $C \ge 0$ and a nonnegative integer N such that

$$\left|\langle u, \varphi\rangle\right| \le C \sum_{\alpha,\beta \le N} \sup_x \left|x^\alpha \partial^\beta \varphi\right|$$

for all $\varphi \in \mathcal{S}(\mathbb{R})$. The set of all tempered distributions is denoted by $\mathcal{S}'(\mathbb{R})$.

If $\varphi \in \mathcal{S}(\mathbb{R})$, then each derivative of φ decreases faster than x^{-N} for all $N > 0$ as $x \to \infty$. It is easy to see that the function $\varphi(x) = \exp(-ax^2)$, where $a > 0$ belongs to $\mathcal{S}(\mathbb{R})$ but $\psi(x) = (1 + x^2)^{-1}$, is not a member of $\mathcal{S}(\mathbb{R})$. For example, every polynomial $p(x) = \sum_{\alpha \le m} a_\alpha x^\alpha$, where $a_\alpha \in \mathbb{C}$, defines a tempered distribution by

$$\langle p(x), \varphi\rangle = \int_{\mathbb{R}} p(x)\varphi(x)\,dx, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

Note that tempered distributions are generalizations of L^p-functions. These are very useful for the study of Fourier transforms in generality, since all tempered distributions have a Fourier transform, but not all distributions have one. Imposing the growth condition on $\|\cdot\|_{\alpha,\beta}$ in (2.1), we get a new space of test functions as follows.

Definition 2.2 [22] We denote by $\mathcal{F}(\mathbb{R})$ the set of all infinitely differentiable functions φ in $\mathbb{R}$ such that

$$\|\varphi\|_{A,B} = \sup_{x,\alpha,\beta} \frac{|x^\alpha \partial^\beta \varphi(x)|}{A^{|\alpha|} B^{|\beta|} \alpha! \beta!} < \infty \tag{2.2}$$

for some positive constants A, B depending only on φ. The strong dual of $\mathcal{F}(\mathbb{R})$, denoted by $\mathcal{F}'(\mathbb{R})$, is called the Fourier hyperfunction.

It is easy to see the following topological inclusions:

$$\mathcal{F}(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}), \qquad \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{F}'(\mathbb{R}). \tag{2.3}$$

To solve the general solution and the stability problem of (1.2) in the space $\mathcal{F}'(\mathbb{R})$, we employ the heat kernel which is the fundamental solution to the heat equation,

$$E_t(x) = E(x,t) = \begin{cases} (4\pi t)^{-1/2} \exp(-|x|^2/4t), & x \in \mathbb{R}, t > 0, \\ 0, & x \in \mathbb{R}, t \leq 0. \end{cases}$$

Since for each $t > 0$, $E(\cdot, t)$ belongs to the space $\mathcal{F}(\mathbb{R})$, the convolution

$$\tilde{u}(x,t) = (u * E)(x,t) = \langle u_y, E_t(x-y) \rangle, \quad x \in \mathbb{R}, t > 0$$

is well defined for all $u \in \mathcal{F}'(\mathbb{R})$, which is called the Gauss transform of u. It is well known that the semigroup property of the heat kernel

$$(E_t * E_s)(x) = E_{t+s}(x)$$

holds for convolution. The semigroup property will be very useful for converting equation (1.3) into the classical functional equation defined on an upper-half plane. We also use the following famous result, the so-called heat kernel method, which is stated as follows.

Theorem 2.3 [23] *Let $u \in \mathcal{S}'(\mathbb{R})$. Then its Gauss transform $\tilde{u}$ is a C^∞-solution of the heat equation*

$$(\partial/\partial t - \Delta)\tilde{u}(x,t) = 0$$

satisfying the following:

(i) *There exist positive constants C, M and N such that*

$$\left|\tilde{u}(x,t)\right| \leq Ct^{-M}\left(1 + |x|\right)^N \quad \text{in } \mathbb{R} \times (0, \delta). \tag{2.4}$$

(ii) $\tilde{u}(x,t) \to u$ *as* $t \to 0^+$ *in the sense that for every* $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\langle u, \varphi \rangle = \lim_{t \to 0^+} \int \tilde{u}(x,t)\varphi(x)\,dx.$$

Conversely, every C^∞-solution $U(x,t)$ of the heat equation satisfying the growth condition (2.4) *can be uniquely expressed as $U(x,t) = \tilde{u}(x,t)$ for some $u \in \mathcal{S}'(\mathbb{R})$.*

Similarly, we can represent Fourier hyperfunctions as initial values of solutions of the heat equation as a special case of the results as in [24]. In this case, the estimate (2.4) is replaced by the following:

For every $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\left|\tilde{u}(x,t)\right| \leq C_\varepsilon \exp\bigl(\varepsilon\bigl(|x| + 1/t\bigr)\bigr) \quad \text{in } \mathbb{R} \times (0,\delta).$$

We are now going to solve the general solutions and the stability problems of (1.2) in the spaces of $\mathcal{S}'$ and $\mathcal{F}'$. Here, we need the following lemma which will be crucial in the proof of the main theorem.

