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On the value distribution and uniqueness of difference polynomials of meromorphic functions

Hong-Yan Xu*

*Correspondence:
xhyhhh@126.com
Department of Informatics and Engineering, Jingdezhen Ceramic Institute, Jingdezhen, Jiangxi 333403, China

Abstract

In this paper, we study the zeros of difference polynomials of meromorphic functions of the forms

$$\left(P(f)\prod_{j=1}^{d}f(z+c_j)^{s_j}\right)^{(k)}-\alpha(z),\qquad \left(P(f)\prod_{j=1}^{d}\left[f(z+c_j)-f(z)\right]^{s_j}\right)^{(k)}-\alpha(z),$$

where $P(f)$ is a nonzero polynomial of degree n, $c_j \in \mathbb{C}\setminus\{0\}$ $(j=1,\dots,d)$ are distinct constants, n,k,d,s_j $(j=1,\dots,d)\in\mathbb{N}_+$, and $\alpha(z)$ is a small function of f. Our results of this paper are an improvement of the previous theorems given by Chen and Chen (Chinese Ann. Math., Ser. A 33(3):359-374, 2012) and Liu *et al.* (Appl. Math. J. Chin. Univ. Ser. B 27(1):94-104, 2012). In addition, we also investigate the uniqueness for difference polynomials of transcendental entire functions sharing one value and obtain some results which extend the recent theorem given by Liu *et al.* (Adv. Differ. Equ. 2011:234215, 2012).

MSC: 30D35; 39A10

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1 Introduction and main results

This purpose of this paper is to study the problem of the zeros and uniqueness of complex difference polynomials of meromorphic functions. The fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions are used (see [1, 2]). In addition, for a meromorphic function f, we use $S(r,f)$ to denote any quantity satisfying $S(r,f)=o(T(r,f))$ for all r outside a possible exceptional set E of finite logarithmic measure $\lim_{r\to\infty}\int_{[1,r)\cap E}\frac{dt}{t}<\infty$. A meromorphic function $a(z)$ is called a small function with respect to f if $T(r,a(z))=S(r,f)$, and the hyper order of a meromorphic function f is defined by

$$\rho_2(f)=\limsup_{r\to\infty}\frac{\log\log T(r,f)}{\log r}.$$

Recently, the topic of a difference equation and a difference product in the complex plane $\mathbb{C}$ has attracted many mathematicians, a number of papers have focused on value distribution and uniqueness of differences and differences operator analogues of Nevanlinna theory (including [3–7]).

For a transcendental meromorphic function f of finite order, a nonzero complex constant c and $\alpha(z) \in \mathbb{S}(f)$, Liu *et al.* [8], Chen *et al.* [9], Luo and Lin [10] studied the zeros distributions of difference polynomials of meromorphic functions and obtained: If $n \geq 6$, then $f(z)^n f(z+c) - \alpha(z)$ has infinitely many zeros [8, Theorem 1.2]; if $n > m$, then $P(f)f(z+c) - a(z)$ has infinitely many zeros (see [10, Theorem 1]), where $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is a nonzero polynomial, where $a_0, a_1, \ldots, a_n \ (\not\equiv 0)$ are complex constants and m is the number of the distinct zeros of $P(z)$.

In 2012, Liu *et al.* [11] studied the zeros distribution of difference-differential polynomials, which are the derivatives of difference products of entire functions, and obtained the result as follows.

Theorem 1.1 (see [11, Theorem 1.1]) *Let f be a transcendental entire function of finite order. If $n \geq k+2$, then the difference-differential polynomial $[f(z)^n f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.*

In the same year, Chen and Chen [12] studied the zeros of difference polynomials $f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{s_j}$, where $c_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, d)$ are distinct constants, n, d, s_j $(j = 1, \ldots, d) \in \mathbb{N}_+$, and obtained the following theorem.

Theorem 1.2 (see [12, Theorem 1.1]) *Let f be a transcendental entire function of finite order. If $n \geq 2$, then $f^n(f^m - 1)\prod_{j=1}^{d} f(z+c_j)^{s_j} - \alpha(z)$ has infinitely many zeros.*

Let

$$F(z) = P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}, \tag{1}$$

$$F_1(z) = P(f)\prod_{j=1}^{d} \bigl[f(z+c_j) - f(z)\bigr]^{s_j}. \tag{2}$$

We investigate the value distribution of difference polynomials of a more general form

$$\bigl(F(z)\bigr)^{(k)} - \alpha(z) = \left(P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)} - \alpha(z),$$

$$\bigl(F_1(z)\bigr)^{(k)} - \alpha(z) = \left(P(f)\prod_{j=1}^{d} \bigl[f(z+c_j) - f(z)\bigr]^{s_j}\right)^{(k)} - \alpha(z),$$

where $P(f)$ is a nonzero polynomial of degree n, m is the number of the distinct zeros of $P(z)$, $c_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, d)$ are distinct constants, n, d, s_j $(j = 1, \ldots, d) \in \mathbb{N}_+$, and $\alpha(z)$ is a small function of f. Set $\lambda = s_1 + s_2 + \cdots + s_d$, we obtain the following results.

