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Positive solutions of second-order linear difference equation with variable delays

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Abstract

In this paper we consider the second-order linear difference equations with variable delays

$$\Delta^2 a(n) + \sum_{i=1}^{m} P_i(n)a\bigl(n - k_i(n)\bigr) = 0, \quad n \geq n_0,$$

where $n_0, n \in \mathbf{N}$, $\mathbf{N}$ is the set of positive integers. Using the method of Riccati transform and the generalized characteristic equations, we give sufficient conditions for the existence of positive solutions.

MSC: Primary 39A11; secondary 39A12

Keywords: second-order linear difference equation; variable delays; existence; positive solutions; non-oscillatory solutions; sufficient conditions

Introduction

In the past few years, oscillation and non-oscillation theory of second-order linear and nonlinear difference equations has attracted considerable attention, and we refer the reader to the papers of Diblik *et al.* [1], El-Sheik *et al.* [2], He [3], Huiqin and Zhen [4], Koplatadze and Kvinikadze [5], Krasznai *et al.* [6], Li *et al.* [7], Liu and Cheng [8], Medina and Pituk [9], Tang [10], Tang and Yu [11], Zhang [12], Zhang and Li [13], and the references therein. For comprehensive treatment, see the papers by Bastinec *et al.* [14–17], Čermak [18, 19], Deng [20], Došly and Fišnarová [21], Hille [22], Jaroš and Stavroulakis [23], Thandapani *et al.* [24], Yang [25] and the monographs [26, 27]. Some sharp conditions of Hille-type criterion on the existence of oscillatory and non-oscillatory solutions of second-order differential equations are given by Kusano *et al.* [28, 29], by Péics and Karsai [30], by Bastinec *et al.* [31] and by Berezansky *et al.* [32]. Some sufficient conditions of oscillation and non-oscillation of second-order difference equations can be found in the papers by Lei [33], by Li and Jiang [34], by Zhou and Zhang [35], and the references therein.

Consider the second-order delay differential equation of the form

$$x''(t) + \sum_{i=1}^{m} p_i(t)x\bigl(t - \tau_i(t)\bigr) = 0 \tag{1}$$

for $t_0 \leq t < T \leq \infty$, where the following hypotheses are satisfied:

(H_1^*) $p_i \in C[[t_0, T), \mathbf{R}]$, $i = 1, 2, \dots, m$;
(H_2^*) $\tau_i \in C[[t_0, T), \mathbf{R}^+]$, $i = 1, 2, \dots, m$.

As a special case we can formulate the following results given in [30] for half-linear delay differential equations.

Theorem A (see Theorem 2 in [30]) *Assume that* (H_1^*) *and* (H_2^*) *hold and there exists a positive function* $\mu(t)$ *for* $t \ge t_0$ *such that*

$$\int_t^{\infty} \left[\mu(t)^2 + \sum_{i=1}^{m} \left| p_i(s) \right| \exp\left(\int_{s-\tau_i(s)}^{s} \mu(\xi)\, d\xi \right) \right] ds \le \mu(t)$$

holds for t *large enough. Then equation* (1) *has a positive solution.*

Theorem B (see Corollary 2 in [30]) *Assume that* (H_1^*) *and* (H_2^*) *hold and the functions* $\tau_i(t)$, *where* $i = 1, 2, \dots, m$, *are bounded. If*

$$\limsup_{t\to\infty} t \int_t^{\infty} \sum_{i=1}^{m} \left| p_i(s) \right| ds \le \frac{1}{4},$$

then equation (1) *has a positive solution.*

Consider the second-order difference equation of the form

$$\Delta^2 a(n) + P(n)a(n) = 0, \quad n \ge n_0, \tag{2}$$

where $n_0 \in \mathbf{N}$, $\mathbf{N}$ is the set of positive integers and $\{P(n)\}$ is a sequence of real numbers.

The following result is a sufficient condition for the existence of non-oscillatory solutions of equation (2) and it is demonstrated in the paper [35].