Lemma 2.4 *Suppose that $f : \mathbb{R} \times (0,\infty) \to \mathbb{C}$ is a continuous function satisfying*

$$\sum_{1\leq i<j\leq n} \bigl(f(x_i + x_j, t_i + t_j) + f(x_i - x_j, t_i + t_j)\bigr) = 2(n-1)\sum_{i=1}^{n} f(x_i, t_i) \tag{2.5}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$. Then the solution f has the form

$$f(x,t) = ax^2 + bt$$

for some constants $a, b \in \mathbb{C}$.

Proof Define a function $h(x,t) := f(x,t) - f(0,t)$ for all $x \in \mathbb{R}$, $t > 0$. Then h satisfies $h(0,t) = 0$ for all $t > 0$ and

$$\sum_{1\leq i<j\leq n} \bigl(h(x_i + x_j, t_i + t_j) + h(x_i - x_j, t_i + t_j)\bigr) = 2(n-1)\sum_{i=1}^{n} h(x_i, t_i) \tag{2.6}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$. Putting $x_2 = \cdots = x_n = 0$ in (2.6) gives

$$\sum_{j=2}^{n} h(x_1, t_1 + t_j) = (n-1)h(x_1, t_1) \tag{2.7}$$

for all $x_1 \in \mathbb{R}$, $t_1, \dots, t_n > 0$. Letting $t_3 = \cdots = t_n \to 0^+$ in (2.7) yields

$$h(x_1, t_1 + t_2) = h(x_1, t_1)$$

for all $x_1 \in \mathbb{R}$, $t_1, t_2 > 0$. This means that $h(x,t)$ is independent of t. Thus we can write $q(x) := h(x,1) = h(x,t)$. It follows from (2.6) that $q(x)$ satisfies

$$\sum_{1\leq i<j\leq n} \bigl(q(x_i + x_j) + q(x_i - x_j)\bigr) = 2(n-1)\sum_{i=1}^{n} q(x_i) \tag{2.8}$$

for all $x_1, \dots, x_n$. Putting $x_3 = \cdots = x_n = 0$ in (2.8), we see that $q(x)$ satisfies the quadratic functional equation

$$q(x_1 + x_2) + q(x_1 - x_2) = 2q(x_1) + 2q(x_2)$$

for all $x_1, x_2 \in \mathbb{R}$. Given the continuity, the function q has the form

$$q(x,t) = ax^2$$

for some constant $a \in \mathbb{C}$.

On the other hand, putting $x_1 = \cdots = x_n = 0$ in (2.5) gives

$$\sum_{1\le i<j\le n} f(0, t_i + t_j) = (n-1)\sum_{i=1}^{n} f(0, t_i) \tag{2.9}$$

for all $t_1, \ldots, t_n > 0$. In view of (2.9), we see that

$$c := \limsup_{t\to 0^+} f(0, t)$$

exists. In (2.9), let $t_1 = t_{1_k} \to 0^+$ so that $f(0, t_1) \to c$ as $k \to \infty$. Then we have

$$\sum_{2\le i<j\le n} f(0, t_i + t_j) = (n-1)c + (n-2)\sum_{i=2}^{n} f(0, t_i) \tag{2.10}$$

for all $t_2, \ldots, t_n > 0$. In the same way, in (2.10), let $t_2 = t_{1_k} \to 0^+$ so that $f(0, t_2) \to c$ as $k \to \infty$. Then we have

$$\sum_{3\le i<j\le n} f(0, t_i + t_j) = (2n-3)c + (n-3)\sum_{i=3}^{n} f(0, t_i) \tag{2.11}$$

for all $t_3, \ldots, t_n > 0$. Applying the same way in (2.11) repeatedly, we obtain $c = 0$. Setting $t_3 = \cdots = t_n = t_{1_k} \to 0^+$ in (2.9) yields