Theorem 1.3 *Let f be a transcendental meromorphic* (*resp. entire*) *function of $\rho_2(f) < 1$, and let $F(z)$ be stated as in* (1). *If $c_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, d)$ are distinct constants, n, d, s_j $(j = 1, \ldots, d) \in \mathbb{N}_+$ satisfy $n > m(k+1) + 2d + 1 + \lambda$* (*resp. $n > m(k+1) + d - \lambda$*), *then $(F(z))^{(k)} - \alpha(z)$ has infinitely many zeros.*

Theorem 1.4 *Let f be a transcendental meromorphic (resp. entire) function of $\rho_2(f) < 1$, and let $F_1(z)$ be stated as in (2). If $c_j \in \mathbb{C} \setminus \{0\}$ $(j = 1, \ldots, d)$ are distinct constants, n, d, s_j $(j = 1, \ldots, d) \in \mathbb{N}_+$ satisfy $n > (m + 2d)(k + 1) + \lambda + d + 1$ (resp. $n > (m + 2d)(k + 1)$), then $(F_1(z))^{(k)} - \alpha(z)$ has infinitely many zeros, provided that $f(z + c_j) \not\equiv f(z)$ for $j = 1, 2, \ldots, d$.*

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions for some $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f(z) - a$ and $g(z) - a$ (if $a = \infty$, zeros of $f(z) - a$ and $g(z) - a$ are the poles of $f(z)$ and $g(z)$ respectively) coincide in locations and multiplicities, we say that $f(z)$ and $g(z)$ share the value a *CM* (counting multiplicities), and if they coincide in locations only, we say that $f(z)$ and $g(z)$ share a *IM* (ignoring multiplicities).

For the uniqueness of difference of meromorphic functions, some results have been obtained (see [8, 10, 13–15]). Here, we only state some newest theorems as follows.

Theorem 1.5 [10, Theorem 2] *Let f and g be transcendental entire functions of finite order, let c be a nonzero complex constant, let $P(z)$ be stated as in Theorem 1.3, and let $n > 2\Gamma_0 + 1$ be an integer, where $\Gamma_0 = m_1 + 2m_2$, m_1 is the number of simple zeros of $P(z)$, and m_2 is the number of multiple zeros of $P(z)$. If $P(f)f(z + c)$ and $P(g)g(z + c)$ share 1 CM, then one of the following results holds:*

(i) *$f \equiv tg$ for a constant t such that $t^l = 1$, where $l = GCD\{\lambda_0, \lambda_1, \ldots, \lambda_n\}$ and*

$$\lambda_i = \begin{cases} i + 1, & a_i \neq 0, \\ n + 1, & a_i = 0, \end{cases} \qquad i = 0, 1, 2, \ldots, n.$$

(ii) *f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = P(\omega_1)\omega_1(z + c) - P(\omega_2)\omega_2(z + c)$;*

(iii) *$f(z) = e^{\alpha(z)}$, $g(z) = e^{\beta(z)}$, where $\alpha(z)$ and $\beta(z)$ are two polynomials, b is a constant satisfying $\alpha + \beta \equiv b$ and $a_n^2 e^{(n+1)b} = 1$.*

In this paper, we investigate the uniqueness problem of difference polynomials of entire functions

$$F^*(z) = f^n\left(f^m - 1\right)\prod_{j=1}^{d}\left(f(z + c_j)\right)^{s_j} \quad \text{and} \quad G^*(z) = g^n\left(g^m - 1\right)\prod_{j=1}^{d}\left(g(z + c_j)\right)^{s_j}$$

and obtain the following results.

Theorem 1.6 *Let f, g be transcendental entire functions of $\rho_2(f), \rho_2(g) < 1$. If $(F^*(z))^{(k)}$ and $(G^*(z))^{(k)}$ share 1 CM and $n > \max\{2(k + d + 2) + m - \lambda, 2d + 2 + \lambda, k + 1\}$, where $c_j \in \mathbb{C}$, n, m, k, d, s_j $(j = 1, 2, \ldots, d) \in \mathbb{N}_+$, then $f \equiv tg$ for a constant t such that $t^\kappa = 1$, where $\kappa = GCD\{m, n + \lambda\}$.*

Theorem 1.7 *Under the assumptions of Theorem 1.6, if $(F^*(z))^{(k)}$ and $(G^*(z))^{(k)}$ share 1 IM and $n > \max\{5(k + d) + 4m + 7 - \lambda, k + 1\}$, then the conclusions of Theorem 1.6 hold.*

2 Some lemmas

To prove our theorems, we require some lemmas as follows.

Lemma 2.1 [2] *Let f be a nonconstant meromorphic function, and let $P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n$, where $a_0, a_1, a_2, \ldots, a_n$ are constants and $a_n \neq 0$. Then*

$$T\bigl(r, P(f)\bigr) = nT(r,f) + S(r,f).$$

Lemma 2.2 [16, Theorem 5.1] *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, $\varsigma < 1$, and let ε be an enough small number. Then*

$$m\left(r, \frac{f(z+c)}{f(z)}\right) = o\left(\frac{T(r,f)}{r^{1-\varsigma-\varepsilon}}\right) = S(r,f)$$

for all r outside of a set of finite logarithmic measure.

Lemma 2.3 [16, Lemma 8.3] *Let $T : [0, +\infty) \to [0, +\infty)$ be a non-decreasing continuous function, and let $s \in (0, +\infty)$. If the hyper order of T is strictly less that one, that is,*

$$\limsup_{r\to\infty} \frac{\log\log T(r)}{\log r} < 1,$$

and $\delta \in (0, 1 - \varsigma)$, then

$$T(r+s) = T(r) + o\left(\frac{T(r)}{r^{\delta}}\right)$$

for all r runs to infinity outside of a set of finite logarithmic measure.