Theorem C (see Lemma 4 in [35]) *Assume that* $\{P(n)\}$ *is a real sequence with* $P(n) \ge 0$ *for all* $n \ge n_0$. *If*

$$n \sum_{i=n+1}^{\infty} P_i(s) \le \frac{1}{4}$$

for all large n, *then equation* (2) *has a non-oscillatory solution.*

Consider now the second-order difference equation with variable delays

$$\Delta^2 a(n) + \sum_{i=1}^{m} P_i(n) a\bigl(n - k_i(n)\bigr) = 0, \quad n \ge n_0, \tag{3}$$

where $n_0 \in \mathbf{N}$, $\mathbf{N}$ is the set of positive integers and the following hypotheses are satisfied:

(H_1) $\{P_i(n)\}$ is a sequence of real numbers for $i = 1, 2, \dots, m$ and $n \ge n_0$;
(H_2) $\{k_i(n)\}$ is a sequence of positive integers such that $k_i(n) \le n$ for $i = 1, 2, \dots, m$ and $n \ge n_0$.

Let $M \geq n_0$ be an arbitrary natural number and set

$$\mathbf{N}_M = \{n | n \in \mathbf{N}, n \geq M\}.$$

By a solution of equation (3) we mean a sequence of real numbers $\{a(n)\}$ defined for $n \in \mathbf{N}_M$, which satisfies equation (3) for all $n \in \mathbf{N}_M$. A nontrivial solution $\{a(n)\}$ of equation (3) is said to be oscillatory if for every $\nu > n_0$, there exists an $n \geq \nu$ such that $a(n)a(n+1) \leq 0$. Otherwise, it is non-oscillatory. Thus, a non-oscillatory solution is either eventually positive or eventually negative.

The basic idea in the paper is to use the Riccati transformation technique by the substitution

$$\lambda(n) = \frac{\Delta a(n)}{a(n)},$$

where $\{a(n)\}$ is a solution sequence of equation (3). This transformation leads to the generalization of the Riccati-type equation associated with equation (3). The aim of this paper is to prove theorems for the existence of positive solutions of equation (3), using the Riccati-type equation associated with equation (3). The obtained results are discrete analogues of results given for some differential equations and generalize results given for second-order linear difference equations with constant delays or without delays.

Let $\mathbf{R}^\infty := \{\{\xi(n)\}_{n=1}^\infty : \xi(n) \in \mathbf{R}, n = 1, 2, \ldots\}$ and

$$\ell^\infty := \left\{\{\xi(n)\} \in \mathbf{R}^\infty : \left|\xi(n)\right| \leq \mu(n), n \in \mathbf{N}_M\right\},$$

where $\{\mu(n)\}$ is a fixed positive bounded sequence.

Let $\{x_p(n)\}$ denote the sequence of sequences defined for all natural numbers $n \in [n_0, \infty)$ and $p = 1, 2, 3, \ldots$.

Theorem D (Schauder-Tychonoff, see [36–38]) *Let $F = \ell^\infty$, and let $\{\mu(n)\}$ be a fixed positive bounded sequence. Let S be a mapping of F into itself with the properties*:

(i) *S is continuous in the sense that if $x_p(n) \in F$ for all natural number $n \geq n_0$, $p = 1, 2, \ldots$, and $x_p(n) \to x(n)$, $p \to \infty$, uniformly on every compact subinterval of $[n_0, \infty)$, then $Sx_p(n) \to Sx(n)$, $p \to \infty$, uniformly on every compact subinterval of $[n_0, \infty)$;*

(ii) *the sequences in the image set SF are bounded at every point of $[n_0, \infty)$.*

Then the mapping S has at least one fixed point in F.

Main results

First, we apply the Riccati transformation and show the relationship between equation (3) and the Riccati-type equation (4).