$$f(0, t_1 + t_2) = f(0, t_1) + f(0, t_2)$$

for all $t_1, t_2 > 0$. Given the continuity, we must have $f(0, t) = bt$ for some $b \in \mathbb{C}$. Therefore, the solution f of (2.5) is of the form

$$f(x, t) = h(x, t) + f(0, t) = ax^2 + bt$$

for some constants $a, b \in \mathbb{C}$. □

According to the above lemma, we solve the general solution of (1.2) in the spaces of $\mathcal{S}'$ and $\mathcal{F}'$. From the inclusions (2.3), it suffices to consider the space $\mathcal{F}'$ instead of $\mathcal{S}'$.

Theorem 2.5 *Every solution u in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$ resp.) of the equation*

$$\sum_{1\le i<j\le n} (u\circ A_{ij} + u\circ B_{ij}) = 2(n-1)\sum_{i=1}^{n} u\circ P_i \tag{2.12}$$

has the form

$$u = ax^2,$$

where $a \in \mathbb{C}$.

Proof Convolving the tensor product $E_{t_1}(x_1)\cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (2.12), we have

$$\begin{aligned}
&\big[(u\circ A_{ij}) * \big(E_{t_1}(x_1)\cdots E_{t_n}(x_n)\big)\big](\xi_1,\dots,\xi_n)\\
&\quad= \big\langle u\circ A_{ij}, E_{t_1}(\xi_1-x_1)\cdots E_{t_n}(\xi_n-x_n)\big\rangle\\
&\quad= \left\langle u, \int E_{t_i}(\xi_i-x_i+x_j)E_{t_j}(\xi_j-x_j)\,dx_j\right\rangle\\
&\quad= \left\langle u, \int E_{t_i}(\xi_i+\xi_j-x_i-x_j)E_{t_j}(x_j)\,dx_j\right\rangle\\
&\quad= \big\langle u, (E_{t_i}*E_{t_j})(\xi_i+\xi_j-x_i)\big\rangle\\
&\quad= \big\langle u, E_{t_i+t_j}(\xi_i+\xi_j-x_i)\big\rangle\\
&\quad= \tilde{u}(\xi_i+\xi_j, t_i+t_j),\\
&\big[(u\circ B_{ij}) * \big(E_{t_1}(x_1)\cdots E_{t_n}(x_n)\big)\big](\xi_1,\dots,\xi_n) = \tilde{u}(\xi_i-\xi_j, t_i+t_j),\\
&\big[(u\circ P_{i}) * \big(E_{t_1}(x_1)\cdots E_{t_n}(x_n)\big)\big](\xi_1,\dots,\xi_n) = \tilde{u}(\xi_i, t_i),
\end{aligned}$$

where $\tilde{u}$ is the Gauss transform of u. Thus, (2.12) is converted into the following classical functional equation:

$$\sum_{1\le i<j\le n}\big(\tilde{u}(x_i+x_j,t_i+t_j)+\tilde{u}(x_i-x_j,t_i+t_j)\big) = 2(n-1)\sum_{i=1}^{n}\tilde{u}(x_i,t_i) \tag{2.13}$$

for all $x_1,\dots,x_n\in\mathbb{R}$, $t_1,\dots,t_n>0$. It follows from Lemma 2.4 that the solution $\tilde{u}$ of (2.13) has the form

$$\tilde{u}(x,t) = ax^2+bt, \tag{2.14}$$

where $a,b\in\mathbb{C}$. Letting $t\to 0^+$ in (2.14), we obtain

$$u = ax^2.$$

This completes the proof. □

From the above theorem, we have the following corollary immediately.

Corollary 2.6 *Every solution u in $\mathcal{F}'(\mathbb{R})$ (or $\mathcal{S}'(\mathbb{R})$ resp.) of the equation*

$$u\circ A_{12}+u\circ B_{12} = 2u\circ P_1 + 2u\circ P_2$$

has the form

$$u = ax^2,$$

where $a\in\mathbb{C}$.

We now solve the stability problem of (1.2) in the spaces of $\mathcal{S}'$ and $\mathcal{F}'$.

