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Existence of mild solutions for fractional impulsive neutral evolution equations with nonlocal conditions

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Abstract
In this paper, by using the fractional power of an operator and some fixed point theorems, we study the existence of mild solutions for the nonlocal problem of Caputo fractional impulsive neutral evolution equations in Banach spaces. In the end, an example is given to illustrate the applications of the abstract results.
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1 Introduction

During the past two decades, fractional differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics, and hence they have gained considerable attention. Some basic theory for the initial value problem of fractional differential (or evolution) equations was discussed in [1–10]. But all these papers did not consider the effect of impulsive conditions in the equations. Recently, Wang *et al.* [11] studied the existence of mild solutions for the fractional impulsive evolution equations

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t)), \quad t \in J = [0, a], t \neq t_k, \\ u(t_k^+) = u(t_k^-) + y_k, \quad k = 1, 2, \ldots, m, \\ u(0) = u_0 \end{cases} \tag{1}$$

in a Banach space X, where $a > 0$ is a constant, D^q denotes the Caputo fractional derivative of order $q \in (0, 1)$, $A : D(A) \subset X \to X$ is a closed linear operator and $-A$ generates a C_0-semigroup $T(t)$ $(t \geq 0)$ in X, $f : J \times X \to X$ is continuous, y_k, u_0 are the elements of X, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$, $u(t_k^+)$ and $u(t_k^-)$ represent the right and left limits of $u(t)$ at $t = t_k$, respectively. By using some fixed point theorems of compact operator, they derive many existence and uniqueness results concerning the mild solutions for problem (1) under the different assumptions on the nonlinear term f. For more articles about the fractional impulsive evolution equations, we refer to [12–14] and the references therein.

On the other hand, the fractional neutral differential equations have also been studied by many authors. Many methods of nonlinear analysis have been employed to research this problem; see [6, 15–18]. But, as far as we know, papers considering the fractional impulsive neutral evolution equations are seldom.

In this paper, we consider the following nonlocal problem of fractional impulsive neutral evolution equations:

$$\begin{cases} D^q[u(t) - h(t,u(t))] + Au(t) = f(t,u(t)), & t \in J, t \neq t_k, \\ \Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-) = I_k(u(t_k)), & k = 1,2,\dots,m, \\ u(0) + g(u) = u_0 \end{cases} \tag{2}$$

in a Banach space X, where $J = [0,a]$, D^q denotes the Caputo fractional derivative of order $q \in (0,1)$, $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ $(t \geq 0)$ in X, I_k $(k = 1,2,\dots,m)$ are the impulsive functions, f, h, g are given functions and will be specified later. By utilizing the fixed point theorems, we derive many existence results concerning the mild solutions for problem (2) under different assumptions on the nonlinear term and nonlocal term.

The rest of this paper is organized as follows. In Section 2, some preliminaries are given on the fractional power of the generator of an analytic semigroup and the fractional calculus. In Section 3, we study the existence of mild solutions of the problem (2). An example is given in Section 4 to illustrate the applications of the abstract results.

2 Preliminaries

In this section, we introduce some basic facts as regards the fractional power of the generator of an analytic semigroup and the fractional calculus.

Let X be a Banach space with norm $\|\cdot\|$. Throughout this paper, we assume that $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$ $(t \geq 0)$ of a uniformly bounded linear operator in X, that is, there exists $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$. Without loss of generality, let $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A. Then for any $\alpha > 0$, we can define $A^{-\alpha}$ by

$$A^{-\alpha} := \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t)\, dt.$$

It follows that each $A^{-\alpha}$ is an injective continuous endomorphism of X. Hence we can define A^α by $A^\alpha := (A^{-\alpha})^{-1}$, which is a closed bijective linear operator in X. It can be shown that each A^α has dense domain and that $D(A^\beta) \subset D(A^\alpha)$ for $0 \leq \alpha \leq \beta$. Moreover, $A^{\alpha+\beta}x = A^\alpha A^\beta x = A^\beta A^\alpha x$ for every $\alpha, \beta \in \mathbb{R}$ and $x \in D(A^\mu)$, where $\mu := \max\{\alpha, \beta, \alpha + \beta\}$. $A^0 = I$, I is the identity in X. (For proofs of these facts we refer to [19, 20].)

We denote by X_α the Banach space of $D(A^\alpha)$ equipped with norm $\|x\|_\alpha = \|A^\alpha x\|$ for $x \in D(A^\alpha)$, which is equivalent to the graph norm of A^α. Then we have $X_\beta \hookrightarrow X_\alpha$ for $0 \leq \alpha \leq \beta \leq 1$ (with $X_0 = X$), and the embedding is continuous. Moreover, A^α has the following basic properties.

Lemma 1 [19] *A^α has the following properties.*

(i) *$T(t): X \to X_\alpha$ for each $t > 0$ and $\alpha \geq 0$.*

(ii) *$A^\alpha T(t)x = T(t)A^\alpha x$ for each $x \in D(A^\alpha)$ and $t \geq 0$.*

(iii) *For every* $t > 0$, $A^\alpha T(t)$ *is bounded in* X *and there exists* $M_\alpha > 0$ *such that*

$$\|A^\alpha T(t)\| \le M_\alpha t^{-\alpha}.$$

(iv) $A^{-\alpha}$ *is a bounded linear operator for* $0 \le \alpha \le 1$ *in* X.

From Lemma 1(iv), there exists a constant C_α such that $\|A^{-\alpha}\| \le C_\alpha$ for $0 \le \alpha \le 1$.

For any $t \ge 0$, denote by $T_\alpha(t)$ the restriction of $T(t)$ to X_α. From Lemma 1(i) and (ii), $T_\alpha(t)$ $(t \ge 0)$ is a strongly continuous semigroup in X_α, and $\|T_\alpha(t)\|_\alpha \le \|T(t)\|$ for all $t \ge 0$. To prove our main results, the following lemma is needed.

Lemma 2 [21] $T_\alpha(t)$ $(t \ge 0)$ *is an immediately compact semigroup in* X_α, *and hence it is immediately norm-continuous.*

Let us recall the following known definitions in fractional calculus. For more details, see [1–10, 22, 23] and the references therein.

Definition 1 The fractional integral of order $\sigma > 0$ with the lower limits zero for a function f is defined by

$$I^\sigma f(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} f(s)\,ds, \quad t > 0,$$

where Γ is the gamma function.

The Riemann-Liouville fractional derivative of order $n-1 < \sigma < n$ with the lower limits zero for a function f can be written as

$${}^L D^\sigma f(t) = \frac{1}{\Gamma(n-\sigma)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\sigma-1} f(s)\,ds, \quad t > 0, n \in \mathbb{N}.$$

Also the Caputo fractional derivative of order $n-1 < \sigma < n$ with the lower limits zero for a function $f \in C^n[0,\infty)$ can be written as

$$D^\sigma f(t) = \frac{1}{\Gamma(n-\sigma)} \int_0^t (t-s)^{n-\sigma-1} f^{(n)}(s)\,ds, \quad t > 0, n \in \mathbb{N}.$$

Remark 1 If f is an abstract function with values in X, then integrals which appear in Definition 1 are taken in Bochner's sense.