From Lemma 2.3, we can get the following lemma easily.

Lemma 2.4 *Let $f(z)$ be a transcendental meromorphic function of hyper order $\rho_2(f) < 1$, and let c be a nonzero complex constant. Then we have*

$$T\bigl(r, f(z+c)\bigr) = T\bigl(r, f(z)\bigr) + S(r,f), \qquad N\bigl(r, f(z+c)\bigr) = N\bigl(r, f(z)\bigr) + S(r,f),$$

and

$$N\left(r, \frac{1}{f(z+c)}\right) = N\left(r, \frac{1}{f}\right) + S(r,f).$$

Lemma 2.5 *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, let $F(z)$ be stated as in* (1). *Then we have*

$$(n-\lambda)T(r,f) + S(r,f) \leq T\bigl(r, F(z)\bigr) \leq (n+\lambda)T(r,f) + S(r,f). \tag{3}$$

If f is a transcendental entire function of $\rho_2(f) < 1$, we have

$$T\bigl(r, F(z)\bigr) = T\bigl(r, P(f)f^{\lambda}\bigr) + S(r,f) = (n+\lambda)T(r,f) + S(r,f), \tag{4}$$

where $\lambda = s_1 + s_2 + \cdots + s_d$.

Proof If f is a transcendental entire function of $\rho_2(f) < 1$, from Lemma 2.1-Lemma 2.3, we have

$$\begin{aligned} T\big(r,F(z)\big) = m\big(r,F(z)\big) &\leq m\big(r,P(f)f^{\lambda}(z)\big) + m\left(r,\frac{\prod_{j=1}^{d} f(z+c_j)^{s_j}}{f^{\lambda}(z)}\right) \\ &\leq m\big(r,P(f)f^{\lambda}(z)\big) + S(r,f) = T\big(r,P(f)f^{\lambda}(z)\big) + S(r,f) \\ &= (n+\lambda)T(r,f) + S(r,f). \end{aligned}$$

On the other hand, from Lemma 2.1, we have

$$\begin{aligned} (n+\lambda)T(r,f) &= T\big(r,P(f)f^{\lambda}(z)\big) + S(r,f) = m\big(r,P(f)f^{\lambda}(z)\big) + S(r,f) \\ &\leq m\big(r,F(z)\big) + m\left(r,\frac{f^{\lambda}(z)}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &= T\big(r,F(z)\big) + S(r,f). \end{aligned}$$

Thus, we can get (4).

If f is a meromorphic function of $\rho_2(f) < 1$, from Lemma 2.1 and Lemma 2.4, we have

$$T\left(r,P(f)\prod_{i=1}^{d} f(z+c_j)^{s_j}\right) \leq T\big(r,P(f)\big) + T\left(r,\prod_{i=1}^{d} f(z+c_j)^{s_j}\right) \leq (n+\lambda)T(r,f) + S(r,f).$$

On the other hand, from Lemma 2.1 and Lemma 2.4, we have

$$\begin{aligned} (n+\lambda)T(r,f) &= T\big(r,P(f)f^{\lambda}\big) + S(r,f) = m\big(r,P(f)f^{\lambda}\big) + N\big(r,P(f)f^{\lambda}\big) + S(r,f) \\ &\leq m\left(r,F(z)\frac{f^{\lambda}(z)}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + N\left(r,F(z)\frac{f^{\lambda}(z)}{\prod_{j=1}^{d} f(z+c_j)^{s_j}}\right) + S(r,f) \\ &\leq T\big(r,F(z)\big) + 2\lambda T(r,f) + S(r,f). \end{aligned}$$

Thus, we can get (3). □

Using the same method as in Lemma 2.5, we can get the following lemma easily.

Lemma 2.6 *Let f be a transcendental meromorphic function of $\rho_2(f) < 1$, and let $F_1(z)$ be stated as in* (2). *Then we have*

$$(n-\lambda)T(r,f) + S(r,f) \leq T\big(r,F_1(z)\big) \leq (n+2\lambda)T(r,f) + S(r,f).$$

If f is a transcendental entire function of $\rho_2(f) < 1$, we have

$$T\big(r,F_1(z)\big) = T\left(r,P(f)\prod_{j=1}^{d}\big[f(z+c_j) - f(z)\big]^{s_j}\right) \geq nT(r,f) + S(r,f).$$

In the following, we explain some definitions and notations which are used in this paper. For $a \in \mathbb{C} \cup \infty$ and k is a positive integer, we denote by $\overline{N}_k(r,\frac{1}{f-a})$ the counting function of those a-points of f whose multiplicities are not less than k; in counting the a-points of f, we ignore the multiplicities (see [1, 17]) and $N_k(r, \frac{1}{f-a}) = \overline{N}(r, \frac{1}{f-a}) + \overline{N}_2(r, \frac{1}{f-a}) + \cdots + \overline{N}_k(r, \frac{1}{f-a})$.

Definition 2.7 [18] When f and g share 1 *IM*, we denote by $\overline{N}_L(r, \frac{1}{f-1})$ the counting function of the 1-points of f whose multiplicities are greater than 1-points of g, where each zero is counted only once; similarly, we have $\overline{N}_L(r, \frac{1}{g-1})$. Let z_0 be a zero of $f-1$ of multiplicity p and a zero of $g-1$ of multiplicity q. We also denote by $N_{11}(r, \frac{1}{f-1})$ the counting function of those 1-points of f where $p = q = 1$.

