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Global stability in n-dimensional discrete Lotka-Volterra predator-prey models

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Dedicated to Professor SangKwon Chung on the occasion of his 60th birthday

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Abstract

There are few theoretical works on global stability of Euler difference schemes for two-dimensional Lotka-Volterra predator-prey models. Furthermore no attempt is made to show that the Euler schemes have positive solutions. In this paper, we consider Euler difference schemes for both the two-dimensional models and n-dimensional models that are a generalization of the two-dimensional models. It is first shown that the difference schemes have positive solutions and equilibrium points which are globally asymptotically stable in the two-dimensional cases. The approaches used in the two-dimensional models are extended to the n-dimensional models for obtaining the positivity and the global stability. Numerical examples are presented to verify the results.

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1 Introduction

Consider the n-dimensional system

$$\frac{dx_i}{dt} = x_i\left(\sigma_i r_i + \sum_{1\le j\le i-1} a_{ij}x_j - \sum_{i\le j\le n} a_{ij}x_j\right), \tag{1.1}$$

where $r_i > 0$, $a_{ij} > 0$ for $1 \le i,j \le n$, $\sigma_1 = 1$, and $\sigma_i \in \{-1,1\}$ for $2 \le i \le n$.

The system equation (1.1) can be seen as a generalization of the two-dimensional Lotka-Volterra predator-prey model

$$\frac{dx}{dt} = x(r_1 - a_{11}x - a_{12}y), \qquad \frac{dy}{dt} = y(-r_2 + a_{21}x - a_{22}y), \tag{1.2}$$

where x and y denote the population sizes of prey and predator, respectively.

There are a number of works on investigating nonstandard finite difference schemes for the Lotka-Volterra competition models (see [1] and the references given there), but relatively few theoretical papers are published on discretized models of equation (1.2). In particular, to my knowledge, Euler difference schemes for equation (1.2) have not theoretically been studied for the global stability of the equilibrium points except a recent paper [2]. In Section 2, it is shown that the Euler difference scheme has positive solutions. In order to show the global asymptotic stability of the equilibrium point whose components are all positive, the paper [2] assumes that $(0,0)$ is globally stable. Without using the assumption, we show the global stability of all of the equilibrium points in Section 3. In addition, we also analyze the Euler difference scheme for equation (1.2) with r_2 replaced by $-r_2$. We are interested in extending the method used in the two-dimensional discrete models to the n-dimensional discrete models for equation (1.1). In Section 4, we demonstrate the positivity and the global stability in the n-dimensional discrete cases. Numerical examples are given to verify the results of this paper.

2 Two-dimensional predator-prey model

In this section, we consider the Euler difference scheme for equation (1.2)

$$x_0 = x(0), \qquad y_0 = y(0), \qquad x_{k+1} = F_{y_k}(x_k), \qquad y_{k+1} = G_{x_k}(y_k), \tag{2.1}$$

where Δt is a time step size, $\hat{r}_i = r_i\Delta t$, $\hat{a}_{ij} = a_{ij}\Delta t$ for $1 \leq i,j \leq 2$, and

$$F_y(x) = x(1 + \hat{r}_1 - \hat{a}_{11}x - \hat{a}_{12}y), \qquad G_x(y) = y(1 - \hat{r}_2 + \hat{a}_{21}x - \hat{a}_{22}y). \tag{2.2}$$

Note that if τ_1 and τ_2 are positive constants such that

$$U_1(\tau_2) = \frac{1 + \hat{r}_1 - \hat{a}_{12}\tau_2}{2\hat{a}_{11}} > 0, \qquad U_2(\tau_1) = \frac{1 - \hat{r}_2 + \hat{a}_{21}\tau_1}{2\hat{a}_{22}} > 0,$$

then

$$F_{\tau_2}(x) \text{ and } G_{\tau_1}(y) \text{ are increasing on } 0 \leq x < U_1(\tau_2) \text{ and } 0 \leq y < U_2(\tau_1). \tag{2.3}$$

Let Δt satisfy $\max\{\hat{r}_1, \hat{r}_2\} < 1$. Take positive constants χ_1 and χ_2 such that

$$-\hat{r}_2 + \hat{a}_{21}\chi_1 \leq 1, \qquad \frac{r_1}{a_{11}} \leq \chi_1 \leq U_1(\chi_2), \qquad \frac{-r_2 + a_{21}\chi_1}{a_{22}} \leq \chi_2 \leq U_2(0). \tag{2.4}$$