Lemma 3 [4, 5] *A measurable function* $h : [0,a] \to X$ *is Bochner integrable if* $\|h\|$ *is Lebesgue integrable.*

For $x \in X$, we define two families $\{U(t)\}_{t\ge 0}$ and $\{V(t)\}_{t\ge 0}$ of operators by

$$U(t)x = \int_0^\infty \eta_q(\theta) T(t^q\theta) x\,d\theta,$$
$$V(t)x = q\int_0^\infty \theta\eta_q(\theta) T(t^q\theta) x\,d\theta, \quad 0 < q < 1,$$

where

$$\eta_q(\theta) = \frac{1}{q}\theta^{-1-\frac{1}{q}}\rho_q\big(\theta^{-\frac{1}{q}}\big),$$
$$\rho_q(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-qn-1}\frac{\Gamma(nq+1)}{n!}\sin(n\pi q), \quad \theta \in (0,\infty).$$

η_q is a probability density function defined on $(0,\infty)$, which has properties $\eta_q(\theta) \geq 0$ for all $\theta \in (0,\infty)$ and $\int_0^\infty \eta_q(\theta)\,d\theta = 1$, $\int_0^\infty \theta\eta_q(\theta)\,d\theta = \frac{1}{\Gamma(q+1)}$. It is not difficult to verify (see [5, Remark 2.8]) that for any $\mu \in [0,1]$, we have

$$\int_0^\infty \theta^\mu \eta_q(\theta)\,d\theta = \frac{\Gamma(1+\mu)}{\Gamma(1+q\mu)}.$$

The following lemma follows from the results in [4–7, 11].

Lemma 4 *The operators $U(t)$ and $V(t)$ have the following properties.*

(i) *For fixed $t \geq 0$ and any $x \in X_\alpha$, we have*

$$\|U(t)x\|_\alpha \leq M\|x\|_\alpha, \qquad \|V(t)x\|_\alpha \leq \frac{qM}{\Gamma(1+q)}\|x\|_\alpha = \frac{M}{\Gamma(q)}\|x\|_\alpha.$$

For fixed $t \geq 0$ and any $x \in X$, we have

$$\|V(t)x\|_\alpha \leq \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}t^{-q\alpha}\|x\|.$$

(ii) *The operators $U(t)$ and $V(t)$ are strongly continuous for all $t \geq 0$.*

(iii) *If $T(t)$ $(t \geq 0)$ is a compact semigroup, then $U(t)$ and $V(t)$ are compact operators in X for $t > 0$.*

(iv) *If $T(t)$ $(t \geq 0)$ is a compact semigroup, then the restriction of $U(t)$ to X_α and the restriction of $V(t)$ to X_α are compact operators in X_α for every $t > 0$.*

Lemma 5 (Krasnoselskii's fixed point theorem) *Let E be a Banach space, B be a bounded closed and convex subset of E and F_1, F_2 be maps of B into E such that $F_1x + F_2y \in B$ for every pair $x, y \in B$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1x + F_2x = x$ has a solution on B.*

We denote by $C(J,X_\alpha)$ the Banach space of all continuous X_α-value functions on interval J with the norm $\|u\|_C = \sup_{t\in J}\|u(t)\|_\alpha$, and by $PC(J,X_\alpha) := \{u : J \to X_\alpha | u$ is continuous on $t \neq t_k$, $u(t_k) = u(t_k^-)$ and right limits exist on $t = t_k$, $k = 1,2,\ldots,m\}$ the norm space endowed with the norm $\|u\|_{PC} = \sup_{t\in J}\|u(t)\|_\alpha$. Let $\{u_n\}_{n=1}^\infty \subset PC(J,X_\alpha)$ be a Cauchy sequence. Then for any $\varepsilon > 0$, there exists a constant $N > 0$ such that for any $m,n \geq N$, we have $\|u_m - u_n\|_{PC} \leq \varepsilon$. Then for any $t \in J$, we have

$$\|u_m(t) - u_n(t)\|_\alpha \leq \|u_m - u_n\|_{PC} \leq \varepsilon.$$

That is, $\{u_m(t)\} \subset X_\alpha$ is a Cauchy sequence. Noticing that X_α is a Banach space, then $\{u_m(t)\}$ is convergence in X_α. That is, $\lim_{m\to\infty} u_m(t) = u_0(t) \in X_\alpha$ for all $t \in J$. Let $m \to \infty$.

Then for $n \geq N$, we have

$$\left\| u_n(t) - u_0(t) \right\|_\alpha \leq \varepsilon, \quad t \in J.$$

This means that $\{u_n(t)\}$ is uniformly convergent to $u_0(t)$ in X_α for all $t \in J$. Hence we get $u_0 \in PC(J, X_\alpha)$ and $u_n \to u_0$ in $PC(J, X_\alpha)$ as $n \to \infty$. That is $PC(J, X_\alpha)$ is a Banach space endowed with norm $\|u\|_{PC}$ for $u \in PC(J, X_\alpha)$.

If the problem (2) is without impulse, we have

$$\begin{cases} D^q[u(t) - h(t, u(t))] + Au(t) = f(t, u(t)), & t \in J, \\ u(0) + g(u) = u_0. \end{cases} \tag{3}$$

A function $u \in C(J, X_\alpha)$ is said to be a mild solution of the problem (3), if u satisfies the integral equation

$$\begin{aligned} u(t) = {} & U(t)\bigl[u_0 - g(u) - h\bigl(0, u(0)\bigr)\bigr] + h\bigl(t, u(t)\bigr) + \int_0^t (t-s)^{(q-1)} AV(t-s) h\bigl(s, u(s)\bigr)\, ds \\ & + \int_0^t (t-s)^{q-1} V(t-s) f\bigl(s, u(s)\bigr)\, ds, \quad t \in J. \end{aligned}$$

Hence, by using a completely similar technique as in [11, Section 3], we obtain the following definition.

Definition 2 By a mild solution of the problem (2), we mean a function $u \in PC(J, X_\alpha)$ satisfying

$$u(t) = \begin{cases} U(t)[u_0 - g(u) - h(0, u(0))] + h(t, u(t)) + \int_0^t (t-s)^{(q-1)} AV(t-s) h(s, u(s))\, ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds, \quad t \in [0, t_1], \\ U(t)[u_0 - g(u) - h(0, u(0))] + U(t-t_1) I_1(u(t_1)) + h(t, u(t)) \\ \quad + \int_0^t (t-s)^{(q-1)} AV(t-s) h(s, u(s))\, ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds, \quad t \in (t_1, t_2], \\ \vdots \\ U(t)[u_0 - g(u) - h(0, u(0))] + \sum_{i=1}^m U(t-t_i) I_i(u(t_i)) + h(t, u(t)) \\ \quad + \int_0^t (t-s)^{(q-1)} AV(t-s) h(s, u(s))\, ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds, \quad t \in (t_m, a]. \end{cases}$$

3 Existence of mild solutions

In this section, we introduce the existence theorems of mild solutions of the problem (2). The discussions are based on fixed point theorems. Our main results are as follows.