Lemma 2.8 [2] and [7, Lemma 2.4] *Let f be a nonconstant meromorphic function, and let p, k be positive integers. Then*

$$
\begin{aligned}
&T\bigl(r,f^{(k)}\bigr) \le T(r,f) + k\overline{N}(r,f) + S(r,f),\\
&N_p\left(r,\frac{1}{f^{(k)}}\right) \le T\bigl(r,f^{(k)}\bigr) - T(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f),\\
&N_p\left(r,\frac{1}{f^{(k)}}\right) \le k\overline{N}(r,f) + N_{p+k}\left(r,\frac{1}{f}\right) + S(r,f).
\end{aligned}
$$

Lemma 2.9 [19] *Let f and g be two meromorphic functions. If f and g share* 1 *CM, then one of the following three cases holds*:

(i)

$$
\begin{aligned}
T(r,f) + T(r,g) \le{}& 2N_2(r,f) + 2N_2(r,g) + 2N_2\left(r,\frac{1}{f}\right) + 2N_2\left(r,\frac{1}{g}\right)\\
&+ S(r,f) + S(r,g);
\end{aligned}
$$

(ii) $f \equiv g$;

(iii) $f \cdot g = 1$.

Lemma 2.10 [20] *Let f and g be two meromorphic functions. If f and g share* 1 *IM, then one of the following cases must occur*:

(i)

$$
\begin{aligned}
T(r,f) + T(r,g) \le{}& 2\left[N_2(r,f) + N_2\left(r,\frac{1}{f}\right) + N_2(r,g) + N_2\left(r,\frac{1}{g}\right)\right]\\
&+ 3\overline{N}_L\left(r,\frac{1}{f-1}\right) + 3\overline{N}_L\left(r,\frac{1}{g-1}\right)\\
&+ S(r,f) + S(r,g);
\end{aligned}
$$

(ii) $f = \frac{(b+1)g+(a-b-1)}{bg+(a-b)}$, *where a $(\neq 0)$, b are two constants.*

Lemma 2.11 [21, Theorem 2.1] *Let $a_0(z), \ldots, a_n(z), b(z)$ be polynomials such that $a_0(z)a_n(z) \not\equiv 0$, $\deg(\sum_{\deg a_j = d} a_j) = d$, where $d = \max_{0\le j\le n}\{\deg a_j\}$. If $f(z)$ is a transcendental meromorphic solution of*

$$a_0(z)f(z) + a_1(z)f(z+c_j) + \cdots + a_n(z)f(z+c_n) = b(z),$$

then $\rho(f) \ge 1$.

Lemma 2.12 *Let $f(z)$ and $g(z)$ be transcendental entire functions of $\rho_2(f) < 1$, $\rho_2(g) < 1$. If $n \geq k+1$ and*

$$\left(f^n(f^m-1)\prod_{j=1}^{d} f(z+c_j)^{s_j}\right)^{(k)}\left(g^n(g^m-1)\prod_{j=1}^{d} g(z+c_j)^{s_j}\right)^{(k)} \equiv 1, \tag{5}$$

then $f \equiv g$.

Proof Suppose that $f \not\equiv g$. Since $f(z)$ and $g(z)$ are transcendental entire functions of $\rho_2(f) < 1$, $\rho_2(g) < 1$ and $n \geq k+1$, we have that f and g have no zeros. Then we can set $f(z) = e^{\alpha(z)}$ and $g(z) = e^{\beta(z)}$, where $\alpha(z)$, $\beta(z)$ are entire functions with order less than one. Substituting f and g into (5), we get

$$\left[e^{n\alpha(z)}\left(e^{m\alpha(z)}-1\right)e^{\sum_{j=1}^{d} s_j\alpha(z+c_j)}\right]^{(k)}\left[e^{n\beta(z)}\left(e^{m\beta(z)}-1\right)e^{\sum_{j=1}^{d} s_j\beta(z+c_j)}\right]^{(k)} = 1.$$

Set

$$A_1(z) = (n+m)\alpha(z) + \sum_{j=1}^{d} s_j\alpha(z+c_j), \qquad A_2(z) = n\alpha(z) + \sum_{j=1}^{d} s_j\alpha(z+c_j),$$

$$B_1(z) = (n+m)\beta(z) + \sum_{j=1}^{d} s_j\beta(z+c_j), \qquad B_2(z) = n\beta(z) + \sum_{j=1}^{d} s_j\beta(z+c_j).$$

If $k = 1$, we have

$$\left[A_1'(z)e^{A_1(z)} - A_2'(z)e^{A_2(z)}\right]\left[B_1'(z)e^{B_1(z)} - B_2'(z)e^{B_2(z)}\right] = 1,$$

which implies that $A_1'(z)e^{A_1(z)} - A_2'(z)e^{A_2(z)}$ has no zeros. If $A_1'(z) \not\equiv 0$, then $A_2'(z) \equiv 0$, and thus $A_2(z)$ has to be a constant, that is,

$$n\alpha(z) + \sum_{j=1}^{d} s_j\alpha(z+c_j) = \xi_1,$$

where $\xi_1 \in \mathbb{C}$. Then from Lemma 2.11, we get $\rho(\alpha(z)) \geq 1$, which implies $\rho_2(f) \geq 1$. Hence, we can get a contradiction. If $A_1'(z) = 0$, then $A_1(z)$ has to be a constant, that is,

$$(n+m)\alpha(z) + \sum_{j=1}^{d} s_j\alpha(z+c_j) = \xi_2,$$

where $\xi_2 \in \mathbb{C}$. Thus, from Lemma 2.11 we can get that $\rho_2(f) \geq 1$, a contradiction.

