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Some new identities of Bernoulli, Euler and Hermite polynomials arising from umbral calculus

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Abstract

In this paper, we derive the identities of higher-order Bernoulli, Euler and Frobenius-Euler polynomials from the orthogonality of Hermite polynomials. Finally, we give some interesting and new identities of several special polynomials arising from umbral calculus.

MSC: 05A10; 05A19

Keywords: Bernoulli polynomial; Euler polynomial; Abel polynomial

1 Introduction

The Hermite polynomials are defined by the generating function to be

$$e^{2xt-t^2} = e^{H(x)t} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{1.1}$$

with the usual convention about replacing $H^n(x)$ by $H_n(x)$ (see [1]). In the special case, $x = 0$, $H_n(0) = H_n$ are called the *nth Hermite numbers*. From (1.1) we have

$$H_n(x) = (H + 2x)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}x^l 2^l. \tag{1.2}$$

Thus, by (1.2), we get

$$\frac{d^k}{dx^k}H_n(x) = 2^k(n)_k H_{n-k}(x) = 2^k\frac{n!}{(n-k)!}H_{n-k}(x), \tag{1.3}$$

where $(x)_k = x(x-1)\cdots(x-k+1)$.

As is well known, the Bernoulli polynomials of order r are defined by the generating function to be

$$\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.4}$$

In the special case, $x = 0$, $B_n^{(r)}(0) = B_n^{(r)}$ are called the nth Bernoulli numbers of order r (see [1–4]).

The Euler polynomials of order r are also defined by the generating function to be

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.5}$$

In the special case, $x = 0$, $E_n^{(r)}(0) = E_n^{(r)}$ are called the *nth Euler numbers* of order r.

For $\lambda(\neq 1) \in \mathbb{C}$, the *Frobenius-Euler polynomials* of order r are given by

$$\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!} \quad (r \in \mathbb{R}). \tag{1.6}$$

In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the *nth Frobenius-Euler numbers* of order r (see [1–16]).

The Stirling numbers of the first kind are defined by the generating function to be

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k \quad \text{(see [11, 14])}, \tag{1.7}$$

and the Stirling numbers of the second kind are given by

$$\left(e^t-1\right)^n = n!\sum_{l=n}^{\infty} S_2(l,n)\frac{t^l}{l!} \quad \text{(see [14])}. \tag{1.8}$$

In [1] it is known that $H_0(x), H_1(x), \ldots, H_n(x)$ from an orthogonal basis for the space

$$\mathbb{P}_n = \left\{p(x) \in \mathbb{Q}[x] \mid \deg p(x) \le n\right\} \tag{1.9}$$

with respect to the inner product

$$\left\langle p_1(x), p_2(x)\right\rangle = \int_{-\infty}^{\infty} e^{-x^2} p_1(x)p_2(x)\,dx \quad \text{(see [1])}. \tag{1.10}$$

For $p(x) \in \mathbb{P}_n$, let us assume that

$$p(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{1.11}$$

Then, from the orthogonality of Hermite polynomials and Rodrigues' formula, we have

$$\begin{aligned} C_k &= \frac{1}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} H_k(x)p(x)\,dx \\ &= \frac{(-1)^k}{2^k k!\sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k}{dx^k} e^{-x^2}\right) p(x)\,dx \quad \text{(see [1])}. \end{aligned} \tag{1.12}$$

In particular, for $p(x) = x^m$ $(m \geq 0)$, we easily get

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{d^n}{dx^n} e^{-x^2} \right) x^m \, dx \\ &\quad = \begin{cases} 0 & \text{if } n > m \text{ or } n \leq m \text{ with } m - n \not\equiv 0 \pmod 2, \\ \frac{(-1)^n m! \sqrt{\pi}}{2^{m-n} (\frac{m-n}{2})!} & \text{if } n \leq m \text{ with } m - n \equiv 0 \pmod 2. \end{cases} \end{aligned} \tag{1.13}$$