Theorem 1 *Assume that the following conditions are satisfied.*

(H_1) $T(t)$ $(t \geq 0)$ *is a compact analytic semigroup*;

(H_2) *The function* $h : J \times X_\alpha \to X_1$ *is continuous and there exists a constant* $L_1 > 0$ *such that*

$$\left\| Ah(t_1, x_1) - Ah(t_2, x_2) \right\| \leq L_1 \bigl(|t_1 - t_2| + \|x_1 + x_2\|_\alpha \bigr)$$

for all $t_1, t_2 \in J$ *and* $x_1, x_2 \in X_\alpha$;

(H_3) *The function $f : J \times X_\alpha \to X$ satisfies the following conditions.*
  (i) *For a.e. $t \in J$, the function $f(t,\cdot) : X_\alpha \to X$ is continuous, and for every $x \in X_\alpha$, the function $f(\cdot, x) : J \to X$ is strongly measurable.*
  (ii) *For each $r > 0$ and $t \in J$, there exists a constant $q_1 \in (0, q)$ and a function $F \in L^{\frac{1}{q_1}}(J, \mathbb{R}^+)$ such that*

$$\sup_{\|x\|_\alpha \le r} \|f(t,x)\| \le F(t).$$

(H_4) *For the functions $I_k : X_\alpha \to X_\alpha$, there exists a constant $L_2 > 0$ such that*

$$\|I_k(x_1) - I_k(x_2)\|_\alpha \le L_2 \|x_1 - x_2\|_\alpha, \quad k = 1, 2, \ldots, m$$

for all $x_1, x_2 \in X_\alpha$.

(H_5) *The function $g : PC(J, X_\alpha) \to X_\alpha$ and there exists a constant $L_3 > 0$ such that*

$$\|g(v_1) - g(v_2)\|_\alpha \le L_3 \|v_1 - v_2\|_{PC}$$

for all $v_1, v_2 \in PC(J, X_\alpha)$.

If $u_0 \in X_\alpha$, then the problem (2) *has a mild solution $u \in PC(J, X_\alpha)$ provided that*

$$M^* \triangleq M(L_2 m + L_3) + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha L_1 \Gamma(2-\alpha) a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} < 1. \tag{4}$$

Proof Let $b = \frac{q(1-\alpha)-1}{1-q_1} \in (-1, 0)$, $\overline{M} = \|F\|_{L^{\frac{1}{q_1}}[0,a]}$. Direct calculation shows that $(t-s)^{q(1-\alpha)-1} \in L^{\frac{1}{1-q_1}}[0,t]$ for $t \in J$. In view of Lemma 4, a similar argument as in the proof of [6, Theorem 3.1] shows that $(t-s)^{q-1}V(t-s)f(s,u(s))$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in J$.

For any $r > 0$, let $B_r = \{u \in PC(J, X_\alpha) : \|u\|_{PC} \le r\}$. Since the function $h : J \times X_\alpha \to X_1$ is continuous, for any $u \in B_r$ and $t \in J$, by Lemma 4, we get

$$\begin{aligned}
&\int_0^t \|(t-s)^{q-1} AV(t-s) h(s,u(s))\|_\alpha \, ds \\
&\quad = \int_0^t \|(t-s)^{q-1} A^\alpha V(t-s) \cdot Ah(s,u(s))\| \, ds \\
&\quad \le \frac{q M_\alpha \Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_0^t (t-s)^{q(1-\alpha)-1} \bigl[\|Ah(s,u(s)) - Ah(0,0)\| + \|Ah(0,0)\|\bigr] \, ds \\
&\quad \le \frac{M_\alpha \Gamma(2-\alpha) a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} \bigl[L_1(a+r) + \|Ah(0,0)\|\bigr].
\end{aligned}$$

Thus, $\|(t-s)^{q-1}AV(t-s)h(s,u(s))\|_\alpha$ is Lebesgue integrable with respect to $s \in [0,t]$ for all $t \in J$. From Lemma 3, it follows that $(t-s)^{q-1}AV(t-s)h(s,u(s))$ is Bochner integrable with respect to $s \in [0,t]$ for all $t \in J$.

Define two operators Q_1 and Q_2 on $PC(J, X_\alpha)$ by

$$(Q_1 u)(t) = \int_0^t (t-s)^{q-1} V(t-s) f(s,u(s)) \, ds, \quad t \in J;$$

$$Q_2(u)(t)=\begin{cases} U(t)[u_0-g(u)-h(0,u(0))]+h(t,u(t)) \\ \quad +\int_0^t (t-s)^{q-1}AV(t-s)h(s,u(s))\,ds, \quad t\in[0,t_1], \\ U(t)[u_0-g(u)-h(0,u(0))]+h(t,u(t)) \\ \quad +\int_0^t (t-s)^{q-1}AV(t-s)h(s,u(s))\,ds+U(t-t_1)I_1(u(t_1)), \quad t\in[t_1,t_2], \\ \vdots \\ U(t)[u_0-g(u)-h(0,u(0))]+h(t,u(t)) \\ \quad +\int_0^t (t-s)^{q-1}AV(t-s)h(s,u(s))\,ds \\ \quad +\sum_{i=1}^m U(t-t_i)I_i(u(t_i)), \quad t\in[t_m,a]. \end{cases}$$

Obviously, u is a mild solution of the problem (2) if and only if u is a solution of the operator equation $u=Q_1u+Q_2u$. We will use Krasnoselskii's fixed point theorem to prove that the operator equation $u=Q_1u+Q_2u$ has a solution on B_r. For this purpose, we first prove that there is a positive number r_0 such that $Q_1u+Q_2u\in B_{r_0}$ for any $u\in B_{r_0}$. If this were not the case, then for each $r>0$, there exist $u_r\in B_r$ and $t_r\in J$ such that $\|(Q_1u_r+Q_2u_r)(t_r)\|_\alpha>r$. It is clear that there is a $0\le k\le m$ such that $t_r\in[t_k,t_{k+1}]$. Thus, from assumptions (H_2)-(H_5), we see that