If $k \geq 2$, then we have

$$\left[\left(U_1' + A_1'U_1\right)e^{A_1} - \left(U_2' + A_2'U_2\right)e^{A_2}\right]\left[\left(V_1' + B_1'V_1\right)e^{B_1} - \left(V_2' + B_2'V_2\right)e^{B_2}\right] = 1, \tag{6}$$

where

$$U_i = \sum_I c_\mu\left(A_i'\right)^{i_1}\left(A_i''\right)^{i_2}\cdots\left(A_i^{(k-1)}\right)^{i_{k-1}}, \qquad V_i = \sum_J d_\mu\left(B_i'\right)^{j_1}\left(B_i''\right)^{j_2}\cdots\left(B_i^{(k-1)}\right)^{j_{k-1}},$$

c_μ, d_μ are positive integers, $i_\iota, j_\iota \in N$ ($\iota = 1, 2, \ldots, k-1$), $I = \{i_1, i_2, \ldots, i_{k-1}\}$ and $J = \{j_1, j_2, \ldots, j_{k-1}\}$ satisfy

$$i_1 + 2i_2 + \cdots + (k-1)i_{k-1} = k-1, \qquad j_1 + 2j_2 + \cdots + (k-1)j_{k-1} = k-1.$$

From (6), we have $(U_1' + A_1'U_1)e^{A_1} - (U_2' + A_2'U_2)e^{A_2}$ has no zeros. If $U_1' + A_1'U_1 \not\equiv 0$, then $U_2' + A_2'U_2 = 0$. If A_2 is transcendental entire, we have

$$m(r, A_2') = m\left(r, \frac{U_2'}{U_2}\right) = S(r, U_2') = o\big(T(r, U_2')\big). \tag{7}$$

Since A_1, A_2 are entire, we have

$$T(r, A_i') = m(r, A_i'), \qquad T\big(r, (A_i')^{(k)}\big) \le T(r, A_i') + S(r, A_i), \quad k \in N, i = 1, 2. \tag{8}$$

Thus, from (7), (8) and the definition of U_2, we can get $T(r, A_2') \le o(T(r, A_2'))$, a contradiction with A_2' is transcendental entire. Hence, A_2 is a polynomial. From Lemma 2.11, we can get $\rho_2(f) \ge 1$, which is a contradiction. If $U_1' + A_1'U_1 = 0$, similar to the above argument, we can get a contradiction.

Therefore, we can get that $f \equiv g$. □

3 Proofs of Theorems 1.3 and 1.4

3.1 The Proof of Theorem 1.3

From (1), by Lemma 2.5 and Lemma 2.8, we have that $F(z)$ is not constant and $S(r, F^{(k)}) = S(r, F) = S(r, f)$. Next, we consider two cases.

Case 1. If f is a transcendental meromorphic function of $\rho_2(f) < 1$, we first suppose that $(P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)} = a(z)$ has finitely solutions. By the second fundamental theorem for three small functions (see [1, Theorem 2.25]) and Lemma 2.8, we have

$$\begin{aligned} T\big(r, F^{(k)}\big) &\le \overline{N}\big(r, F^{(k)}\big) + \overline{N}\left(r, \frac{1}{F^{(k)}}\right) + \overline{N}\left(r, \frac{1}{F^{(k)} - a(z)}\right) + S\big(r, F^{(k)}\big) \\ &\le \overline{N}(r, f) + \sum_{j=1}^{d} \overline{N}\big(r, f(z+c_j)\big) + N_1\left(r, \frac{1}{F^{(k)}}\right) + S\big(r, F^{(k)}\big) \\ &\le (d+1)T(r, f) + T\big(r, F^{(k)}\big) - T(r, F) + N_{k+1}\left(r, \frac{1}{F}\right) + S\big(r, F^{(k)}\big). \end{aligned}$$

By Lemma 2.4, Lemma 2.5 and Lemma 2.8, we obtain

$$\begin{aligned} (n-\lambda)T(r, f) + S(r, f) \le T(r, F) &\le (d+1)T(r, f) + N_{k+1}\left(r, \frac{1}{F}\right) + S(r, f) \\ &\le (d+1)T(r, f) + m(k+1)\overline{N}\left(r, \frac{1}{f}\right) + \sum_{j=1}^{d} \overline{N}\left(r, \frac{1}{f(z+c_j)}\right) \\ &\quad + S(r, f) \\ &\le \big[m(k+1) + 2d + 1\big]T(r, f) + S(r, f). \end{aligned} \tag{9}$$

From f is transcendental and $n > m(k+1) + 2d + 1 + \lambda$, we can get a contradiction to (10). Then $F(z)^{(k)} - a(z)$ has infinitely many zeros.

Case 2. If f is a transcendental entire function of $\rho_2(f) < 1$. Suppose that $(P(f)\prod_{j=1}^{d} f(z+c_j)^{s_j})^{(k)} = a(z)$ has finitely solutions. By using the same argument as in Case 1 and (4), we have

$$(n+\lambda)T(r,f) + S(r,f) \leq \bigl[m(k+1) + d\bigr]T(r,f) + S(r,f),$$

which is a contradiction with $n > m(k+1) + d - \lambda$.

Thus, we complete the proof of Theorem 1.3.

3.2 The Proof of Theorem 1.4

We consider two cases as follows.