Lemma 1 *Assume that conditions* (H_1) *and* (H_2) *hold. Then the following statements are equivalent.*

(a) *Equation* (3) *has an eventually positive solution.*

(b) *There is a sequence* $\{\lambda(n)\}$, $n \in \mathbf{N}_M$, *for some* $M \geq n_0$ *such that* $\lambda(n) + 1 > 0$ *for all* $n \in \mathbf{N}_M$ *and* $\{\lambda(n)\}$ *satisfies the Riccati-type equation*

$$\Delta\lambda(n) + \lambda(n+1)\lambda(n) + \sum_{i=1}^{m} P_i(n) \prod_{j=n-k_i(n)}^{n-1} \frac{1}{\lambda(j)+1} = 0 \quad \text{for } n \in \mathbf{N}_M. \tag{4}$$

Proof (a) ⇒ (b): Let $\{a(n)\}$ be the solution of difference equation (3) and suppose, according to the hypotheses, that $a(n) > 0$ for $n \geq M \geq n_0$. It will be shown that the sequence $\{\lambda(n)\}$ defined by

$$\lambda(n) = \frac{\Delta a(n)}{a(n)}, \quad n \in \mathbf{N}_M, \tag{5}$$

is a solution of the Riccati-type equation (4) for $n \in \mathbf{N}_M$. Expressing $\Delta a(n)$ from (5) and applying the difference operator to the transformed equality, we get that

$$\Delta a(n) = a(n)\lambda(n) \quad \text{and} \quad \Delta\big[\Delta a(n)\big] = a(n)\lambda(n+1)\lambda(n) + a(n)\Delta\lambda(n). \tag{6}$$

It follows from the first equality of (6) that

$$a(n+1) = a(n)\big(\lambda(n)+1\big) \quad \text{and} \quad a(n) = a(M)\prod_{j=M}^{n-1}\big(\lambda(j)+1\big) \quad \text{for } n \in \mathbf{N}_M,$$

and hence the following equalities are valid:

$$\frac{a(n-k_i(n))}{a(n)} = \frac{a(M)\prod_{j=M}^{n-k_i(n)-1}(\lambda(j)+1)}{a(M)\prod_{j=M}^{n-1}(\lambda(j)+1)} = \prod_{j=n-k_i(n)}^{n-1} \frac{1}{\lambda(j)+1}. \tag{7}$$

By dividing both sides of difference equation (3) by $a(n)$, we obtain that

$$\frac{\Delta[\Delta a(n)]}{a(n)} + \sum_{i=1}^{m} P_i(n) \frac{a(n-k_i(n))}{a(n)} = 0. \tag{8}$$

In virtue of the equalities (6) and (7), we obtain that

$$\frac{a(n)\lambda(n+1)\lambda(n) + a(n)\Delta\lambda(n)}{a(n)} + \sum_{i=1}^{m} P_i(n) \prod_{j=n-k_i(n)}^{n-1} \frac{1}{\lambda(j)+1} = 0.$$

After reducing the first term, we get that

$$\lambda(n+1)\lambda(n) + \Delta\lambda(n) + \sum_{i=1}^{m} P_i(n) \prod_{j=n-k_i(n)}^{n-1} \frac{1}{\lambda(j)+1} = 0,$$

and we conclude that the sequence $\{\lambda(n)\}$ satisfies the Riccati-type equation (4), and the first part of the proof is complete.

(b) $\Rightarrow$ (a): Let now $\{\lambda(n)\}$ be a solution sequence of Riccati-type equation (4) for $n \in \mathbf{N}_M$ such that $\lambda(n) + 1 > 0$ for all $n \in \mathbf{N}_M$. We show by direct substitution that the sequence defined by

$$a(n) = \prod_{j=M}^{n-1} \bigl(\lambda(j) + 1\bigr) \quad \text{for } n \in \mathbf{N}_M$$

is the positive solution of difference equation (3). Completing the following transformations, we get

$$\begin{aligned}
\Delta\bigl[\Delta a(n)\bigr] &= a(n)\bigl(\lambda(n+1)\lambda(n) + \Delta\lambda(n)\bigr) \\
&= a(n)\left(-\sum_{i=1}^{m} P_i(n) \prod_{j=n-k_i(n)}^{n-1} \frac{1}{\lambda(j)+1}\right) \\
&= -a(n)\sum_{i=1}^{m} P_i(n)\frac{a(n-k_i(n))}{a(n)} \\
&= -\sum_{i=1}^{m} P_i(n)a\bigl(n-k_i(n)\bigr)
\end{aligned}$$

for $n \in \mathbf{N}_M$ and the proof of Lemma 1 is complete. □

Now, we introduce an antidifference equation associated with the Riccati-type equation (4), and we show in the main results that it is very useful to study the antidifference equation instead of equation (4).