Theorem 2.7 *Suppose that* $u \in \mathcal{F}'(\mathbb{R})$ (*or* $\mathcal{S}'(\mathbb{R})$ *resp.*) *satisfies the inequality*

$$\left\| \sum_{1\le i<j\le n} (u \circ A_{ij} + u \circ B_{ij}) - 2(n-1)\sum_{i=1}^{n} u \circ P_i \right\| \le \epsilon. \tag{2.15}$$

Then there exists a unique quadratic function $q(x) = ax^2$ *such that*

$$\left\| u - q(x) \right\| \le \frac{n^2-n-1}{n^2-n}\epsilon.$$

Proof Convolving the tensor product $E_{t_1}(x_1)\cdots E_{t_n}(x_n)$ of the heat kernels on both sides of (2.15), we have

$$\left| \sum_{1\le i<j\le n} \bigl(f(x_i+x_j, t_i+t_j) + f(x_i-x_j, t_i+t_j)\bigr) - 2(n-1)\sum_{i=1}^{n} f(x_i, t_i) \right| \le \epsilon \tag{2.16}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$, where f is the Gauss transform of u. Putting $x_1 = \cdots = x_n = 0$ in (2.16) yields

$$\left| \sum_{1\le i<j\le n} f(0, t_i+t_j) - (n-1)\sum_{i=1}^{n} f(0, t_i) \right| \le \frac{\epsilon}{2} \tag{2.17}$$

for all $t_1, \dots, t_n > 0$. Then by the triangle inequality we have

$$\left|(n-1)f(0,t_1)\right| \le \frac{\epsilon}{2} + \left| \sum_{1\le i<j\le n} f(0, t_i+t_j) - (n-1)\sum_{i=2}^{n} f(0, t_i) \right|$$

for all $t_1, \dots, t_n > 0$. It follows from the continuity of f and the inequality above that

$$c := \limsup_{t_1\to 0^+} f(0, t_1)$$

exists. Choose a sequence t_{1_k}, $k = 1, 2, \dots$, of positive numbers, which tends to 0 as $k \to \infty$, such that $f(0, t_{1_k}) \to c$ as $k \to \infty$. Letting $t_1 = \cdots = t_n = t_{1_k} \to 0^+$ in (2.17) gives

$$|c| \le \frac{\epsilon}{n^2-n}. \tag{2.18}$$

Setting $t_1 = t_2 = t$, $t_3 = \cdots = t_n = t_{1_k} \to 0^+$ in (2.17) and using (2.18), we have

$$\left| \frac{f(0,2t)}{2} - f(0,t) \right| \le \frac{n^2-n-1}{2n(n-1)}\epsilon. \tag{2.19}$$

Using an induction argument in (2.19), we obtain

$$\left| \frac{f(0,2^k t)}{2^k} - f(0,t) \right| \le \frac{n^2-n-1}{n^2-n}\epsilon \tag{2.20}$$

for all $k \in \mathbb{N}$, $t > 0$. From (2.20) we see that $2^{-k}f(0, 2^k t)$ is a Cauchy sequence, and hence

$$h(t) := \lim_{k\to\infty} 2^{-k} f\bigl(0, 2^k t\bigr)$$

exists. Replacing t_i by $2^k t_i$ in (2.17), $i = 1, 2, \dots, n$, dividing by 2^k and letting $k \to \infty$, we have

$$\sum_{1\le i<j\le n} h(t_i + t_j) = (n-1)\sum_{i=1}^{n} h(t_i)$$

for all $t_1, \dots, t_n > 0$. As in the proof of Lemma 2.4, the function h satisfies

$$h(t_1 + t_2) = h(t_1) + h(t_2) \tag{2.21}$$

for all $t_1, t_2 > 0$. Given the continuity, the function h is of the form $h(t) = bt$ for some constant $b \in \mathbb{C}$. Letting $k \to \infty$ in (2.20), we obtain

$$\bigl|f(0,t) - h(t)\bigr| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon \tag{2.22}$$

for all $t > 0$. It follows from (2.21) and (2.22) that

$$\left|\sum_{j=1}^{k} 4^{-j} f\bigl(0, 2^j t\bigr) - \bigl(1 - 2^{-k}\bigr)h(t)\right| \le \frac{n^2 - n - 1}{3(n^2 - n)}\epsilon. \tag{2.23}$$

On the other hand, putting $x_1 = x_2 = x$, $x_3 = \cdots = x_n = 0$ and letting $t_1 = t_2 = t$, $t_3 = \cdots = t_n \to 0^+$ in (2.16), we have

$$\left|\frac{f(2x, 2t)}{4} + \frac{f(0, 2t)}{4} - f(x,t)\right| \le \frac{n^2 - n - 1}{2(n^2 - n)}\epsilon. \tag{2.24}$$

