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On certain univalent functions with missing coefficients

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Abstract
The main object of the present paper is to show certain sufficient conditions for univalency of analytic functions with missing coefficients.
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1 Introduction

Let $A(n)$ be the class of functions of the form

$$f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (n = 2, 3, \ldots), \tag{1.1}$$

which are analytic in the unit disk $U = \{z : |z| < 1\}$. We write $A(2) = A$.

A function $f(z) \in A$ is said to be starlike in $|z| < r$ $(r \leq 1)$ if and only if it satisfies

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0 \quad \bigl(|z| < r\bigr). \tag{1.2}$$

A function $f(z) \in A$ is said to be close-to-convex in $|z| < r$ $(r \leq 1)$ if and only if there is a starlike function $g(z)$ such that

$$\operatorname{Re} \frac{zf'(z)}{g(z)} > 0 \quad \bigl(|z| < r\bigr). \tag{1.3}$$

Let $f(z)$ and $g(z)$ be analytic in U. Then we say that $f(z)$ is subordinate to $g(z)$ in U, written $f(z) \prec g(z)$, if there exists an analytic function $w(z)$ in U, such that $|w(z)| \leq |z|$ and $f(z) = g(w(z))$ $(z \in U)$. If $g(z)$ is univalent in U, then the subordination $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$.

Recently, several authors showed some new criteria for univalency of analytic functions (see, *e.g.*, [1–7]). In this note, we shall derive certain sufficient conditions for univalency of analytic functions with missing coefficients.

For our purpose, we shall need the following lemma.

Lemma (see [8, 9]) *Let $f(z)$ and $g(z)$ be analytic in U with $f(0) = g(0)$. If $h(z) = zg'(z)$ is starlike in U and $zf'(z) \prec h(z)$, then*

$$f(z) \prec f(0) + \int_0^z \frac{h(t)}{t}\,dt. \tag{1.4}$$

2 Main results

Our first theorem is given by the following.

Theorem 1 *Let* $f(z) = z + a_n z^n + \cdots \in A(n)$ *with* $f(z) \neq 0$ *for* $0 < |z| < 1$. *If*

$$\left|\left(\frac{z}{f(z)}\right)^{(n)}\right| \leq \beta \quad (z \in U), \tag{2.1}$$

where $0 < \beta \leq 2[1 - (n-2)|a_n|]$, *then* $f(z)$ *is univalent in* U.

Proof Let

$$p(z) = \left(\frac{z}{f(z)}\right)^{(n)} \quad (z \in U), \tag{2.2}$$

then $p(z)$ is analytic in U. By integration from 0 to z n-times, we obtain

$$\frac{z}{f(z)} = 1 - a_n z^{n-1} + \int_0^z dw_n \int_0^{w_n} dw_{n-1} \cdots \int_0^{w_2} p(w_1)\, dw_1 \quad (z \in U). \tag{2.3}$$

Thus, we have

$$f(z) = \frac{z}{1 - a_n z^{n-1} + \varphi(z)} \quad (z \in U), \tag{2.4}$$

where

$$\varphi(z) = \int_0^z dw_n \int_0^{w_n} dw_{n-1} \cdots \int_0^{w_2} p(w_1)\, dw_1 \quad (z \in U). \tag{2.5}$$

It is easily seen from (2.1), (2.2) and (2.5) that

$$\left|\varphi^{(n)}(z)\right| \leq \beta \quad (z \in U) \tag{2.6}$$

and, in consequence,

$$\left|\varphi''(z)\right| \leq \beta \quad (z \in U).$$

Since

$$\left(\frac{\varphi(z)}{z}\right)' = \frac{1}{z^2} \int_0^z w\varphi''(w)\, dw \quad (z \in U),$$

we get

$$\left|\left(\frac{\varphi(z)}{z}\right)'\right| = \left|\frac{1}{z^2} \int_0^z w\varphi''(w)\, dw\right| \leq \frac{\beta}{2} \quad (z \in U)$$

and so

$$\left|\frac{\varphi(z_2)}{z_2} - \frac{\varphi(z_1)}{z_1}\right| = \left|\int_{z_1}^{z_2} \left(\frac{\varphi(w)}{w}\right)' dw\right| \leq \frac{\beta}{2}|z_2 - z_1| \tag{2.7}$$

for $z_1, z_2 \in U$ and $z_1 \neq z_2$.

Now it follows from (2.4) and (2.7) that

$$\begin{aligned}
&\left|f(z_2)-f(z_1)\right| \\
&\quad = \frac{|(z_2-z_1)+a_nz_1z_2(z_2^{n-2}-z_1^{n-2})-z_1z_2(\frac{\varphi(z_2)}{z_2}-\frac{\varphi(z_1)}{z_1})|}{|1-a_nz_1^{n-1}+\varphi(z_1)||1-a_nz_2^{n-1}+\varphi(z_2)|} \\
&\quad > \frac{|z_2-z_1|(1-(n-2)|a_n|-\frac{\beta}{2})}{|1-a_nz_1^{n-1}+\varphi(z_1)||1-a_nz_2^{n-1}+\varphi(z_2)|} \\
&\quad \geq 0.
\end{aligned}$$

Hence, $f(z)$ is univalent in U. The proof of the theorem is complete. □

Let $S_n(\beta)$ denote the class of functions $f(z)=z+a_nz^n+\cdots\in A(n)$ with $f(z)\neq 0$ for $0<|z|<1$, which satisfy the condition (2.1) given by Theorem 1.

Next we derive the following.