Let $\mathcal{F}$ be the set of all formal power series in the variable t over $\mathbb{C}$ with

$$\mathcal{F} = \left\{ f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \,\middle|\, a_k \in \mathbb{C} \right\}. \tag{1.14}$$

Let us assume that $\mathbb{P}$ is the algebra of polynomials in the variable x over $\mathbb{C}$ and that $\mathbb{P}^*$ is the vector space of all linear functionals on $\mathbb{P}$. $\langle L|p(x)\rangle$ denotes the action of the linear functional L on polynomials $p(x)$, and we remind that the vector space structure on $\mathbb{P}^*$ is defined by

$$\begin{aligned} &\langle L + M|p(x)\rangle = \langle L|p(x)\rangle + \langle M|p(x)\rangle, \\ &\langle cL|p(x)\rangle = c\langle L|p(x)\rangle, \end{aligned}$$

where c is a complex constant (see [2, 11, 14]).

The formal power series

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{k!} t^k \in \mathcal{F} \tag{1.15}$$

defines a linear functional on $\mathbb{P}$ by setting

$$\langle f(t)|x^n\rangle = a_n \quad \text{for all } n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{1.16}$$

Thus, by (1.15) and (1.16), we get

$$\langle t^k|x^n\rangle = n!\delta_{n,k} \quad (n, k \geq 0), \tag{1.17}$$

where $\delta_{n,k}$ is the Kronecker symbol (see [2, 11, 14]).

Let $f_L(t) = \sum_{k=0}^{\infty} \frac{\langle L|x^k\rangle}{k!} t^k$. By (1.16), we get

$$\langle f_L(t)|x^n\rangle = \langle L|x^n\rangle, \quad n \geq 0. \tag{1.18}$$

Thus, by (1.18), we see that $f_L(t) = L$. The map $L \mapsto f_L(t)$ is a vector space isomorphism from $\mathbb{P}^*$ onto $\mathcal{F}$. Henceforth, $\mathcal{F}$ will be thought of as both a formal power series and a linear functional. We call $\mathcal{F}$ the *umbral algebra*. The umbral calculus is the study of umbral algebra (see [2, 11, 14]).

The order $o(f(t))$ of the nonzero power series $f(t)$ is the smallest integer k for which the coefficient of t^k does not vanish. A series $f(t)$ having $o(f(t)) = 1$ is called a *delta series*, and a series $f(t)$ having $o(f(t)) = 0$ is called an *invertible series* (see [2, 11, 14]). By (1.16) and (1.17), we see that $\langle e^{yt}|p(x)\rangle = p(y)$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$f(t) = \sum_{k=0}^{\infty} \frac{\langle f(t)|x^k\rangle}{k!} t^k, \qquad p(x) = \sum_{k=0}^{\infty} \frac{\langle t^k|p(x)\rangle}{k!} x^k. \tag{1.19}$$

Let $f(t), g(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$. Then we easily see that

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t)|g(t)p(x)\rangle = \langle g(t)|f(t)p(x)\rangle. \tag{1.20}$$

From (1.19), we can derive the following equation:

$$p^{(k)}(0) = \langle t^k|p(x)\rangle \quad \text{and} \quad \langle 1|p^{(k)}(x)\rangle = p^{(k)}(0). \tag{1.21}$$

Thus, by (1.21), we get

$$t^k p(x) = p^{(k)}(x) = \frac{d^k p(x)}{dx^k} \quad \text{(see [2, 11, 14])}. \tag{1.22}$$