Using the iterative method in (2.24) gives

$$\left|f(x,t) - 4^{-k} f\bigl(2^k x, 2^k t\bigr) - \sum_{j=1}^{k} 4^{-j} f\bigl(0, 2^j t\bigr)\right| \le \frac{2(n^2 - n - 1)}{3(n^2 - n)}\epsilon. \tag{2.25}$$

Adding (2.23) to (2.25) and letting $F(x,t) := f(x,t) - h(t)$, we obtain

$$\bigl|F(x,t) - 4^{-k} F\bigl(2^k x, 2^k t\bigr)\bigr| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon. \tag{2.26}$$

From (2.26) we see that $4^{-k}F(2^k x, 2^k t)$ is a Cauchy sequence, and hence

$$G(x,t) := \lim_{k\to\infty} 4^{-k} F\bigl(2^k x, 2^k t\bigr)$$

exists. By the definition of G and (2.16), we have $G(0,t) = 0$ for all $t > 0$ and

$$\sum_{1\le i<j\le n} \bigl(G(x_i + x_j, t_i + t_j) + G(x_i - x_j, t_i + t_j)\bigr) = 2(n-1)\sum_{i=1}^{n} G(x_i, t_i) \tag{2.27}$$

for all $x_1, \dots, x_n \in \mathbb{R}$, $t_1, \dots, t_n > 0$. As in the proof of Lemma 2.4, we obtain

$$G(x,t) = ax^2$$

for some constant $a \in \mathbb{C}$. Letting $k \to \infty$ in (2.16) yields

$$\left|f(x,t) - ax^2 - bt\right| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon \tag{2.28}$$

for some constants $a, b \in \mathbb{C}$. Letting $t \to 0^+$ in (2.28), we have

$$\left\|u - ax^2\right\| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon.$$

This completes the proof. □

Corollary 2.8 *Suppose that* $u \in \mathcal{F}'(\mathbb{R})$ (*or* $\mathcal{S}'(\mathbb{R})$ *resp.*) *satisfies the inequality*

$$\|u \circ A_{12} + u \circ B_{12} - 2u \circ P_1 - 2u \circ P_2\| \le \epsilon.$$

Then there exists a unique quadratic function $q(x) = ax^2$ *such that*

$$\left\|u - q(x)\right\| \le \frac{1}{2}\epsilon.$$

3 Stability in $\mathcal{D}'$

In this section, we extend the previous results to the space of distributions. Recall that a distribution u is a linear functional on $C_c^\infty(\mathbb{R})$ of infinitely differentiable functions on $\mathbb{R}$ with compact supports such that for every compact set $K \subset \mathbb{R}$, there exist constants $C > 0$ and $N \in \mathbb{N}_0$ satisfying

$$\left|\langle u, \varphi\rangle\right| \le C \sum_{|\alpha| \le N} \sup\left|\partial^\alpha \varphi\right|$$

for all $\varphi \in C_c^\infty(\mathbb{R})$ with supports contained in K. The set of all distributions is denoted by $\mathcal{D}'(\mathbb{R})$. It is well known that the following topological inclusions hold:

$$C_c^\infty(\mathbb{R}) \hookrightarrow \mathcal{S}(\mathbb{R}), \qquad \mathcal{S}'(\mathbb{R}) \hookrightarrow \mathcal{D}'(\mathbb{R}).$$

As we see in [14, 16], by virtue of the semigroup property of the heat kernel, equation (1.3) can be controlled easily in the space $\mathcal{S}'$. But we cannot employ the heat kernel in the space $\mathcal{D}'$. Instead of the heat kernel, we use the function $\psi_t(x) := t^{-1}\psi(\frac{x}{t})$, $x \in \mathbb{R}$, $t > 0$, where $\psi(x) \in C_c^\infty(\mathbb{R})$ such that

$$\psi(x) \ge 0, \qquad \operatorname{supp}\psi(x) \subset \left\{x \in \mathbb{R} : |x| \le 1\right\}, \qquad \int \psi(x)\,dx = 1.$$