Case 1. Suppose that f is a transcendental meromorphic function of $\rho_2(f) < 1$. Similar to the proof of Case 1 in Theorem 1.3, and using Lemma 2.6, we have

$$\begin{aligned}
(n-\lambda)T(r,f) + S(r,f) &\leq T(r,F_1) \leq (d+1)\overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{F_1}\right) + S(r,f) \\
&\leq (d+1)T(r,f) + m(k+1)\overline{N}\left(r, \frac{1}{f}\right) \\
&\quad + (k+1)\sum_{j=1}^{d}\overline{N}\left(r, \frac{1}{f(z+c_j) - f(z)}\right) + S(r,f) \\
&\leq \bigl[(m+2d)(k+1) + d + 1\bigr]T(r,f) + S(r,f).
\end{aligned}$$

Since $n > (m+2d)(k+1) + d + 1 + \lambda$ and f is transcendental, we can get a contradiction.

Case 2. Suppose that f is a transcendental entire function of $\rho_2(f) < 1$. Similar to the proof of Case 1 in Theorem 1.4, by Lemma 2.6, we can get

$$nT(r,f) \leq (m+2d)(k+1)T(r,f) + S(r,f),$$

which is a contradiction with $n > (m+2d)(k+1)$ and f is transcendental.

Thus, from Cases 1 and 2, we prove Theorem 1.4.

4 Proofs of Theorems 1.6 and 1.7

4.1 The proof of Theorem 1.6

Let $H(z) = (F^*(z))^{(k)}$ and $G(z) = (G^*(z))^{(k)}$. From the assumptions of Theorem 1.6, we have that $H(z)$, $G(z)$ share 1 CM. By Lemma 2.8, we have

$$T\bigl(r, H(z)\bigr) \leq T\left(r, f^n\bigl(f^m - 1\bigr)\prod_{j=1}^{d}\bigl(f(z+c_j)\bigr)^{s_j}\right) + S\left(r, f^n\bigl(f^m - 1\bigr)\prod_{j=1}^{d}\bigl(f(z+c_j)\bigr)^{s_j}\right).$$

By Lemma 2.4 and Lemma 2.5, we can get $S(r,H) = S(r,f)$. Similarly, we have $S(r,G) = S(r,g)$. Then the following three cases are considered.

Case 1. Suppose that $H(z)$, $G(z)$ satisfy Lemma 2.9(i). Since $f(z)$, $g(z)$ are entire functions of $\rho_2(f)$, $\rho_2(g) < 1$, from Lemma 2.4 and Lemma 2.5, we have

$$\begin{aligned} N_2\left(r, \frac{1}{H}\right) &\le N_{k+2}\left(r, \frac{1}{f^n(f^m-1)\prod_{j=1}^d (f(z+c_j))^{s_j}}\right) + S(r,f) \\ &\le (k+m+d+2)T(r,f) + S(r,f) \end{aligned} \tag{10}$$

and

$$\begin{aligned} N_2\left(r, \frac{1}{H}\right) &\le N_2\left(r, \frac{1}{(f^n(f^m-1)\prod_{j=1}^d (f(z+c_j))^{s_j})^{(k)}}\right) \\ &\le T(r,H) - T\left(r, f^n\left(f^m-1\right)\prod_{j=1}^d \left(f(z+c_j)\right)^{s_j}\right) \\ &\quad + N_{k+2}\left(r, \frac{1}{f^n(f^m-1)\prod_{j=1}^d (f(z+c_j))^{s_j}}\right) + S(r,f). \end{aligned} \tag{11}$$

From (11) and by Lemma 2.5, we have

$$\begin{aligned} (n+m+\lambda)T(r,f) &= T\left(r, f^n\left(f^m-1\right)\prod_{j=1}^d \left(f(z+c_j)\right)^{s_j}\right) + S(r,f) \\ &\le T(r,H) - N_2\left(r, \frac{1}{H}\right) \\ &\quad + N_{k+2}\left(r, \frac{1}{f^n(f^m-1)\prod_{j=1}^d (f(z+c_j))^{s_j}}\right) + S(r,f). \end{aligned} \tag{12}$$

Similarly, we have

$$N_2\left(r, \frac{1}{G}\right) \le (k+m+d+2)T(r,g) + S(r,g) \tag{13}$$

and

$$\begin{aligned} (n+m+\lambda)T(r,g) &\le T(r,G) - N_2\left(r, \frac{1}{G}\right) \\ &\quad + N_{k+2}\left(r, \frac{1}{g^n(g^m-1)\prod_{j=1}^d (g(z+c_j))^{s_j}}\right) + S(r,g). \end{aligned} \tag{14}$$

Then, from Lemma 2.9(i) and (10)-(14), we have

$$\begin{aligned} &(n+m+\lambda)\left[T(r,f) + T(r,g)\right] \\ &\quad \le 2N_{k+2}\left(r, \frac{1}{f^n(f^m-1)\prod_{j=1}^d (f(z+c_j))^{s_j}}\right) \\ &\qquad + 2N_{k+2}\left(r, \frac{1}{g^n(g^m-1)\prod_{j=1}^d (g(z+c_j))^{s_j}}\right) + S(r,f) + S(r,g) \\ &\quad \le 2(k+m+d+2)\left[T(r,f) + T(r,g)\right] + S(r,f) + S(r,g). \end{aligned} \tag{15}$$

Since f, g are transcendental and $n > 2(k+d+2)+m-\lambda$, we can get a contradiction.