$$\begin{aligned} r &< \|(Q_1u_r+Q_2u_r)(t_r)\|_\alpha \\ &\le \int_0^{t_r}(t_r-s)^{q-1}\|V(t_r-s)f(s,u_r(s))\|_\alpha\,ds \\ &\quad + \|U(t_r)[u_0-g(u_r)-h(0,u_r(0))]\|_\alpha + \|h(t_r,u_r(t_r))\|_\alpha \\ &\quad + \int_0^{t_r}(t_r-s)^{q-1}\|AV(t_r-s)h(s,u_r(s))\|_\alpha\,ds + \sum_{i=1}^k \|U(t_r-t_i)I_i(u_r(t_i))\|_\alpha \\ &\le \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\int_0^{t_r}(t_r-s)^{q(1-\alpha)-1}\cdot F(s)\,ds + M\|u_0\|_\alpha \\ &\quad + M[\|g(u_r)-g(0)\|_\alpha+\|g(0)\|_\alpha] \\ &\quad + MC_{1-\alpha}[\|Ah(0,u_r(0))-Ah(0,0)\|+\|Ah(0,0)\|]+C_{1-\alpha}\|Ah(t_r,u_r(t_r))\| \\ &\quad + \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\int_0^{t_r}(t_r-s)^{q(1-\alpha)-1}[\|Ah(s,u_r(s))-Ah(0,0)\|+\|Ah(0,0)\|]\,ds \\ &\quad + M\sum_{i=1}^m[\|I_i(u_r(t_i))-I_i(0)\|_\alpha+\|I_i(0)\|_\alpha] \\ &\le \frac{qM_\alpha\overline{M}\Gamma(2-\alpha)a^{(1+b)(1-q_1)}}{\Gamma(1+q(1-\alpha))(1+b)^{1-q_1}} + M\|u_0\|_\alpha + ML_3r + M\|g(0)\|_\alpha \\ &\quad + (M+1)L_1C_{1-\alpha}r + (M+1)C_{1-\alpha}\|Ah(0,0)\|_\alpha + L_1aC_{1-\alpha} \\ &\quad + \frac{M_\alpha\Gamma(2-\alpha)a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}[L_1(a+r)+\|Ah(0,0)\|] \\ &\quad + ML_2mr + M\sum_{i=1}^m\|I_i(0)\|_\alpha. \end{aligned}$$

Dividing on both sides by r and taking the limits as $r\to+\infty$, we have

$$M(L_2m+L_3)+(M+1)L_1C_{1-\alpha}+\frac{M_\alpha L_1\Gamma(2-\alpha)a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\ge 1,$$

which contradicts (4). Hence there exists a positive constant r_0 such that $Q_1u + Q_2u \in B_{r_0}$ for any $u \in B_{r_0}$.

Next, we will show that Q_1 is a completely continuous operator and Q_2 is a contraction on B_{r_0}. Our proof will be divided into three steps.

Step I. Q_1 is continuous on B_{r_0}.

For any $u_n, u \in B_{r_0}$, $n = 1, 2, \ldots$ with $\lim_{n\to\infty} \|u_n - u\|_{PC} = 0$, we get $\lim_{n\to\infty} u_n(t) = u(t)$ for all $t \in J$. Hence, by the assumption (H_3), we have

$$\lim_{n\to\infty} f\bigl(t, u_n(t)\bigr) = f\bigl(t, u(t)\bigr), \quad t \in J.$$

Noting that $\|f(s, u_n(s)) - f(s, u(s))\| \le 2F(t)$, by the dominated convergence theorem, we have

$$\begin{aligned} &\bigl\|(Q_1u_n)(t) - (Q_1u)(t)\bigr\|_\alpha \\ &\quad\le \int_0^t \bigl\|(t-s)^{q-1}A^\alpha V(t-s)\bigl[f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr]\bigr\|\,ds \\ &\quad\le \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\int_0^t (t-s)^{q(1-\alpha)-1}\bigl\|f\bigl(s,u_n(s)\bigr) - f\bigl(s,u(s)\bigr)\bigr\|\,ds \\ &\quad\to 0, \end{aligned}$$

as $n \to \infty$, which implies that Q_1 is continuous.

Step II. $\{Q_1u : u \in B_{r_0}\}$ is relatively compact. It suffices to show that the family of functions $\{Q_1u : u \in B_{r_0}\}$ is uniformly bounded and equicontinuous, and for any $t \in J$, $\{(Q_1u)(t) : u \in B_{r_0}\}$ is relatively compact in X_α.

For any $u \in B_{r_0}$, we see from above that $\|Q_1u\|_{PC} \le r_0$, which means that $\{Q_1u : u \in B_{r_0}\}$ is uniformly bounded. In the following, we will show that $\{Q_1u : u \in B_{r_0}\}$ is a family of equicontinuous functions.

For any $u \in B_{r_0}$ and $0 \le t' < t'' \le a$, we get

$$\begin{aligned} &\bigl\|(Q_1u)\bigl(t''\bigr) - (Q_1u)\bigl(t'\bigr)\bigr\|_\alpha \\ &\quad\le \biggl\|\int_0^{t'}\bigl[\bigl(t''-s\bigr)^{q-1} - \bigl(t'-s\bigr)^{q-1}\bigr]V\bigl(t''-s\bigr)f\bigl(s,u(s)\bigr)\,ds\biggr\|_\alpha \\ &\qquad + \biggl\|\int_0^{t'}\bigl(t'-s\bigr)^{q-1}\bigl[V\bigl(t''-s\bigr) - V\bigl(t'-s\bigr)\bigr]f\bigl(s,u(s)\bigr)\,ds\biggr\|_\alpha \\ &\qquad + \biggl\|\int_{t'}^{t''}\bigl(t''-s\bigr)^{q-1}V\bigl(t''-s\bigr)f\bigl(s,u(s)\bigr)\,ds\biggr\|_\alpha \\ &\quad\le \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\int_0^{t'}\bigl|\bigl(t''-s\bigr)^{q-1} - \bigl(t'-s\bigr)^{q-1}\bigr|\bigl(t''-s\bigr)^{-q\alpha}\cdot F(s)\,ds \\ &\qquad + \int_0^{t'}\bigl(t'-s\bigr)^{q-1}\bigl\|A^\alpha V\bigl(t''-s\bigr) - A^\alpha V\bigl(t'-s\bigr)\bigr\|\cdot F(s)\,ds \\ &\qquad + \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\int_{t'}^{t''}\bigl(t''-s\bigr)^{q(1-\alpha)-1}\cdot F(s)\,ds \\ &\quad\triangleq A_1 + A_2 + A_3. \end{aligned}$$