For example, let

$$\psi(x) = \begin{cases} A\exp(-(1 - |x|^2)^{-1}), & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

where

$$A = \left(\int_{|x|<1} \exp\bigl(-\bigl(1 - |x|^2\bigr)^{-1}\bigr)\, dx \right)^{-1},$$

then it is easy to see $\psi(x)$ is an infinitely differentiable function with support $\{x : |x| \le 1\}$. Now we employ the function $\psi_t(x) := t^{-1}\psi(x/t)$, $t > 0$. If $u \in \mathcal{D}'(\mathbb{R})$, then for each $t > 0$, $(u * \psi_t)(x) = \langle u_y, \psi_t(x - y)\rangle$ is a smooth function in $\mathbb{R}$ and $(u * \psi_t)(x) \to u$ as $t \to 0^+$ in the sense of distributions, that is, for every $\varphi \in C_c^\infty(\mathbb{R})$,

$$\langle u, \varphi\rangle = \lim_{t\to 0^+} \int (u * \psi_t)(x)\varphi(x)\, dx.$$

Theorem 3.1 *Every solution u in $\mathcal{D}'(\mathbb{R})$ of the equation*

$$\sum_{1\le i<j\le n} (u \circ A_{ij} + u \circ B_{ij}) = 2(n-1) \sum_{i=1}^{n} u \circ P_i \tag{3.1}$$

has the form

$$u = ax^2,$$

where $a \in \mathbb{C}$.

Proof Convolving the tensor product $\psi_{t_1}(\xi_1)\cdots\psi_{t_n}(\xi_n)$ of the regularizing functions on both sides of (3.1), we have

$$\begin{aligned}
&\bigl[(u \circ A_{ij}) * \bigl(\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n)\bigr)\bigr](\xi_1, \ldots, \xi_n) \\
&\quad = \bigl\langle u \circ A_{ij}, \psi_{t_1}(\xi_1 - x_1)\cdots\psi_{t_n}(\xi_n - x_n)\bigr\rangle \\
&\quad = \left\langle u, \int \psi_{t_i}(\xi_i - x_i + x_j)\psi_{t_j}(\xi_j - x_j)\, dx_j \right\rangle \\
&\quad = \left\langle u, \int \psi_{t_i}(\xi_i + \xi_j - x_i - x_j)\psi_{t_j}(x_j)\, dx_j \right\rangle \\
&\quad = \bigl\langle u, (\psi_{t_i} * \psi_{t_j})(\xi_i + \xi_j - x_i)\bigr\rangle \\
&\quad = (u * \psi_{t_i} * \psi_{t_j})(\xi_i + \xi_j), \\
&\bigl[(u \circ B_{ij}) * \bigl(\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n)\bigr)\bigr](\xi_1, \ldots, \xi_n) = (u * \psi_{t_i} * \psi_{t_j})(\xi_i - \xi_j), \\
&\bigl[(u \circ P_i) * \bigl(\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n)\bigr)\bigr](\xi_1, \ldots, \xi_n) = (u * \psi_{t_i})(\xi_i).
\end{aligned}$$

Thus, (3.1) is converted into the following functional equation:

$$\sum_{1\le i<j\le n} \bigl((u * \psi_{t_i} * \psi_{t_j})(x_i + x_j) + (u * \psi_{t_i} * \psi_{t_j})(x_i - x_j)\bigr) = 2(n-1) \sum_{i=1}^{n} (u * \psi_{t_i})(x_i) \tag{3.2}$$

for all $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n > 0$. In view of (3.2), it is easy to see that

$$f(x) := \limsup_{t\to 0^+} (u * \psi_t)(x)$$

exists. Putting $x_1 = \cdots = x_n = 0$ and letting $t_1 = \cdots = t_n \to 0^+$ in (3.2) yields $f(0) = 0$. Setting $x_1 = x$, $x_2 = y$, $x_3 = \cdots = x_n = 0$ and letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (3.2), we have

$$(u * \psi_t * \psi_s)(x+y) + (u * \psi_t * \psi_s)(x-y) = 2(u * \psi_t)(x) + 2(u * \psi_s)(y) \tag{3.3}$$

for all $x, y \in \mathbb{R}$, $t, s > 0$. Letting $t \to 0^+$ in (3.3) gives

$$(u * \psi_s)(x+y) + (u * \psi_s)(x-y) = 2f(x) + 2(u * \psi_s)(y) \tag{3.4}$$

for all $x, y \in \mathbb{R}$, $s > 0$. Putting $y = 0$ in (3.4) yields

$$f(x) = (u * \psi_s)(x) - (u * \psi_s)(0). \tag{3.5}$$

Applying (3.5) to (3.4), we see that f satisfies the following quadratic functional equation:

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in \mathbb{R}$. Since f is a smooth function, in view of (3.5), it follows that $f(x) = ax^2$, where $a \in \mathbb{C}$. Thus, from (3.5), we have

$$(u * \psi_s)(x) = ax^2 + (u * \psi_s)(0). \tag{3.6}$$

Letting $s \to 0^+$ in (3.6), we obtain

$$u = ax^2.$$

This completes the proof. □

In a similar manner, we have the following corollary immediately.