Case 2. If $H(z) \equiv G(z)$, then we have

$$f^n\left(f^m-1\right)\prod_{j=1}^{d}\left(f(z+c_j)\right)^{s_j} = g^n\left(g^m-1\right)\prod_{j=1}^{d}\left(g(z+c_j)\right)^{s_j} + Q(z), \tag{16}$$

where $Q(z)$ is a polynomial of degree at most $k-1$. If $Q(z) \not\equiv 0$, by the second fundamental theorem and Lemma 2.5, we have

$$\begin{aligned}(n+m+\lambda)T(r,f) &= T\left(r, \frac{f^n(f^m-1)\prod_{j=1}^{d}(f(z+c_j))^{s_j}}{Q(z)}\right) + S(r,f)\\ &\le \overline{N}\left(r, \frac{f^n(f^m-1)\prod_{j=1}^{d}(f(z+c_j))^{s_j}}{Q(z)}\right)\\ &\quad + \overline{N}\left(r, \frac{Q(z)}{f^n(f^m-1)\prod_{j=1}^{d}(f(z+c_j))^{s_j}}\right)\\ &\quad + \overline{N}\left(r, \frac{Q(z)}{g^n(g^m-1)\prod_{j=1}^{d}(g(z+c_j))^{s_j}}\right) + S(r,f)\\ &\le (m+d+1)\left[T(r,f)+T(r,g)\right] + S(r,f) + S(r,g).\end{aligned} \tag{17}$$

Similarly, we have

$$(n+m+\lambda)T(r,g) \le (m+d+1)\left[T(r,f)+T(r,g)\right] + S(r,f) + S(r,g). \tag{18}$$

From (17), (18) and $n > 2(k+d+2)+m-\lambda > 2d+m+2-\lambda$, we can get a contradiction. Thus, $Q(z) \equiv 0$, that is,

$$f^n\left(f^m-1\right)\prod_{j=1}^{d}\left(f(z+c_j)\right)^{s_j} \equiv g^n\left(g^m-1\right)\prod_{j=1}^{d}\left(g(z+c_j)\right)^{s_j}. \tag{19}$$

Set $h = \frac{f}{g}$. If h is a constant. Substituting $f = gh$ into (19), we can get

$$\prod_{j=1}^{d} g(z+c_j)^{s_j}\left[g^{n+m}\left(h^{n+m+\lambda}-1\right) + g^n\left(h^{n+\lambda}-1\right)\right] \equiv 0. \tag{20}$$

Since g is a transcendental entire function, we have $\prod_{j=1}^{d} g(z+c_j)^{s_j} \not\equiv 0$. Then, from (20), we have

$$g^m\left(h^{n+m+\lambda}-1\right) + \left(h^{n+\lambda}-1\right) \equiv 0. \tag{21}$$

Suppose that $h^{n+\lambda} \neq 1$, by Lemma 2.4 and (21), we have $T(r,g) = S(r,g)$, which is contradiction with a transcendental function g. Then $h^{n+\lambda} = 1$. Similar to this discussion, we can get that $h^{n+m+\lambda} = 1$. If h is not a constant, from (19) we have

$$g(z)^m = \frac{h(z)^n \prod_{j=1}^{d}(h(z+c_j))^{s_j} - 1}{h(z)^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j} - 1}. \tag{22}$$

If 1 is a Picard value of $h(z)^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}$, then by the second fundamental theorem and Lemma 2.5,

$$\begin{aligned}
&T\left(r, h^{n+m}\prod_{j=1}^{d}\bigl(h(z+c_j)\bigr)^{s_j}\right)\\
&\quad\le \overline{N}\left(r, h^{n+m}\prod_{j=1}^{d}\bigl(h(z+c_j)\bigr)^{s_j}\right)+\overline{N}\left(r, \frac{1}{h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}}\right)\\
&\qquad+\overline{N}\left(r, \frac{1}{h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}-1}\right)+S(r,h)\\
&\quad\le (2d+2)T(r,h)+S(r,h).
\end{aligned}$$

From the above inequality and $n > 2d+2+\lambda > \lambda+2d+2-m$, by Lemma 2.5, we can get a contradiction. Therefore, 1 is a Picard value of $h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}$. Now, two cases are considered as follows.

Subcase 2.1. $h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}\not\equiv 1$. Then from (22), we have

$$\overline{N}\left(r, \frac{1}{h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}}-1\right)\le \overline{N}\left(r, \frac{1}{h^m-1}\right)\le mT(r,h)+S(r,h). \tag{23}$$

Then from (23) and by the second fundamental theorem and Lemma 2.5, we have

$$\begin{aligned}
&T\left(r, h^{n+m}\prod_{j=1}^{d}\bigl(h(z+c_j)\bigr)^{s_j}\right)\\
&\quad\le \overline{N}\left(r, h^{n+m}\prod_{j=1}^{d}\bigl(h(z+c_j)\bigr)^{s_j}\right)+\overline{N}\left(r, \frac{1}{h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}}\right)\\
&\qquad+\overline{N}\left(r, \frac{1}{h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}-1}\right)+S(r,h)\\
&\quad\le (m+2d+2)T(r,h)+S(r,h),
\end{aligned}$$

which is a contradiction with $n > 2d+2+\lambda$.

Subcase 2.2. $h^{n+m}\prod_{j=1}^{d}(h(z+c_j))^{s_j}\equiv 1$. Thus, we have

$$(n+m)T(r,h)\le \lambda T(r,h)+S(r,h),$$

which is a contradiction with $n > 2d+2+\lambda > \lambda-m$.