Theorem 2 *Let* $f(z)=z+a_nz^n+\cdots\in S_n(\beta)$. *Then, for* $z\in U$,

$$\left|\frac{z}{f(z)}-1\right| \leq |z|^{n-1}\left(|a_n|+\frac{\beta}{2}|z|\right); \tag{2.8}$$

$$1-|z|^{n-1}\left(|a_n|+\frac{\beta}{2}|z|\right) \leq \operatorname{Re}\frac{z}{f(z)} \leq 1+|z|^{n-1}\left(|a_n|+\frac{\beta}{2}|z|\right); \tag{2.9}$$

$$\left|f(z)\right| \geq \frac{|z|}{1+|a_n||z|^{n-1}+\frac{\beta}{2}|z|^n}. \tag{2.10}$$

Proof In view of (2.1), we have

$$z\left(\frac{z}{f(z)}\right)^{(n)} \prec \beta z \quad (z\in U). \tag{2.11}$$

Applying Lemma to (2.11), we get

$$\left(\frac{z}{f(z)}\right)^{(n-1)} + (n-1)!a_n \prec \beta z \quad (z\in U). \tag{2.12}$$

By using the lemma repeatedly, we finally have

$$\left(\frac{z}{f(z)}\right)' + (n-1)a_nz^{n-2} \prec \beta z \quad (z\in U). \tag{2.13}$$

According to a result of Hallenbeck and Ruscheweyh [1, Theorem 1], (2.13) gives

$$\frac{1}{z}\int_0^z\left[\left(\frac{t}{f(t)}\right)' + (n-1)a_nt^{n-2}\right]dt \prec \frac{\beta}{2}z \quad (z\in U), \tag{2.14}$$

i.e.,

$$\frac{z}{f(z)} = 1-a_nz^{n-1}+\frac{\beta}{2}zw(z) \quad (z\in U), \tag{2.15}$$

where $w(z)$ is analytic in U and $|w(z)|\leq|z|^{n-1}$ $(z\in U)$.

Now, from (2.15), we can easily derive the inequalities (2.8), (2.9) and (2.10). □

Finally, we discuss the following theorem.

Theorem 3 *Let $f(z) \in S_n(\beta)$ and have the form*

$$f(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \cdots \quad (z \in U). \tag{2.16}$$

(i) *If $\frac{2}{\sqrt{5}} \le \beta \le 2$, then $f(z)$ is starlike in $|z| < \sqrt[n]{\frac{2}{\beta}} \cdot \frac{1}{\sqrt[2n]{5}}$;*

(ii) *If $\sqrt{3} - 1 \le \beta \le 2$, then $f(z)$ is close-to-convex in $|z| < \sqrt[n]{\frac{\sqrt{3}-1}{\beta}}$.*

Proof If we put

$$p(z) = \frac{z^2 f'(z)}{f^2(z)} = 1 + p_n z^n + \cdots \quad (z \in U), \tag{2.17}$$

then by (2.1) and the proof of Theorem 2 with $a_n = 0$, we have

$$zp'(z) = -z^2 \left(\frac{z}{f(z)}\right)'' \prec \beta z. \tag{2.18}$$

It follows from the lemma that

$$p(z) \prec 1 + \beta z, \tag{2.19}$$

which implies that

$$\left|\frac{z^2 f'(z)}{f^2(z)} - 1\right| \le \beta |z|^n \quad (z \in U). \tag{2.20}$$

(i) Let $\frac{2}{\sqrt{5}} \le \beta \le 2$ and

$$|z| < r_1 = \sqrt[n]{\frac{2}{\beta}} \cdot \frac{1}{\sqrt[2n]{5}}. \tag{2.21}$$

Then by (2.20), we have

$$\left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| < \arcsin \frac{2}{\sqrt{5}}. \tag{2.22}$$

Also, from (2.8) in Theorem 2 with $a_n = 0$, we obtain

$$\left|\frac{z}{f(z)} - 1\right| < \frac{\beta}{2} r_1^n \tag{2.23}$$

and so

$$\left|\arg \frac{z}{f(z)}\right| < \frac{1}{\sqrt{5}}. \tag{2.24}$$

Therefore, it follows from (2.22) and (2.24) that

$$\begin{aligned} \left|\arg \frac{zf'(z)}{f(z)}\right| &\leq \left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| + \left|\arg \frac{z}{f(z)}\right| \\ &< \arcsin \frac{2}{\sqrt{5}} + \arcsin \frac{1}{\sqrt{5}} \\ &= \frac{\pi}{2} \end{aligned}$$

for $|z| < r_1$. This proves that $f(z)$ is starlike in $|z| < r_1$.

(ii) Let $\sqrt{3} - 1 \leq \beta \leq 2$ and

$$|z| < r_2 = \sqrt[n]{\frac{\sqrt{3}-1}{\beta}}. \tag{2.25}$$

Then we have

$$\begin{aligned} \left|\arg f'(z)\right| &\leq \left|\arg \frac{z^2 f'(z)}{f^2(z)}\right| + 2\left|\arg \frac{z}{f(z)}\right| \\ &< \arcsin\left(\beta r_2^n\right) + 2\arcsin\left(\frac{\beta}{2} r_2^n\right) \\ &= \arcsin(\sqrt{3}-1) + 2\arcsin\left(\frac{\sqrt{3}-1}{2}\right) \\ &= \frac{\pi}{2}. \end{aligned}$$

Thus, $\operatorname{Re} f'(z) > 0$ for $|z| < r_2$. This shows that $f(z)$ is close-to-convex in $|z| < r_2$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have made the same contribution. All authors read and approved the final manuscript.

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