Corollary 3.2 *Every solution u in $\mathcal{D}'(\mathbb{R})$ of the equation*

$$u \circ A_{12} + u \circ B_{12} = 2u \circ P_1 + 2u \circ P_2$$

has the form

$$u = ax^2,$$

where $a \in \mathbb{C}$.

We are now going to state and prove the main result of this paper.

Theorem 3.3 *Suppose that $u \in \mathcal{D}'(\mathbb{R})$ satisfies the inequality*

$$\left\| \sum_{1 \le i < j \le n} (u \circ A_{ij} + u \circ B_{ij}) - 2(n-1) \sum_{i=1}^{n} u \circ P_i \right\| \le \epsilon. \tag{3.7}$$

Then there exists a unique quadratic function $q(x) = ax^2$ *such that*

$$\|u - q(x)\| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon.$$

Proof It suffices to show that every distribution satisfying (3.7) belongs to the space $\mathcal{S}'(\mathbb{R})$. Convolving the tensor product $\psi_{t_1}(x_1)\cdots\psi_{t_n}(x_n)$ on both sides of (3.7), we have

$$\left|\sum_{1\le i<j\le n}\big((u*\psi_{t_i}*\psi_{t_n})(x_i+x_j)+(u*\psi_{t_i}*\psi_{t_n})(x_i-x_j)\big)-2(n-1)\sum_{i=1}^{n}(u*\psi_{t_i})(x_i)\right|\le\epsilon \quad (3.8)$$

for all $x_1,\ldots,x_n \in \mathbb{R}$, $t_1,\ldots,t_n > 0$. In view of (3.8), it is easy to see that for each fixed x,

$$f(x) := \limsup_{t\to 0^+}(u * \psi_t)(x)$$

exists. Putting $x_1 = \cdots = x_n = 0$ and letting $t_1 = \cdots = t_n \to 0^+$ in (3.8) yields

$$|f(0)| \le \frac{\epsilon}{n^2 - n}. \quad (3.9)$$

Setting $x_1 = x$, $x_2 = y$, $x_3 = \cdots = x_n = 0$, letting $t_1 = t$, $t_2 = s$, $t_3 = \cdots = t_n \to 0^+$ in (3.8) and using (3.9), we have

$$\begin{aligned}&\big|(u * \psi_t * \psi_s)(x + y) + (u * \psi_t * \psi_s)(x - y) - 2(u * \psi_t)(x) - 2(u * \psi_s)(y)\big|\\ &\quad\le \frac{2(n^2 - n - 1)}{n^2 - n}\epsilon. \end{aligned}\quad (3.10)$$

Putting $y = 0$ in (3.10) and dividing the result by 2, we obtain

$$\big|(u * \psi_t * \psi_s)(x) - (u * \psi_t)(x) - (u * \psi_s)(0)\big| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon. \quad (3.11)$$

Letting $t \to 0^+$ in (3.11) gives

$$\big|(u * \psi_s)(x) - f(x) - (u * \psi_s)(0)\big| \le \frac{n^2 - n - 1}{n^2 - n}\epsilon. \quad (3.12)$$

From (3.10), (3.11) and (3.12) we have

$$\big|f(x + y) + f(x - y) - 2f(x) - 2f(y)\big| \le \frac{10(n^2 - n - 1)}{n^2 - n}\epsilon$$

for all $x, y \in \mathbb{R}$. According to the result as in [8], there exists a unique quadratic function $q : \mathbb{R} \to \mathbb{C}$ satisfying

$$q(x + y) + q(x - y) = 2q(x) + 2q(y)$$

such that

$$\big|f(x) - q(x)\big| \le \frac{5(n^2 - n - 1)}{n^2 - n}\epsilon. \quad (3.13)$$

It follows from (3.12) and (3.13) that

$$\left|(u * \psi_s)(x) - q(x) - (u * \psi_s)(0)\right| \le \frac{6(n^2 - n - 1)}{n^2 - n}\epsilon. \tag{3.14}$$

Letting $s \to 0^+$ in (3.14), we have

$$\left\|u - q(x)\right\| \le \frac{6n^2 - 6n - 5}{n^2 - n}\epsilon. \tag{3.15}$$

Inequality (3.15) implies that $h(x) := u - q(x)$ belongs to $(L^1)' = L^\infty$. Thus, we conclude that $u = q(x) + h(x) \in \mathcal{S}'(\mathbb{R})$. □

From the above theorem, we have the following corollary immediately.