From the expressions of A_1 and A_3, it is easy to see that $A_1 \to 0$ and $A_3 \to 0$ as $t''-t' \to 0$ independently of $u \in B_{r_0}$. For $t' = 0$, $0 < t'' \le a$, it is easy to see that $A_2 = 0$. For $t' > 0$, let $\varepsilon \in (0,t')$ be small enough. Then, from the expression of A_2, we have

$$\begin{aligned} A_2 &\le \int_0^{t'-\varepsilon} (t'-s)^{q-1} \| A^\alpha V(t''-s) - A^\alpha V(t'-s) \| \cdot F(s)\,ds \\ &\quad + \int_{t'-\varepsilon}^{t'} (t'-s)^{q-1} \| A^\alpha V(t''-s) - A^\alpha V(t'-s) \| \cdot F(s)\,ds \\ &\le \frac{\overline{M}[(t')^{1+b_1} - \varepsilon^{1+b_1}]^{1-q_1}}{\Gamma(1+q)(1+b_1)^{1-q_1}} \cdot \sup_{s\in[0,t'-\varepsilon]} \| T((t''-s)^q\theta) - T((t'-s)^q\theta) \|_\alpha \\ &\quad + \frac{2M_\alpha q\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))} \int_{t'-\varepsilon}^{t'} (t'-s)^{q(1-\alpha)-1} \cdot F(s)\,ds \end{aligned}$$

for $\theta \in (0,\infty)$, where $b_1 = \frac{q-1}{1-q_1}$. Since Lemma 2 implies the continuity of $T_\alpha(t)$ in $t > 0$ in the uniformly operator topology, it is easy to see that $A_2 \to 0$ as $t''-t' \to 0$ independently of $u \in B_{r_0}$. Thus, $\|(Q_1u)(t'') - (Q_1u)(t')\|_\alpha \to 0$ as $t''-t' \to 0$ independently of $u \in B_{r_0}$, which means that the set $\{Q_1u : u \in B_{r_0}\}$ is equicontinuous.

It remains to prove that for any $t \in J$, the set $W(t) := \{(Q_1u)(t) : u \in B_{r_0}\}$ is relatively compact in X_α.

Obviously, $W(0)$ is relatively compact in X_α. Let $0 < t \le a$ be fixed. For $\forall\varepsilon \in (0,t)$ and $\forall\delta > 0$, define an operator $Q_1^{\varepsilon,\delta}$ on B_{r_0} by

$$\begin{aligned} (Q_1^{\varepsilon,\delta}u)(t) &= q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\eta_q(\theta)T((t-s)^q\theta)f(s,u(s))\,d\theta\,ds \\ &= T(\varepsilon^q\delta)q\int_0^{t-\varepsilon}\int_\delta^\infty \theta(t-s)^{q-1}\eta_q(\theta)T((t-s)^q\theta - \varepsilon^q\delta)f(s,u(s))\,d\theta\,ds. \end{aligned}$$

Then from the compactness of $T(\varepsilon^q\delta)$, we find that the set $W_{\varepsilon,\delta}(t) := \{(Q_1^{\varepsilon,\delta}u)(t) : u \in B_{r_0}\}$ is relatively compact in X_α for $\forall\varepsilon \in (0,t)$ and $\forall\delta > 0$. Moreover, for every $u \in B_{r_0}$, we have

$$\begin{aligned} &\| (Q_1u)(t) - (Q_1^{\varepsilon,\delta}u)(t) \|_\alpha \\ &\quad \le q\left\| \int_0^t\int_0^\delta \theta(t-s)^{q-1}\eta_q(\theta)T((t-s)^q\theta)f(s,u(s))\,d\theta\,ds \right\|_\alpha \\ &\qquad + q\left\| \int_{t-\varepsilon}^t\int_\delta^\infty \theta(t-s)^{q-1}\eta_q(\theta)T((t-s)^q\theta)f(s,u(s))\,d\theta\,ds \right\|_\alpha \\ &\quad \le qM_\alpha\left(\int_0^t (t-s)^b\,ds\right)^{1-q_1} \cdot \|F\|_{L^{\frac{1}{q_1}}[0,t]} \cdot \int_0^\delta \theta^{1-\alpha}\eta_q(\theta)\,d\theta \\ &\qquad + \frac{qM_\alpha\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))}\left(\int_{t-\varepsilon}^t (t-s)^b\,ds\right)^{1-q_1} \cdot \|F\|_{L^{\frac{1}{q_1}}[t-\varepsilon,t]} \\ &\quad \le \frac{qM_\alpha\overline{M}}{(1+b)^{1-q_1}} a^{(1+b)(1-q_1)} \cdot \int_0^\delta \theta^{1-\alpha}\eta_q(\theta)\,d\theta \\ &\qquad + \frac{qM_\alpha\overline{M}\Gamma(2-\alpha)}{\Gamma(1+q(1-\alpha))(1+b)^{1-q_1}}\varepsilon^{(1+b)(1-q_1)}, \end{aligned}$$

where $b = \frac{q(1-\alpha)-1}{1-q_1} \in (-1, 0)$. Therefore, there are relatively compact sets arbitrarily close to the set $W(t)$, $t > 0$. Hence the set $W(t)$, $t > 0$ is also relatively compact in X_α.

Therefore, the set $\{Q_1 u : u \in B_{r_0}\}$ is relatively compact by the Ascoli-Arzela theorem. Thus, the continuity of Q_1 and relative compactness of the set $\{Q_1 u : u \in B_{r_0}\}$ imply that Q_1 is a completely continuous operator.

Step III. Q_2 is a contraction on B_{r_0}.

For any $u, v \in B_{r_0}$ and $t \in J$, if $t \in [0, t_1]$, by the assumptions (H_2), (H_4), and (H_5), we have

$$\begin{aligned}
&\bigl\|(Q_2 u)(t) - (Q_2 v)(t)\bigr\|_\alpha \\
&\quad\le \bigl\|U(t)\bigl[\bigl(g(v) - g(u)\bigr) + \bigl(h\bigl(0, v(0)\bigr) - h\bigl(0, u(0)\bigr)\bigr)\bigr]\bigr\|_\alpha \\
&\qquad + \bigl\|h\bigl(t, u(t)\bigr) - h\bigl(t, v(t)\bigr)\bigr\|_\alpha + \left\| \int_0^t (t-s)^{q-1} AV(t-s)\bigl[h\bigl(s, u(s)\bigr) - h\bigl(s, v(s)\bigr)\bigr]\, ds \right\|_\alpha \\
&\quad\le M\bigl[\bigl\|g(u) - g(v)\bigr\|_\alpha + C_{1-\alpha}\bigl\|Ah\bigl(0, u(0)\bigr) - Ah\bigl(0, v(0)\bigr)\bigr\|\bigr] \\
&\qquad + C_{1-\alpha}\bigl\|Ah\bigl(t, u(t)\bigr) - Ah\bigl(t, v(t)\bigr)\bigr\| \\
&\qquad + \int_0^t (t-s)^{q-1}\bigl\|A^\alpha V(t-s)\bigr\| \cdot \bigl\|Ah\bigl(s, u(s)\bigr) - Ah\bigl(s, v(s)\bigr)\bigr\|\, ds \\
&\quad\le \left[ML_3 + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha L_1 \Gamma(2-\alpha) a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\right] \cdot \|u - v\|_{PC}.
\end{aligned}$$