Lemma 2 *Assume that conditions* (H_1) *and* (H_2) *hold. Then the following statements are equivalent.*

(a) *There is a solution sequence* $\{\lambda(n)\}$, $n \in \mathbf{N}_M$, *of the Riccati-type equation* (4) *for some* $M \ge n_0$ *such that* $\lambda(n) + 1 > 0$ *for all* $n \in \mathbf{N}_M$ *and*

$$\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m} P_i(\ell) \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\lambda(j)+1}\right) < \infty. \tag{9}$$

(b) *There is a sequence* $\{\omega(n)\}$, $n \in \mathbf{N}_M$, *for some* $M \ge n_0$ *such that* $\omega(n) + 1 > 0$ *for all* $n \in \mathbf{N}_M$ *and*

$$\omega(n) = \sum_{\ell=n}^{\infty} \omega(\ell+1)\omega(\ell) + \sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m} P_i(\ell) \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\omega(j)+1}\right). \tag{10}$$

Proof (a) $\Rightarrow$ (b): Let the sequence $\{\lambda(n)\}$, defined by formula $\lambda(n) = \omega(n)$ for $n \in \mathbf{N}_M$, be a solution of the Riccati-type equation (4) for $n \in \mathbf{N}_M$ with the property (9). Let $n \ge M$ be a natural number fixed arbitrarily, and let us sum up both sides of the Riccati-type equation (4) from n to $M_1 - 1$, where $M_1 > n$. Then we get the equality

$$\omega(M_1) - \omega(n) + \sum_{\ell=n}^{M_1-1} \omega(\ell+1)\omega(\ell) + \sum_{\ell=n}^{M_1-1}\left(\sum_{i=1}^{m} P_i(\ell) \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\omega(j)+1}\right) = 0. \tag{11}$$

We claim that

$$\sum_{\ell=n}^{\infty} \omega(\ell+1)\omega(\ell) < \infty. \tag{12}$$

To this end, assume the contrary statement that $\sum_{\ell=n}^{\infty} \omega(\ell+1)\omega(\ell) = \infty$. Now, in view of (11) and the statement $\sum_{\ell=n}^{\infty} \omega(\ell+1)\omega(\ell) = \infty$, there are natural numbers M_1 and M^* large enough such that $M_1 > M^*$ and

$$\begin{aligned}
&\omega(M_1) + \sum_{\ell=M^*}^{M_1-1} \omega(\ell+1)\omega(\ell) \\
&\quad = \omega(n) - \sum_{\ell=n}^{M^*-1} \omega(\ell+1)\omega(\ell) - \sum_{\ell=n}^{M_1-1} \left(\sum_{i=1}^{m} P_i(\ell) \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\omega(j)+1} \right) \leq -1
\end{aligned}$$

for $M_1 > M^* \geq n$, or, equivalently,

$$-\omega(M_1) \geq \sum_{\ell=M^*}^{M_1-1} \omega(\ell+1)\omega(\ell) + 1.$$

Because of the substitution (5), it follows that

$$-\frac{a(M_1+1)}{a(M_1)} = -\omega(M_1) - 1 \geq \sum_{\ell=M^*}^{M_1-1} \omega(\ell+1)\omega(\ell).$$

Since $M_1 > M$ is an arbitrary large number, then the inequality

$$\sum_{\ell=M^*}^{M_1-1} \omega(\ell+1)\omega(\ell) < 0$$

holds. If $M_1 \to \infty$, then the inequality $\sum_{\ell=M^*}^{\infty} \omega(\ell+1)\omega(\ell) < 0$ must hold, which contradicts the assumption

$$\sum_{\ell=n}^{\infty} \omega(\ell+1)\omega(\ell) = \infty.$$

We now let $M_1 \to \infty$ in (11). Using inequalities (12) and (9), we find that $\omega(M_1)$ tends to a finite limit ω_∞. But ω_∞ must be zero because in the other case inequality (12) would fail to hold. These argumentations complete the first part of the proof.