Let $f(t)$ be a delta series, and let $g(t)$ be an invertible series. Then there exists a unique sequence $S_n(x)$ of polynomials with $\langle g(t)f(t)^k|S_n(x)\rangle = n!\delta_{n,k}$, where $n, k \geq 0$ (see [2, 11, 14]). The sequence $S_n(x)$ is called *Sheffer sequence* for $(g(t), f(t))$, which is denoted by $S_n(x) \sim (g(t), f(t))$. For $f(t) \in \mathcal{F}$ and $p(x) \in \mathbb{P}$, we have

$$\left\langle \frac{e^{yt}-1}{t} \middle| p(x) \right\rangle = \int_0^y p(u)\,du, \qquad \langle e^{yt}-1|p(x)\rangle = p(y) - p(0), \tag{1.23}$$

and

$$\langle f(t)|xp(x)\rangle = \langle f'(t)|p(x)\rangle. \tag{1.24}$$

In this paper, we introduce the identities of several special polynomials which are derived from the orthogonality of Hermite polynomials. Finally, we give some new and interesting identities of the higher-order Bernoulli, Euler and Frobenius-Euler polynomials arising from umbral calculus.

2 Some identities of several special polynomials

From (1.5), we note that

$$\left(\frac{2}{e^t+1}\right)^r = \left(1 + \frac{e^t-1}{2}\right)^{-r} = \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j. \tag{2.1}$$

By (2.1), we get

$$\begin{aligned} \left(\frac{2}{e^t+1}\right)^r e^{xt} &= \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j e^{xt} \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} \binom{-r}{j} \left(\frac{e^t-1}{2}\right)^j x^n \right) \frac{t^n}{n!}. \end{aligned} \tag{2.2}$$

From (1.5) and (2.2), we have

$$E_n^{(r)}(x) = \sum_{j=0}^{n} \binom{-r}{j} 2^{-j} \left(e^t - 1\right)^j x^n. \tag{2.3}$$

By (1.8) and (1.9), we get

$$\begin{aligned}\left(e^t - 1\right)^j x^n &= \sum_{k=0}^{n-j} \frac{\langle t^k | (e^t - 1)^j x^n \rangle}{k!} = \sum_{k=0}^{n-j} \frac{\langle (e^t - 1)^j | t^k x^n \rangle}{k!} x^k \\ &= j! \sum_{k=0}^{n-j} \binom{n}{k} \frac{\langle (e^t - 1)^j | x^{n-k} \rangle}{j!} x^k = j! \sum_{k=0}^{n-j} \binom{n}{j} S_2(n-k, j) x^k \\ &= j! \sum_{k=j}^{n} \binom{n}{k} S_2(k, j) x^{n-k}. \end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.1 *For $n \geq 0$, we have*

$$\begin{aligned} E_n^{(r)}(x) &= \sum_{0 \leq j \leq n} \sum_{j \leq k \leq n} \binom{n}{k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j) x^{n-k} \\ &= \sum_{0 \leq k \leq n} \binom{n}{k} \left[\sum_{0 \leq j \leq k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j) \right] x^{n-k}. \end{aligned}$$

By (1.5), we easily see that

$$E_n^{(r)}(x) = \sum_{k=0}^{n} \binom{n}{k} E_k^{(r)} x^{n-k}. \tag{2.5}$$

Therefore, by Theorem 2.1 and (2.5), we obtain the following corollary.

Corollary 2.2 *For $k \geq 0$, we have*

$$E_k^{(r)} = \sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{2^j} S_2(k, j).$$

Let us take $p(x) = E_n^{(r)}(x) \in \mathbb{P}_n$. Then, by (1.11), we get

$$E_n^{(r)}(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.6}$$

From (1.12), we can derive the computation of C_k as follows:

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) E_n^{(r)}(x)\, dx, \tag{2.7}$$