If $t \in (t_k, t_{k+1}]$, $k = 1, 2, \ldots, m$, we have

$$\begin{aligned}
&\bigl\|(Q_2 u)(t) - (Q_2 v)(t)\bigr\|_\alpha \\
&\quad\le \bigl\|U(t)\bigl[\bigl(g(v) - g(u)\bigr) + \bigl(h\bigl(0, v(0)\bigr) - h\bigl(0, u(0)\bigr)\bigr)\bigr]\bigr\|_\alpha \\
&\qquad + \bigl\|h\bigl(t, u(t)\bigr) - h\bigl(t, v(t)\bigr)\bigr\|_\alpha + \left\| \int_0^t (t-s)^{q-1} AV(t-s)\bigl[h\bigl(s, u(s)\bigr) - h\bigl(s, v(s)\bigr)\bigr]\, ds \right\|_\alpha \\
&\qquad + \left\| \sum_{i=1}^k U(t - t_i)\bigl[I_i\bigl(u(t_i)\bigr) - I_i\bigl(v(t_i)\bigr)\bigr] \right\|_\alpha \\
&\quad\le \left[M(L_2 m + L_3) + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha L_1 \Gamma(2-\alpha) a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\right] \cdot \|u - v\|_{PC}.
\end{aligned}$$

Thus, for any $u, v \in B_{r_0}$, it follows from the above that

$$\|Q_2 u - Q_2 v\|_{PC} = \sup_{t \in J}\bigl\|(Q_2 u)(t) - (Q_2 v)(t)\bigr\|_\alpha \le M^* \|u - v\|_{PC}.$$

Since $M^* < 1$, we know that Q_2 is a contraction on B_{r_0}. Hence, Krasnoselskii's fixed point theorem guarantees that the operator equation $Q_1 u + Q_2 u = u$ has a solution on B_{r_0}, which is the mild solution of the problem (2) on B_{r_0}. □

Theorem 2 *Assume that* (H_1)-(H_3) *hold. Further, the following conditions are also satisfied.*

(H_6) *The functions* $I_k : X_\alpha \to X_\alpha$ $(k = 1, 2, \ldots, m)$ *are completely continuous and there exist constants* $L_4, L_5 > 0$ *such that*

$$\bigl\|I_k(x)\bigr\|_\alpha \le L_4 \|x\|_\alpha + L_5, \quad x \in X_\alpha.$$

(H_7) *The function $g : PC(J, X_\alpha) \to X_\alpha$ is completely continuous and there exist constants $L_6, L_7 > 0$ such that*

$$\|g(u)\|_\alpha \leq L_6\|u\|_{PC} + L_7, \quad u \in PC(J, X_\alpha).$$

If $u_0 \in X_\alpha$, then the problem (2) *has at least one mild solution provided that*

$$M(L_4 m + L_6) + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha \Gamma(2-\alpha) L_1 a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} < 1. \tag{5}$$

Proof Define two operators F_1 and F_2 on $PC(J, X_\alpha)$ by

$$\begin{aligned}(F_1 u)(t) &= U(t)\big[u_0 - h\big(0, u(0)\big)\big] + h\big(t, u(t)\big) \\ &\quad + \int_0^t (t-s)^{q-1} AV(t-s) h\big(s, u(s)\big)\, ds, \quad t \in J;\end{aligned}$$

$$F_2(u)(t) = \begin{cases} -U(t)g(u) + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds, & t \in [0, t_1], \\ -U(t)g(u) + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds \\ \quad + U(t-t_1) I_1(u(t_1)), & t \in [t_1, t_2], \\ \vdots \\ -U(t)g(u) + \int_0^t (t-s)^{q-1} V(t-s) f(s, u(s))\, ds \\ \quad + \sum_{i=1}^m U(t-t_i) I_i(u(t_i)), & t \in [t_m, a]. \end{cases}$$

From (5), a similar proof as in Theorem 1 shows that there is a positive number r_0 such that $F_1 u + F_2 u \in B_{r_0}$ for any $u \in B_{r_0}$, and F_1 is a contraction. From (H_2), (H_3), (H_6), and (H_7), it is easy to see that F_2 is continuous. Next, we will prove that the set $\{F_2 u : u \in B_{r_0}\}$ is relatively compact. From the proof of Theorem 1, we only need to prove that the set $\{-U(t)g(u) + F_3 u : u \in B_{r_0}\}$ is relatively compact, where F_3 is defined by

$$F_3(u)(t) = \begin{cases} 0, & t \in [0, t_1], \\ U(t-t_1) I_1(u(t_1)), & t \in [t_1, t_2], \\ \vdots \\ \sum_{i=1}^m U(t-t_i) I_i(u(t_i)), & t \in [t_m, a]. \end{cases}$$

A similar proof as in [24, Theorem 3.1] shows that the set $\{-U(t)g(u) + F_3 u : u \in B_{r_0}\}$ is relatively compact. Hence F_2 is a completely continuous operator. By Krasnoselskii's fixed point theorem, the equation $F_1 u + F_2 u = u$ has a solution on B_{r_0}, which is the mild solution of the problem (2) on B_{r_0}. □

Theorem 3 *Assume that* (H_2), (H_4), (H_5) *hold. Further, the following condition is also satisfied.*

(H_8) *The function $f : J \times X_\alpha \to X$ is Lipschitz continuous, i.e., there exists a constant $L > 0$ such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|_\alpha$$

for any $t \in J$ and $x_1, x_2 \in X_\alpha$.

If $u_0 \in X_\alpha$, then the problem (2) *has a unique mild solution provided that*

$$M^{**} \triangleq M(L_2 m + L_3) + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha (L+L_1)\Gamma(2-\alpha)a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))} < 1. \tag{6}$$

Proof Define an operator Q on $PC(J, X_\alpha)$ by

$$(Qu)(t) = \begin{cases} U(t)[u_0 - g(u) - h(0,u(0))] + h(t,u(t)) + \int_0^t (t-s)^{(q-1)} AV(t-s)h(s,u(s))\,ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s,u(s))\,ds, \quad t \in [0,t_1], \\ U(t)[u_0 - g(u) - h(0,u(0))] + U(t-t_1)I_1(u(t_1)) + h(t,u(t)) \\ \quad + \int_0^t (t-s)^{(q-1)} AV(t-s)h(s,u(s))\,ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s,u(s))\,ds, \quad t \in (t_1,t_2], \\ \vdots \\ U(t)[u_0 - g(u) - h(0,u(0))] + \sum_{i=1}^m U(t-t_i)I_i(u(t_i)) + h(t,u(t)) \\ \quad + \int_0^t (t-s)^{(q-1)} AV(t-s)h(s,u(s))\,ds \\ \quad + \int_0^t (t-s)^{q-1} V(t-s) f(s,u(s))\,ds, \quad t \in (t_m,a]. \end{cases}$$