(b) ⇒ (a): Assume that there is a sequence $\{\omega(n)\}$ which satisfies equation (10) for $n \in \mathbf{N}_M$, where $M \geq n_0$ is an arbitrary natural number such that $\omega(n) + 1 > 0$ for all $n \in \mathbf{N}_M$. Applying the difference operator to both sides of equation (10), we show that the sequence $\{\lambda(n)\}$, defined by formula $\lambda(n) = \omega(n)$ for $n \in \mathbf{N}_M$, is a solution of the Riccati-type equation (4) for $n \in \mathbf{N}_M$, and it satisfies assumption (9). The proof of the theorem is complete. □

Now we can formulate the main theorem.

Theorem 1 *Assume that conditions* (H_1) *and* (H_2) *hold. Then the following statements are equivalent.*

(a) *There exists a natural number* $M \geq n_0$ *and there exist the real sequences* $\{\beta(n)\}$ *and* $\{\gamma(n)\}$ *such that* $-1 < \beta(n) \leq \gamma(n)$, $\sup_n \gamma(n) < \infty$ *for* $n \in \mathbf{N}_M$,

$$\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m}\left|P_i(\ell)\right| \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\beta(j)+1}\right) < \infty \tag{13}$$

and such that

$$\beta(n) \leq \xi(n) \leq \gamma(n) \quad \text{implies that} \quad \beta(n) \leq S\xi(n) \leq \gamma(n) \tag{14}$$

for $n \in \mathbf{N}_M$ *and for real sequences* $\{\xi(n)\}$, *where*

$$S\xi(n) = \sum_{\ell=n}^{\infty} \xi(\ell+1)\xi(\ell) + \sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m} P_i(\ell) \prod_{j=\ell-k_i(\ell)}^{\ell-1} \frac{1}{\xi(j)+1}\right). \tag{15}$$

(b) *There exists a real solution sequence* $\{\omega(n)\}$ *of equation* (10) *which satisfies the inequality* $\beta(n) \leq \omega(n) \leq \gamma(n)$ *for* $n \in \mathbf{N}_M$.

Proof (a) ⇒ (b): We have to show that equation (10) has a solution sequence $\{\omega(n)\}$ for $n \in \mathbf{N}_M$. To this end, using Theorem D, we prove that operator S, defined by (15), has a fixed point $\{\omega(n)\}$, which is a solution sequence of equation (10), and obviously satisfies the estimate $\beta(n) \leq \omega(n) \leq \gamma(n)$ for $n \in \mathbf{N}_M$.

Let now M_1 and M_2 be natural numbers such that $M \leq M_1 < M_2 < \infty$. Then the set

$$\mathbf{N}_{1,2} = \{n \mid n \in \mathbf{N}, M_1 \leq n \leq M_2\}$$

is an arbitrary compact subset of the set $\mathbf{N}_M$. Set

$$L := \max_{M \leq n \leq M_2} \sum_{i=1}^{m}\left|P_i(n)\right|, \qquad G := \sup_{n \geq M} \gamma(n),$$

$$B := \min_{M \leq n \leq M_2} \beta(n), \qquad K := \max_{i=1}^{m} \max_{M \leq n \leq M_2} k_i(n),$$

$$K_B := \begin{cases} \max_{i=1}^{m} \max_{M \leq n \leq M_2} k_i(n), & -1 < B \leq 0, \\ \min_{i=1}^{m} \min_{M \leq n \leq M_2} k_i(n), & B > 0, \end{cases}$$

$$K_G := \begin{cases} \max_{i=1}^{m} \max_{M \leq n \leq M_2} k_i(n), & G > 1, \\ \min_{i=1}^{m} \min_{M \leq n \leq M_2} k_i(n), & G < 1. \end{cases}$$