where

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\left(\frac{d^k e^{-x^2}}{dx^k}\right)E_n^{(r)}(x)\,dx \\
&\quad = (-n)\bigl(-(n-1)\bigr)\cdots\bigl(-(n-k+1)\bigr)\int_{-\infty}^{\infty} e^{-x^2}E_{n-k}^{(r)}(x)\,dx \\
&\quad = \frac{(-1)^k n!}{(n-k)!}\int_{-\infty}^{\infty} e^{-x^2}\sum_{l=0}^{n-k}\binom{n-k}{l}E_{n-k-l}^{(r)}x^l\,dx \\
&\quad = \frac{(-1)^k n!}{(n-k)!}\sum_{l=0}^{n-k}\binom{n-k}{l}E_{n-k-l}^{(r)}\int_{-\infty}^{\infty} e^{-x^2}x^l\,dx \\
&\quad = (-1)^k n!\sqrt{\pi}\sum_{0\le l\le n-k,l:\text{even}}\frac{1}{(n-k-l)!2^l(\frac{l}{2})!}\sum_{j=0}^{n-k-l}\binom{-r}{j}\frac{j!}{2^j}S_2(n-k-l,j).
\end{aligned}
\tag{2.8}
$$

From (2.7) and (2.8), we can derive the following equation:

$$
\begin{aligned}
C_k &= n!\sum_{0\le l\le n-k,l:\text{even}}\frac{E_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!} \\
&= n!\sum_{0\le l\le n-k,l:\text{even}}\sum_{j=0}^{n-k-l}\frac{\binom{-r}{j}j!S_2(n-k-l,j)}{k!(n-k-l)!2^{k+l+j}(\frac{l}{2})!}.
\end{aligned}
\tag{2.9}
$$

Therefore, by Corollary 2.2, (2.6) and (2.9), we obtain the following theorem.

Theorem 2.3 *For $n \geq 0$, we have*

$$
\begin{aligned}
E_n^{(r)}(x) &= n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k,l:\text{even}}\frac{E_{n-k-l}^{(r)}}{k!(n-k-l)!2^{k+l}(\frac{l}{2})!}\right\}H_k(x) \\
&= n!\sum_{k=0}^{n}\left\{\sum_{0\le l\le n-k,l:\text{even}}\sum_{j=0}^{n-k-l}\frac{\binom{-r}{j}j!S_2(n-k-l,j)}{k!(n-k-l)!2^{k+l+j}(\frac{l}{2})!}\right\}H_k(x).
\end{aligned}
$$

By (1.4), we easily see that

$$
\left(\frac{t}{e^t-1}\right)^r = \left(1+\frac{e^t-t-1}{t}\right)^{-r} = \sum_{j=0}^{\infty}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j.
\tag{2.10}
$$

Thus, by (2.10), we get

$$
\left(\frac{t}{e^t-1}\right)^r e^{xt} = \sum_{n=0}^{\infty}\left(\sum_{j=0}^{n}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j x^n\right)\frac{t^n}{n!}.
\tag{2.11}
$$

From (1.4) and (2.11), we have

$$
B_n^{(r)}(x) = \sum_{j=0}^{n}\binom{-r}{j}\left(\frac{e^t-t-1}{t}\right)^j x^n.
\tag{2.12}
$$

By (1.19), we easily get

$$\begin{aligned}\left(\frac{e^t-t-1}{t}\right)^j x^n &= \sum_{k=0}^{n-j}\frac{\langle t^k|(\frac{e^t-t-1}{t})^j x^n\rangle}{k!}x^k \\ &= \sum_{k=0}^{n-j}\frac{\langle(\frac{e^t-t-1}{t})^j|t^k x^n\rangle}{k!}x^k = \sum_{k=0}^{n-j}\binom{n}{k}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\left\langle\left(\frac{e^t-1}{t}\right)^l\middle|x^{n-k}\right\rangle x^k \\ &= \sum_{k=0}^{n-j}\binom{n}{k}\sum_{l=0}^{j}\binom{j}{l}(-1)^{j-l}\left\langle t^0\middle|\left(\frac{e^t-1}{t}\right)^l x^{n-k}\right\rangle x^k. \end{aligned} \tag{2.13}$$

From (1.8), (1.21) and (2.13), we have

$$\left(\frac{e^t-t-1}{t}\right)^j x^n = \sum_{k=0}^{n-j}\sum_{l=0}^{j}\binom{n}{k}\binom{j}{l}(-1)^{j-l}\frac{(n-k)!l!}{(n-k+l)!}S_2(n-k+l,l)x^k. \tag{2.14}$$