For any $t \in J$ and $u, v \in PC(J, X_\alpha)$, if $t \in [0,t_1]$, by assumptions (H_2), (H_5), and (H_8), we have

$$\begin{aligned} &\big\|(Qu)(t) - (Qv)(t)\big\|_\alpha \\ &\quad \le \big\|U(t)\big[\big(g(v) - g(u)\big) + \big(h\big(0,v(0)\big) - h\big(0,u(0)\big)\big)\big]\big\|_\alpha \\ &\qquad + \big\|h\big(t,u(t)\big) - h\big(t,v(t)\big)\big\|_\alpha + \left\|\int_0^t (t-s)^{q-1} AV(t-s)\big[h\big(s,u(s)\big) - h\big(s,v(s)\big)\big]\,ds\right\|_\alpha \\ &\qquad + \left\|\int_0^t (t-s)^{q-1} V(t-s)\big[f\big(s,u(s)\big) - f\big(s,v(s)\big)\big]\,ds\right\|_\alpha \\ &\quad \le \left[ML_3 + (M+1)L_1 C_{1-\alpha} + \frac{M_\alpha (L+L_1)\Gamma(2-\alpha)a^{q(1-\alpha)}}{(1-\alpha)\Gamma(1+q(1-\alpha))}\right] \cdot \|u-v\|_{PC}. \end{aligned}$$

If $t \in (t_k, t_{k+1}]$, from the above and the assumption (H_4), we have

$$\begin{aligned} &\big\|(Qu)(t) - (Qv)(t)\big\|_\alpha \\ &\quad \le \big\|U(t)\big[\big(g(v) - g(u)\big) + \big(h\big(0,v(0)\big) - h\big(0,u(0)\big)\big)\big]\big\|_\alpha \\ &\qquad + \big\|h\big(t,u(t)\big) - h\big(t,v(t)\big)\big\|_\alpha + \left\|\int_0^t (t-s)^{q-1} AV(t-s)\big[h\big(s,u(s)\big) - h\big(s,v(s)\big)\big]\,ds\right\|_\alpha \\ &\qquad + \left\|\int_0^t (t-s)^{q-1} V(t-s)\big[f\big(s,u(s)\big) - f\big(s,v(s)\big)\big]\,ds\right\|_\alpha \\ &\qquad + \left\|\sum_{i=1}^k U(t-t_i)\big[I_i\big(u(t_i)\big) - I_i\big(v(t_i)\big)\big]\right\|_\alpha \\ &\quad \le M^{**}\|u-v\|_{PC}. \end{aligned}$$

Thus, for any $u, v \in PC(J, X_\alpha)$, we have

$$\|Qu - Qv\|_{PC} = \sup_{t\in J}\big\|(Qu)(t) - (Qv)(t)\big\|_\alpha \le M^{**}\|u-v\|_{PC}.$$

Since $M^{**} < 1$, it follows that Q is a contraction on $PC(J, X_\alpha)$. By the Banach contraction principle, Q has a unique fixed point in $PC(J, X_\alpha)$, which is the unique mild solution of the problem (2). □

4 An example

Let $X = (L^2([0,1],\mathbb{R}), \|\cdot\|_2)$. We consider the following fractional partial differential equations in X:

$$\begin{cases} \partial_t^q [u(t,x) - \int_0^1 d(x,y)u(t,y)\,dy] - \frac{\partial^2}{\partial t^2}u(t,x) = G(t,x,u(t,x)), \\ \quad t \in [0,a], t \neq t_k, x \in [0,1], \\ u(t,0) = u(t,1) = 0, \quad 0 \le t \le a, \\ \Delta u|_{t=t_k} = I(x, \int_0^1 u(t_k,y)\,dy), \quad k = 1,2,\ldots,m, \\ u(0,x) + \sum_{i=0}^n \int_0^1 \ell(x,y)u(\tau_i,y)\,dy = u_0(x), \quad x \in [0,1], \end{cases} \tag{7}$$

where ∂_t^q is a Caputo fractional partial derivative of order $q \in (0,1)$, $0 < \tau_1 < \tau_2 < \cdots < \tau_n < a$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$.

We define an operator A by $Av = -v''$ with the domain

$$D(A) = \{v(\cdot) \in X : v, v'' \in X, v(0) = v(1) = 0\}.$$

Then $-A$ generates a compact and analytic semigroup $T(t)$ $(t \ge 0)$, and $\|T(t)\| \le e^{-t} \le 1$. It is well known that $0 \in \rho(A)$, and so the fractional powers of A are well defined. Moreover, the eigenvalues of A are $n^2\pi^2$ and the corresponding normalized eigenvectors are $e_n(x) = \sqrt{2}\sin(n\pi x)$, $n = 1,2,\ldots$. We define $A^{\frac{1}{2}}$ by $A^{\frac{1}{2}}z = \sum_{n=1}^\infty n\langle z, e_n\rangle e_n$ for each $z \in D(A^{\frac{1}{2}}) := \{z(\cdot) \in X : \sum_{n=1}^\infty n\langle z, e_n\rangle e_n \in X\}$. From [25] we know that if $z \in D(A^{\frac{1}{2}})$, then z is absolutely continuous with $z' \in X$ and $\|z'\|_2 = \|A^{\frac{1}{2}}z\|_2$.

We define the Banach space $X_{\frac{1}{2}}$ by $X_{\frac{1}{2}} = (D(A^{\frac{1}{2}}), \|\cdot\|_{\frac{1}{2}})$, where $\|z\|_{\frac{1}{2}} = \|A^{\frac{1}{2}}z\|_2 = \|z'\|_2$ for any $z \in D(A^{\frac{1}{2}})$. It is well known that $\|A^{-\frac{1}{2}}\| = 1$.

For solving the problem (7), we need the following assumptions.

(P_1) The function $d : [0,1] \times [0,1] \to \mathbb{R}$ satisfies the following conditions.
  (i) $(x,y) \mapsto \frac{\partial^2}{\partial x^2}d(x,y)$ is well defined and measurable with

$$\int_0^1 \int_0^1 \left(\frac{\partial^2}{\partial x^2}d(x,y)\right)^2 dy\,dx < +\infty.$$

  (ii) $d(0,y) = d(1,y) = 0$, $\forall y \in [0,1]$.

(P_2) The function $I : [0,1] \times \mathbb{R} \to \mathbb{R}$ satisfies the following conditions.
  (i) For each $\xi \in \mathbb{R}$, the function $I(\cdot,\xi)$ is differentiable and $\frac{\partial}{\partial x}I(x,\xi) \in X$.
  (ii) There exists a constant $N_1 > 0$ such that

$$\left|\frac{\partial}{\partial x}I(x,\xi_1) - \frac{\partial}{\partial x}I(x,\xi_2)\right| \le N_1|\xi_1 - \xi_2|$$

  for any $x \in [0,1]$ and $\xi_1, \xi_2 \in \mathbb{R}$.
  (iii) $I(0,y) = I(1,y) = 0$, $\forall y \in \mathbb{R}$.