Case 3. If $H(z)G(z)\equiv 1$. From Lemma 2.12, we can get that $f\equiv g$.

Thus, this completes the proof of Theorem 1.6.

4.2 The proof of Theorem 1.7

From the assumptions of Theorem 1.7, we have $H(z)$, $G(z)$ share 1 *IM*. From the definitions of $H(z)$, $G(z)$, by Lemma 2.4, Lemma 2.5 and Lemma 2.8, we have

$$\begin{aligned}\overline{N}_L\left(r,\frac{1}{H-1}\right) &\leq N\left(r,\frac{H}{H'}\right) = N\left(r,\frac{H'}{H}\right) + S(r,H) \leq \overline{N}\left(r,\frac{1}{H}\right) + S(r,H)\\ &\leq (k+m+d+1)T(r,f) + S(r,f),\end{aligned}\tag{24}$$

similarly, we have

$$\overline{N}_L\left(r,\frac{1}{G-1}\right) \leq (k+m+d+1)T(r,g) + S(r,f).\tag{25}$$

Case 1. Suppose that $H(z)$, $G(z)$ satisfy Lemma 2.10(i). From (9), (11)-(13), (24) and (25), we have

$$(n+m+\lambda)\big[T(r,f)+T(r,g)\big] \leq \big[5(k+m+d)+7\big]\big[T(r,f)+T(r,g)\big] + S(r,f) + S(r,g).$$

Since $n > 5(k+d)+4m+7-\lambda$ and f, g are transcendental, we can get a contradiction.

Case 2. If $H(z)$, $G(z)$ satisfy Lemma 2.10(ii), that is,

$$H = \frac{(b+1)G+(a-b-1)}{bG+(a-b)},\tag{26}$$

where $a\ (\neq 0)$, b are two constants.

We consider three cases as follows.

Subcase 2.1. $b \neq 0,-1$. If $a-b-1 \neq 0$, then by (26) we know

$$\overline{N}\left(r,\frac{1}{G+\frac{a-b-1}{b+1}}\right) = \overline{N}\left(r,\frac{1}{H}\right).$$

Since f, g are entire functions of $\rho_2(f) < 1$, $\rho_2(g) < 1$, by the second fundamental theorem and Lemma 2.5, we have

$$\begin{aligned}(n+m+\lambda)T(r,g) &\leq T(r,G) + N_k\left(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{d}(g(z+c_j))^{s_j}}\right) - N\left(r,\frac{1}{G}\right) + S(r,g)\\ &\leq N_k\left(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{d}(g(z+c_j))^{s_j}}\right) + \overline{N}\left(r,\frac{1}{G+\frac{a-b-1}{b+1}}\right) + S(r,g)\\ &\leq (k+m+d)T(r,g) + (k+m+d+1)T(r,f) + S(r,f) + S(r,g),\end{aligned}$$

that is,

$$(n+\lambda-k-d)T(r,g) \leq (k+m+d+1)T(r,f) + S(r,f) + S(r,g).\tag{27}$$

Similarly, we have

$$(n+\lambda-k-d)T(r,f) \leq (k+m+d+1)T(r,g) + S(r,f) + S(r,g).\tag{28}$$

From (27) and (28), we have

$$(n+\lambda-2k-2d-m-1)\big[T(r,f)+T(r,g)\big]\le S(r,f)+S(r,g),$$

which is a contradiction with $n>5(k+d)+4m+7-\lambda$.

If $a-b-1=0$, then by (26) we know $H=((b+1)G)/(bG+1)$. Since f, g are entire functions, we get that $-\frac{1}{b}$ is a Picard exceptional value of $G(z)$. By the second fundamental theorem, we have

$$\begin{aligned}(n+m+\lambda)T(r,g)&\le T(r,G)+N_k\left(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{d}(g(z+c_j))^{s_j}}\right)-N\left(r,\frac{1}{G}\right)+S(r,g)\\&\le N_k\left(r,\frac{1}{g^n(g^m-1)\prod_{j=1}^{d}(g(z+c_j))^{s_j}}\right)+\overline{N}\left(r,\frac{1}{G+\frac{1}{b}}\right)+S(r,g)\\&\le (k+m+d)T(r,g)+S(r,g),\end{aligned}$$

which is a contradiction with $n>5(k+d)+4m+7-\lambda$.

Subcase 2.2. $b=-1$. Then (26) becomes $H=a/(a+1-G)$.

If $a+1\neq 0$, then $a+1$ is a Picard exceptional value of G. Similar to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a+1=0$, then $HG\equiv 1$. From Lemma 2.12, we can get that $f\equiv g$.

Subcase 2.3. $b=0$. Then (26) becomes $H=(G+a-1)/a$.

If $a-1\neq 0$, then $\overline{N}(r,\frac{1}{G+a-1})=\overline{N}(r,\frac{1}{H})$. Similar to the discussion in Subcase 2.1, we can deduce a contradiction again.

If $a-1=0$, then $H\equiv G$. Using the same argument as in the proof of Case 2 in Theorem 1.6, we can get that f, g satisfy $f\equiv tg$ for a constant t such that $t^m=1$ and $t^{n+\lambda}=1$.

Thus, we complete the proof of Theorem 1.7.

Competing interests
The author declares that they have no competing interests.

Author's contributions
HYX completed the main part of this article, HYX corrected the main theorems. The author read and approved the final manuscript.

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