Thus, by (2.12) and (2.14), we get

$$\begin{aligned}B_n^{(r)}(x) &= \sum_{j=0}^{n}\sum_{k=0}^{n-j}\sum_{l=0}^{j}\binom{-r}{j}\binom{n}{k}\binom{j}{l}(-1)^{j-l}\frac{S_2(n-k+l,l)}{\binom{n-k+l}{l}}x^k \\ &= \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}(-1)^{j-l}\right]x^{n-k}. \end{aligned} \tag{2.15}$$

Therefore, by (2.12) and (2.15), we obtain the following theorem.

Theorem 2.4 *For $n \geq 0$, we have*

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}\left[\sum_{j=0}^{k}\sum_{l=0}^{j}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}(-1)^{j-l}\right]x^{n-k}.$$

By (1.4), we easily get

$$B_n^{(r)}(x) = \sum_{k=0}^{n}\binom{n}{k}B_k^{(r)}x^{n-k}. \tag{2.16}$$

Therefore, by Theorem 2.4 and (2.16), we obtain the following corollary.

Corollary 2.5 *For $k \geq 0$, we have*

$$B_k^{(r)} = \sum_{j=0}^{k}\sum_{l=0}^{j}(-1)^{j-l}\binom{-r}{j}\binom{j}{l}\frac{S_2(k+l,l)}{\binom{k+l}{l}}.$$

Let us consider $p(x) = B_n^{(r)}(x) \in \mathbb{P}_n$. Then, by (1.11), $B_n^{(r)}(x)$ can be written as

$$B_n^{(r)}(x) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.17}$$

Now, we compute C_k's for $B_k^{(r)}(x)$ as follows:

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n^{(r)}(x)\, dx, \tag{2.18}$$

where

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n^{(r)}(x)\, dx \\ &\quad = (-n)\bigl(-(n-1)\bigr) \cdots \bigl(-(n-k+1)\bigr) \int_{-\infty}^{\infty} e^{-x^2} B_{n-k}^{(r)}(x)\, dx \\ &\quad = \frac{(-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} B_{n-k-l}^{(r)} \int_{-\infty}^{\infty} e^{-x^2} x^l\, dx \\ &\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)! 2^l (\frac{l}{2})!}. \end{aligned} \tag{2.19}$$

By Corollary 2.5 and (2.19), we get

$$\begin{aligned} &\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) B_n^{(r)}(x)\, dx \\ &\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} s \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m} \binom{-r}{j} \binom{j}{m} S_2(n-k-l+m, m)}{(n-k-l)! 2^l (\frac{l}{2})! \binom{n-k-l+m}{m}}. \end{aligned} \tag{2.20}$$

From (2.18) and (2.20), we have

$$\begin{aligned} C_k &= n! \sum_{0 \le l \le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)! k! 2^{k+l} (\frac{l}{2})!} \\ &= n! \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m} \binom{-r}{j} \binom{j}{m} S_2(n-k-l+m, m)}{(n-k-l)! k! 2^{k+l} (\frac{l}{2})! \binom{n-k-l+m}{m}}. \end{aligned} \tag{2.21}$$

Therefore, by (2.17) and (2.21), we obtain the following theorem.

Theorem 2.6 *For $n \ge 0$, we have*

$$\begin{aligned} B_n^{(r)}(x) &= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \frac{B_{n-k-l}^{(r)}}{(n-k-l)! k! 2^{k+l} (\frac{l}{2})!} \right\} H_k(x) \\ &= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \sum_{m=0}^{j} \frac{(-1)^{j-m} \binom{-r}{j} \binom{j}{m} S_2(n-k-l+m, m)}{(n-k-l)! k! 2^{k+l} (\frac{l}{2})! \binom{n-k-l+m}{m}} \right\} H_k(x). \end{aligned}$$