Let

$$F := \left\{\left\{\xi(n)\right\} \in \mathbf{R}^{\infty} : \left|\xi(n)\right| \leq \mu(n), n \in \mathbf{N}_M\right\}, \quad \text{where } \mu(n) = \max_{n \geq n_0}\left\{\left|\gamma(n)\right|, 1\right\}.$$

It follows from assumptions (13) and (14) that the operator S, defined for $\{\xi(n)\} \in F$, satisfies the inequality $\sum_{\ell=n}^{\infty} \xi(\ell+1)\xi(\ell) < \infty$ and maps F to F. It follows immediately from assumption (14) that the sequences in the image set SF are uniformly bounded on any subset of $\mathbf{N}_M$.

Let the sequence of sequences $\{\xi_p(n)\} \in F$ tend to the sequence $\{\xi(n)\}$, $p \to \infty$, uniformly on any finite interval of N_M, which means this convergence is uniform for all $n \in \mathbf{N}_{1,2}$.

In virtue of the following transformations:

$$\begin{aligned} &\left|\xi(\ell+1)\xi(\ell) - \xi_p(\ell+1)\xi_p(\ell)\right| \\ &\quad = \left|\xi(\ell+1)\xi(\ell) - \xi_p(\ell+1)\xi(\ell) + \xi_p(\ell+1)\xi(\ell) - \xi_p(\ell+1)\xi_p(\ell)\right| \\ &\quad \le \left|\xi(\ell+1) - \xi_p(\ell+1)\right|\left|\xi(\ell)\right| + \left|\xi(\ell) - \xi_p(\ell)\right|\left|\xi_p(\ell+1)\right|, \end{aligned}$$

we obtain that

$$\sum_{\ell=n}^{M_2}\left|\xi(\ell+1)\xi(\ell) - \xi_p(\ell+1)\xi_p(\ell)\right| \le 2G\sum_{\ell=n}^{M_2+1}\left|\xi(\ell) - \xi_p(\ell)\right|,$$

and also we obtain that the inequalities

$$\begin{aligned} \prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{\xi(j)+1} - \prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{\xi_p(j)+1} &= \frac{\prod_{j=\ell-k_i(\ell)}^{\ell-1}(\xi_p(j)+1) - \prod_{j=\ell-k_i(\ell)}^{\ell-1}(\xi(j)+1)}{\prod_{j=\ell-k_i(\ell)}^{\ell-1}(\xi(j)+1)(\xi_p(j)+1)} \\ &\le \frac{((K-1)G^{K_G-1}+1)\sum_{j=\ell-k_i(\ell)}^{\ell-1}|\xi(\ell) - \xi_p(\ell)|}{\prod_{j=\ell-k_i(\ell)}^{\ell-1}(\beta(j)+1)(\beta^p(j)+1)} \\ &\le \frac{(KG^{K_G}+1)\sum_{j=\ell-K}^{\ell-1}|\xi(\ell) - \xi_p(\ell)|}{(B+1)^{2K_B}} \end{aligned}$$

are valid. Using the previous inequalities, we can get the following transformations:

$$\begin{aligned} &\left|S\xi(n) - S\xi_p(n)\right| \\ &\quad \le \lim_{M_2\to\infty}\left(\sum_{\ell=n}^{M_2}\left|\xi(\ell+1)\xi(\ell) - \xi_p(\ell+1)\xi_p(\ell)\right|\right. \\ &\qquad \left. + \sum_{\ell=n}^{M_2}\sum_{i=1}^{m}\left|P_i(\ell)\right|\left(\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{\xi(j)+1} - \prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{\xi_p(j)+1}\right)\right) \\ &\quad \le \lim_{M_2\to\infty}\left(2G\sum_{\ell=n}^{M_2+1}\left|\xi(\ell) - \xi_p(\ell)\right| + \frac{L(KG^{K_G}+1)}{(B+1)^{2K_B}}\sum_{\ell=n}^{M_2}\sum_{j=\ell-K}^{\ell-1}\left|\xi(\ell) - \xi_p(\ell)\right|\right). \end{aligned}$$