Theorem 2.1 *Let* Δt, χ_1, χ_2 *satisfy equation* (2.4) *with* $\max\{\hat{r}_1, \hat{r}_2\} < 1$. *If* $(x_0, y_0) \in (0, \chi_1) \times (0, \chi_2)$, *then* $(x_k, y_k) \in (0, \chi_1) \times (0, \chi_2)$ *for all* k *with* $k \geq 1$.

Proof Since

$$0 < x_0 < \chi_1 \leq U_1(\chi_2) < U_1(y_0), \qquad 0 < y_0 < \chi_2 \leq U_2(0) < U_2(x_0),$$

it follows from equation (2.3) that

$$x_1 = F_{y_0}(x_0) > F_{y_0}(0) = 0, \qquad y_1 = G_{x_0}(y_0) > G_{x_0}(0) = 0.$$

If $r_1 - a_{11}x_0 - a_{12}y_0 \leq 0$, then $x_1 = F_{y_0}(x_0) \leq x_0 < \chi_1$, and otherwise,

$$0 < x_0 < \frac{r_1 - a_{12}y_0}{a_{11}} < \frac{\hat{r}_1 + (\hat{r}_1 - \hat{a}_{12}y_0)}{2\hat{a}_{11}} < \frac{1 + \hat{r}_1 - \hat{a}_{12}y_0}{2\hat{a}_{11}} = U_1(y_0)$$

by the condition $\hat{r}_1 < 1$. Hence equation (2.3) implies that

$$x_1 = F_{y_0}(x_0) < F_{y_0}\left(\frac{r_1 - a_{12}y_0}{a_{11}}\right) = \frac{r_1 - a_{12}y_0}{a_{11}} < \frac{r_1}{a_{11}} \leq \chi_1.$$

Similarly if $-r_2 + a_{21}x_0 - a_{22}y_0 \le 0$, then $y_1 = G_{x_0}(y_0) \le y_0 < \chi_2$, and otherwise,

$$0 < y_0 < \frac{-r_2 + a_{21}x_0}{a_{22}} < \frac{(-\hat{r}_2 + \hat{a}_{21}\chi_1) + (-\hat{r}_2 + \hat{a}_{21}x_0)}{2\hat{a}_{22}} \le \frac{1 - \hat{r}_2 + \hat{a}_{21}x_0}{2\hat{a}_{22}} = U_2(x_0)$$

by the condition $-\hat{r}_2 + \hat{a}_{21}\chi_1 \le 1$. Thus equation (2.3) gives

$$y_1 = G_{x_0}(y_0) < G_{x_0}\left(\frac{-r_2 + a_{21}x_0}{a_{22}}\right) = \frac{-r_2 + a_{21}x_0}{a_{22}} < \frac{-r_2 + a_{21}\chi_1}{a_{22}} \le \chi_2.$$

Finally we obtain, if $(x_0, y_0) \in (0, \chi_1) \times (0, \chi_2)$, then

$$(x_1, y_1) \in (0, \chi_1) \times (0, \chi_2).$$

By the principle of mathematical induction, the proof is completed. □

From now on, we assume that $(x_0, y_0) \in (0, \chi_1) \times (0, \chi_2)$ and (x_k, y_k) for $k \ge 1$ denote the solutions of equation (2.1). For simplicity of notation, we write *for all k* instead of *for all k with k* ≥ 1 when there is no confusion.