Corollary 3.4 *Suppose that* $u \in \mathcal{D}'(\mathbb{R})$ *satisfies the inequality*

$$\|u \circ A_{12} + u \circ B_{12} - 2u \circ P_1 - 2u \circ P_2\| \le \epsilon.$$

Then there exists a unique quadratic function $q(x) = ax^2$ *such that*

$$\left\|u - q(x)\right\| \le \frac{1}{2}\epsilon.$$

Competing interests
The author declares that he has no competing interests.

Received: 16 August 2012 Accepted: 4 March 2013 Published: 22 March 2013

References
1. Ulam, SM: Problems in Modern Mathematics. Wiley, New York (1964)
2. Hyers, DH: On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222-224 (1941)
3. Rassias, TM: On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297-300 (1978)
4. Czerwik, S: Functional Equations and Inequalities in Several Variables. World Scientific, River Edge (2002)
5. Hyers, DH, Isac, G, Rassias, TM: Stability of Functional Equations in Several Variables. Birkhäuser, Boston (1998)
6. Kannappan, P: Functional Equations and Inequalities with Applications. Springer, Berlin (2009)
7. Aczél, J, Dhombres, J: Functional Equations in Several Variables. Cambridge University Press, Cambridge (1989)
8. Skof, F: Local properties and approximation of operators. Rend. Semin. Mat. Fis. Milano **53**, 113-129 (1983)
9. Borelli, C, Forti, GL: On a general Hyers-Ulam-stability result. Int. J. Math. Math. Sci. **18**, 229-236 (1995)
10. Cholewa, PW: Remarks on the stability of functional equations. Aequ. Math. **27**, 76-86 (1984)
11. Jun, K-W, Lee, Y-H: On the Hyers-Ulam-Rassias stability of a pexiderized quadratic inequality. Math. Inequal. Appl. **4**, 93-118 (2001)
12. Czerwik, S: On the stability of the quadratic mapping in normed spaces. Abh. Math. Semin. Univ. Hamb. **62**, 59-64 (1992)
13. Eungrasamee, T, Udomkavanich, P, Nakmahachalasint, P: On generalized stability of an *n*-dimensional quadratic functional equation. Thai J. Math. **8**, 43-50 (2010)
14. Chung, J: Stability of functional equations in the spaces of distributions and hyperfunctions. J. Math. Anal. Appl. **286**, 177-186 (2003)
15. Chung, J, Lee, S: Some functional equations in the spaces of generalized functions. Aequ. Math. **65**, 267-279 (2003)
16. Chung, J, Chung, S-Y, Kim, D: The stability of Cauchy equations in the space of Schwartz distributions. J. Math. Anal. Appl. **295**, 107-114 (2004)
17. Chung, J: A distributional version of functional equations and their stabilities. Nonlinear Anal. **62**, 1037-1051 (2005)
18. Lee, Y-S, Chung, S-Y: The stability of a general quadratic functional equation in distributions. Publ. Math. (Debr.) **74**, 293-306 (2009)
19. Lee, Y-S, Chung, S-Y: Stability of quartic functional equations in the spaces of generalized functions. Adv. Differ. Equ. **2009**, Article ID 838347 (2009)
20. Hörmander, L: The Analysis of Linear Partial Differential Operators I. Springer, Berlin (1983)
21. Schwartz, L: Théorie des Distributions. Hermann, Paris (1966)
22. Chung, J, Chung, S-Y, Kim, D: A characterization for Fourier hyperfunctions. Publ. Res. Inst. Math. Sci. **30**, 203-208 (1994)
23. Matsuzawa, T: A calculus approach to hyperfunctions III. Nagoya Math. J. **118**, 133-153 (1990)
24. Kim, KW, Chung, S-Y, Kim, D: Fourier hyperfunctions as the boundary values of smooth solutions of heat equations. Publ. Res. Inst. Math. Sci. **29**, 289-300 (1993)

doi:10.1186/1687-1847-2013-72
Cite this article as: Lee: Stability of quadratic functional equations in generalized functions. *Advances in Difference Equations* 2013 **2013**:72.