The uniform convergence of the sequence $\xi_p(n) \to \xi(n)$, $p \to \infty$, for all $n \in \mathbf{N}_{1,2}$ implies that

$$\left|\xi(n) - \xi_p(n)\right| < \delta \quad \text{for sufficiently large } p \text{ and } n \in \mathbf{N}_{1,2}.$$

If we use the following form of δ:

$$\delta = \frac{\varepsilon}{C(M_2+2)}, \quad \text{where } C = 2G + \frac{LK(KG^{K_G}+1)}{(B+1)^{2K_B}},$$

we obtain

$$\begin{aligned}\left|S\xi(n)-S\xi_p(n)\right| &\le \lim_{M_2\to\infty}\left(2G\delta(M_2+2-n)+\frac{L(KG^{K_G}+1)}{(B+1)^{2K_B}}K\delta(M_2+1-n)\right)\\ &< \lim_{M_2\to\infty}\left(2G+\frac{L(KG^{K_G}+1)}{(B+1)^{2K_B}}K\right)\delta(M_2+2)\\ &= \lim_{M_2\to\infty} C\frac{\varepsilon}{C(M_2+2)}(M_2+2)=\lim_{M_2\to\infty}\varepsilon=\varepsilon\end{aligned}$$

for $n \in \mathbf{N}_{1,2}$, if p is sufficiently large. Thus, $S\xi_p(n) \to S\xi(n)$, $p \to \infty$, uniformly on any finite subset N_M.

We obtained that the conditions of the Schauder-Tychonoff theorem are satisfied, and hence the mapping S has at least one fixed point $\{\omega(n)\}$ in F. Moreover, because of the equality $\omega(n) = S\omega(n)$ for $n \in \mathbf{N}_M$, we conclude that $\{\omega(n)\}$ is the solution sequence of equation (10) with the property that $\beta(n) \le \omega(n) \le \gamma(n)$ for $n \in \mathbf{N}_M$. The first part of the proof is complete.

(b) $\Rightarrow$ (a): If $\{\omega(n)\}$ is a solution sequence of (10), then taking $\beta(n) = \gamma(n) = \omega(n)$ for $n \in \mathbf{N}_M$, the conditions of Theorem 1 are satisfied because of the fact that $\omega(n) = S\omega(n)$. The proof is complete. □

Existence of positive solutions

Let the sequence $\{\mu(n)\}$ be such that $0 < \mu(n) < 1$, and set $\beta(n) = -\mu(n)$ and $\gamma(n) = \mu(n)$ in Theorem 1. Now we can formulate the conditions for the existence of positive solutions of equation (3).

Theorem 2 *Assume that conditions* (H_1) *and* (H_2) *hold and there exists a positive sequence* $\{\mu(n)\}$ *for* $n \in \mathbf{N}_M$ *for some natural number* $M \ge n_0$ *such that* $0 < \mu(n) < 1$ *for* $n \in \mathbf{N}_M$ *and*

$$\sum_{\ell=n}^{\infty}\left(\mu(\ell+1)\mu(\ell)+\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{1-\mu(j)}\right)\le\mu(n) \tag{16}$$

holds for n large enough. Then equation (3) *has a positive solution* $\{a(n)\}$ *for* $n \in \mathbf{N}_M$.

Proof Let the sequence $\{\mu(n)\}$ be given such that the conditions of the theorem hold. Since $0 < \mu(n) < 1$, hence $-1 < -\mu(n) < 0$ and $0 < 1 - \mu(n) < 1$. We show that the conditions of Theorem 1 are satisfied with $\beta(n) = -\mu(n)$ and $\gamma(n) = \mu(n)$ for n large enough.