It is easy to show that

$$\left(\frac{1-\lambda}{e^t - \lambda} \right)^r = \left(1 + \frac{e^t - 1}{1-\lambda} \right)^{-r} = \sum_{j=0}^{\infty} \binom{-r}{j} \left(\frac{1}{1-\lambda} \right)^j (e^t - 1)^j. \tag{2.22}$$

From (1.6) and (2.22), we have

$$H_n^{(r)}(x|\lambda) = \sum_{j=0}^{n} \binom{-r}{j} (1-\lambda)^{-j} \left(e^t - 1\right)^j x^n, \tag{2.23}$$

where

$$\begin{aligned} \left(e^t - 1\right)^j x^n &= j! \sum_{k=j}^{\infty} S_2(k,j) \frac{t^k}{k!} x^n \\ &= j! \sum_{k=j}^{n} \binom{n}{k} S_2(k,j) x^{n-k}. \end{aligned} \tag{2.24}$$

Thus, by (2.24), we get

$$\left(e^t - 1\right)^j x^n = j! \sum_{k=j}^{n} \binom{n}{k} S_2(k,j) x^{n-k}. \tag{2.25}$$

From (2.23) and (2.25), we can derive the following equation:

$$\begin{aligned} H_n^{(r)}(x|\lambda) &= \sum_{j=0}^{n} \sum_{k=j}^{n} \binom{n}{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) x^{n-k} \\ &= \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) x^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} \left[\sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j) \right] x^{n-k}. \end{aligned} \tag{2.26}$$

By (1.6), we easily see that

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(r)}(\lambda) x^{n-k}. \tag{2.27}$$

Therefore, by (2.26) and (2.27), we obtain the following theorem.

Theorem 2.7 *For $k \geq 0$, we have*

$$H_k^{(r)}(\lambda) = \sum_{j=0}^{k} \binom{-r}{j} \frac{j!}{(1-\lambda)^j} S_2(k,j).$$

Let us take $p(x) = H_n^{(r)}(x|\lambda) \in \mathbb{P}_n$. Then, by (1.11), $H_n^{(r)}(x|\lambda)$ is given by

$$H_n^{(r)}(x|\lambda) = \sum_{k=0}^{n} C_k H_k(x). \tag{2.28}$$

By (1.12), we get

$$C_k = \frac{(-1)^k}{2^k k! \sqrt{\pi}} \int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n^{(r)}(x|\lambda)\, dx, \tag{2.29}$$

where

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{d^k e^{-x^2}}{dx^k} \right) H_n^{(r)}(x|\lambda)\, dx \\
&\quad = \frac{(-1)^k n!}{(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} H_{n-k-l}^{(r)}(\lambda) \int_{-\infty}^{\infty} e^{-x^2} x^l \, dx \\
&\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! 2^l (\frac{l}{2})!} \\
&\quad = (-1)^k n! \sqrt{\pi} \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! 2^l (1-\lambda)^j (\frac{l}{2})!}.
\end{aligned} \tag{2.30}$$

By (2.29) and (2.30), we get

$$\begin{aligned}
C_k &= n! \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! k! 2^{l+k} (\frac{l}{2})!} \\
&= n! \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! k! 2^{k+l} (1-\lambda)^j (\frac{l}{2})!}.
\end{aligned} \tag{2.31}$$

Therefore, by (2.28) and (2.31), we obtain the following theorem.

Corollary 2.8 *For $n \ge 0$, we have*

$$\begin{aligned}
H_n^{(r)}(x|\lambda) &= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \frac{H_{n-k-l}^{(r)}(\lambda)}{(n-k-l)! k! 2^{l+k} (\frac{l}{2})!} \right\} H_k(x) \\
&= n! \sum_{k=0}^{n} \left\{ \sum_{0 \le l \le n-k, l:\text{even}} \sum_{j=0}^{n-k-l} \frac{\binom{-r}{j} j! S_2(n-k-l, j)}{(n-k-l)! k! 2^{k+l} (1-\lambda)^j (\frac{l}{2})!} \right\} H_k(x).
\end{aligned}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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