(P_3) The function $\ell \in L^2([0,1] \times [0,1], \mathbb{R})$ satisfies the following conditions.

(i) $(x,y) \mapsto \frac{\partial}{\partial x}\ell(x,y)$ belongs to $L^2([0,1] \times [0,1], \mathbb{R})$ and

$$\int_0^1 \int_0^1 \left[\frac{\partial}{\partial x}\ell(x,y)\right]^2 dy\,dx < +\infty.$$

(ii) $\ell(0,y) = \ell(1,y) = 0$, $\forall y \in [0,1]$.

Let $PC([0,a], X_{\frac{1}{2}})$ be the Banach space equipped with supnorm

$$\|u\|_{PC} = \sup_{0\le t\le a} \left\|u(t)(\cdot)\right\|_{\frac{1}{2}} = \sup_{0\le t\le a} \left\|\left(u(t)\right)'(\cdot)\right\|_2,$$

and let $f : [0,a] \times X_{\frac{1}{2}} \to X$ be defined by $f(t,\phi)(\cdot) = G(t,\cdot,\phi(\cdot))$, $h : [0,a] \times X_{\frac{1}{2}} \to X$ be defined by

$$h(t,\phi)(\cdot) = \int_0^1 d(\cdot,y)\phi(y)\,dy,$$

$I_k : X_{\frac{1}{2}} \to X$ be defined by

$$I_k(\phi)(\cdot) = I\left(\cdot, \int_0^1 \phi(y)\,dy\right),$$

and $g : PC([0,a], X_{\frac{1}{2}}) \to X$ be defined by

$$g(u)(\cdot) = \left(\sum_{i=0}^{n} \ell_g u(\tau_i)\right)(\cdot),$$

where $\ell_g : X_{\frac{1}{2}} \to X$ is defined by

$$\ell_g(\psi)(x) = \int_0^1 \ell(x,y)\psi(y)\,dy, \quad \forall \psi \in X_{\frac{1}{2}}.$$

Moreover, if $u : [0,a] \times [0,1] \to \mathbb{R}$, we defined $u : [0,a] \to \mathbb{R}$ by $u(t)(\cdot) = u(t,\cdot)$. Thus, the system (7) can be reformed as the nonlocal problem (2).

By the definition of h and assumption (P_1), a similar computation as in [26, Theorem 4.2(a)] shows that $h \in D(A)$ and

$$\left\|Ah(t_1,\phi_1) - Ah(t_2,\phi_2)\right\|_2^2 \le \int_0^1 \int_0^1 \left(\frac{\partial^2}{\partial x^2} b(x,y)\right)^2 dy\,dx \cdot \|\phi_1 - \phi_2\|_{\frac{1}{2}}^2$$

for each $(t_1,\phi_1), (t_2,\phi_2) \in [0,a] \times X_{\frac{1}{2}}$. Hence h satisfies the hypothesis (H_2).

For each $\phi \in X_{\frac{1}{2}}$, by the assumption (P_2), we see that

$$\begin{aligned}
\langle I_k(\phi), e_n\rangle &= \int_0^1 \left(I\left(x, \int_0^1 \phi(y)\,dy\right)\right) \cdot \sqrt{2}\sin(n\pi x)\,dx \\
&= \frac{1}{n\pi} \int_0^1 \left(\frac{\partial}{\partial x} I\left(x, \int_0^1 \phi(y)\,dy\right)\right) \cdot \sqrt{2}\cos(n\pi x)\,dx.
\end{aligned}$$

Hence, I_k is a function from $X_{\frac{1}{2}}$ into $X_{\frac{1}{2}}$. By (P_2)(ii) and the Hölder inequality, we have

$$\begin{aligned}\|I_k(\phi_1) - I_k(\phi_2)\|_{\frac{1}{2}}^2 &= \int_0^1 \left| \frac{\partial}{\partial x} I\left(x, \int_0^1 \phi_1(y)\,dy\right) - \frac{\partial}{\partial x} I\left(x, \int_0^1 \phi_2(y)\,dy\right)\right|^2 dx \\ &\le N_1^2 \left| \int_0^1 \phi_1(y)\,dy - \int_0^1 \phi_2(y)\,dy\right|^2 \\ &\le N_1^2 \|\phi_1 - \phi_2\|_{\frac{1}{2}}^2\end{aligned}$$

for each $\phi_1, \phi_2 \in X_{\frac{1}{2}}$. This implies that the assumptions (H_4) and (H_6) hold.

By the assumption (P_3), a similar computation as above shows that g is a function from $PC([0,a], X_{\frac{1}{2}})$ into $X_{\frac{1}{2}}$. And

$$\begin{aligned}\|g(u_1) - g(u_2)\|_{\frac{1}{2}}^2 &= \int_0^1 \left| \frac{\partial}{\partial x} \sum_{i=0}^n \ell_g u_1(\tau_i) - \frac{\partial}{\partial x} \sum_{i=0}^n \ell_g u_2(\tau_i)\right|^2 dx \\ &= \int_0^1 \left| \sum_{i=0}^n \int_0^1 \frac{\partial}{\partial x} \ell(x,y)\big[u_1(\tau_i, y) - u_2(\tau_i, y)\big]\,dy\right|^2 dx \\ &\le \int_0^1 \left| \sum_{i=0}^n \left[\left(\int_0^1 \left|\frac{\partial}{\partial x}\ell(x,y)\right|^2 dy\right)^{\frac{1}{2}} \cdot \big\|u_1(\tau_i, y) - u_2(\tau_i, y)\big\|_2\right]\right|^2 dx \\ &\le (n+1) \int_0^1 \int_0^1 \left(\frac{\partial}{\partial x}\ell(x,y)\right)^2 dy\,dx \cdot \|u_1 - u_2\|_{PC}^2\end{aligned}$$

for each $u_1, u_2 \in PC([0,a], X_{\frac{1}{2}})$. By [26, Theorem 4.3(b)], g is a compact operator. Thus, the assumptions (H_5) and (H_7) hold.

We can take $q = \frac{1}{2}$ and $f(t, u(t)) = \frac{1}{t^{\frac{1}{3}}} \sin u(t)$. Since for any $t \in J$, we have $\|f(t, u(t))\| = \|\frac{1}{t^{\frac{1}{3}}} \sin u(t)\| \le \frac{1}{t^{\frac{1}{3}}}$. So, we choose $F(t) = \frac{1}{t^{\frac{1}{3}}}$, then the assumptions ($\mathrm{H}_3$) and ($\mathrm{H}_8$) hold. Hence, if $u_0 \in X_{\frac{1}{2}}$, according to Theorem 1 or Theorem 2, the system (7) has at least one mild solution provided that (4) or (5) holds. From Theorem 3, the system (7) has a unique mild solution provided that (6) holds.

Competing interests

The author declares that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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