Let $\{\xi(n)\}$ be a real sequence such that $|\xi(n)| \le \mu(n)$. Because of the property that $-\mu(n) \le \xi(n) \le \mu(n)$, the inequality

$$1-\mu(n)\le\xi(n)+1 \quad \text{implies that} \quad \frac{1}{\xi(n)+1}\le\frac{1}{1-\mu(n)} \quad \text{for } n\in\mathbf{N}_M.$$

Because of assumption (16) and some transformations, it follows that

$$\begin{aligned}\left|S\xi(n)\right| &\le \sum_{\ell=n}^{\infty}\left(\left|\xi(\ell+1)\right|\left|\xi(\ell)\right|+\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{\xi(j)+1}\right)\\ &\le \sum_{\ell=n}^{\infty}\left(\mu_{\ell+1}\mu(\ell)+\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{1-\mu(j)}\right)\le\mu(n).\end{aligned}$$

Therefore, in virtue of Theorem 1, equation (3) has a positive solution and the proof is complete. □

Remark 1 The result of Theorem 2 is the discrete analogue of the result presented in Theorem A and generalizes the result given in [39] for first-order linear difference equations with variable delays.

Now we would like to find the sequence $\{\mu(n)\}$ in the form $\mu(n) = \frac{A}{n}$, where A is some constant. Since

$$\sum_{\ell=n}^{\infty}\mu(\ell+1)\mu(\ell) = \sum_{\ell=n}^{\infty}\frac{A^2}{\ell(\ell+1)} = \lim_{M\to\infty}\sum_{\ell=n}^{M}\frac{A^2}{\ell(\ell+1)} = \lim_{M\to\infty}\sum_{\ell=n}^{M}\left(\frac{A^2}{\ell} - \frac{A^2}{\ell+1}\right)$$
$$= \lim_{M\to\infty}\left(\frac{A^2}{n} - \frac{A^2}{n+1} + \frac{A^2}{n+1} - \frac{A^2}{n+2} + \cdots + \frac{A^2}{M} - \frac{A^2}{M+1}\right) = \frac{A^2}{n}$$

and

$$\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{1-\mu(j)} = \prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{1}{1-\frac{A}{j}} = \prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{j}{j-A},$$

condition (16) takes the form

$$\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{j}{j-A}\right) \le \frac{A}{n} - \frac{A^2}{n}.$$

Choosing A such that the function $f(A) = A - A^2$ takes the maximum value, we obtain $A = \frac{1}{2}$ and we can formulate the following corollary of Theorem 2.

Corollary 1 *Assume that conditions* (H_1) *and* (H_2) *hold and*

$$n\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{2j}{2j-1}\right) \le \frac{1}{4} \tag{17}$$

holds for n large enough. Then equation (3) *has an eventually positive solution.*

In particular, if the sequences of delays $\{k_i(n)\}$ $(i = 1, 2, \ldots, m)$ are bounded by K, we have

$$\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m}\left|P_i(\ell)\right|\prod_{j=\ell-k_i(\ell)}^{\ell-1}\frac{2j}{2j-1}\right) \le \left(\frac{2(n-K)}{2(n-K)-1}\right)^{K}\sum_{\ell=n}^{\infty}\left(\sum_{i=1}^{m}\left|P_i(\ell)\right|\right)$$

and

$$\limsup_{n\to\infty}\left(\frac{2(n-K)}{2(n-K)-1}\right)^{K} = 1.$$

Hence, we obtain the following non-oscillation criterion.

Corollary 2 *Assume that conditions* (H_1) *and* (H_2) *hold and the sequences of delays* $\{k_i(n)\}$ ($i = 1, 2, \dots, m$) *are bounded. If*

$$\limsup_{n\to\infty} n \sum_{\ell=n}^{\infty} \sum_{i=1}^{m} \left|P_i(\ell)\right| \le \frac{1}{4} \tag{18}$$

holds, *then equation* (3) *has an eventually positive solution*.

Remark 2 The result of Corollary 2 is the discrete analogue of the result presented in Theorem B and at the same time generalizes the result given in Theorem C for second-order linear difference equations with variable delays.

Competing interests
The author declares that she has no competing interests.

Authors' contributions
Since there is one author, she completed all tasks necessary for the article.

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