Remark 2.2 Theorem 2.1 gives for all k

$$0 < x_k < \chi_1 < U_1(\chi_2) < U_1(y_k), \qquad 0 < y_k < \chi_2 < U_2(0) < U_2(x_k).$$

Hence it follows from equation (2.3) that for every fixed (x_k, y_k)

$$F_{y_k}(x) \text{ and } G_{x_k}(y) \text{ are increasing on } 0 \le x < \chi_1 \text{ and } 0 \le y < \chi_2. \tag{2.5}$$

Let $f(x) = \frac{r_1 - a_{11}x}{a_{12}}$ and $g(x) = \frac{-r_2 + a_{21}x}{a_{22}}$. Since f and g are decreasing and increasing, respectively, it follows from equation (2.4) that for all k

$$f^{-1}(y_k) < f^{-1}(0) \le \chi_1, \qquad \max\{g(f^{-1}(0)), g(x_k)\} \le g(\chi_1) \le \chi_2. \tag{2.6}$$

Set $D = (0, \chi_1) \times (0, \chi_2)$, and let S_i for $1 \le i \le 4$ denote the four areas

$$S_1 = \{(x, y) \in D | g(x) \le y < f(x)\}, \qquad S_2 = \{(x, y) \in D | y \le f(x), y < g(x)\},$$
$$S_3 = \{(x, y) \in D | f(x) < y \le g(x)\}, \qquad S_4 = \{(x, y) \in D | y \ge f(x), y > g(x)\}.$$

Remark 2.3 Let $(x_k, y_k) \in \bigcup_{1 \le i \le 4} S_i$ for some k. The following can be obtained by using equation (2.5), equation (2.6), and the definitions of $F_{y_k}(x_k)$ and $G_{x_k}(y_k)$.

(a) Suppose $(x_k, y_k) \in S_1$. Since $g(x_k) \le y_k < f(x_k)$, we have

$$x_{k+1} = F_{y_k}(x_k) > F_{f(x_k)}(x_k) = x_k, \qquad x_{k+1} = F_{y_k}(x_k) < F_{y_k}(f^{-1}(y_k)) = f^{-1}(y_k),$$
$$y_{k+1} = G_{x_k}(y_k) \le G_{g^{-1}(y_k)}(y_k) = y_k < f(x_{k+1}).$$

(b) Suppose $(x_k, y_k) \in S_2$. This gives $y_k \le f(x_k)$ and $y_k < g(x_k)$, and then

$$x_{k+1} = F_{y_k}(x_k) \ge F_{f(x_k)}(x_k) = x_k,$$
$$y_{k+1} = G_{x_k}(y_k) > G_{g^{-1}(y_k)}(y_k) = y_k, \qquad y_{k+1} < G_{x_k}\big(g(x_k)\big) = g(x_k) < g(x_{k+1}).$$

(c) Suppose $(x_k, y_k) \in S_3$. Since $f(x_k) < y_k \le g(x_k)$, we have

$$x_{k+1} = F_{y_k}(x_k) < F_{f(x_k)}(x_k) = x_k, \qquad x_{k+1} > F_{y_k}\big(f^{-1}(y_k)\big) = f^{-1}(y_k),$$
$$y_{k+1} = G_{x_k}(y_k) \ge G_{g^{-1}(y_k)}(y_k) = y_k > f(x_{k+1}).$$

(d) Suppose $(x_k, y_k) \in S_4$, which means that $y_k \ge f(x_k)$ and $y_k > g(x_k)$. Then

$$x_{k+1} = F_{y_k}(x_k) \le F_{f(x_k)}(x_k) = x_k,$$
$$y_{k+1} = G_{x_k}(y_k) < G_{g^{-1}(y_k)}(y_k) = y_k, \qquad y_{k+1} > G_{x_k}\big(g(x_k)\big) = g(x_k) > g(x_{k+1}).$$

Therefore (x_k, y_k) in S_1, S_2, S_3, and S_4 moves to

$$\text{the lower right, upper right, upper left, and lower left parts} \tag{2.7}$$

of $S_1 \cup S_2$, $S_2 \cup S_3$, $S_3 \cup S_4$, and $S_4 \cup S_1$, respectively (see Figures 1 and 2).

Set $S_5 = S_1$, and use the notation $[\![x_N, y_N]\!] \in S_{i+1}$ for $1 \le i \le 4$ and a positive integer N to denote both $(x_k, y_k) \in S_i$ for all k with $0 \le k < N$ and $(x_N, y_N) \in S_{i+1}$. Then equation (2.7) implies the following theorem.

Theorem 2.4 *Let the assumptions of Theorem* 2.1 *hold. Suppose* $1 \le i \le 4$.

If $(x_0, y_0) \in S_i$, *then* $(x_k, y_k) \in S_i$ *for all* k *or* $[\![x_{N_i}, y_{N_i}]\!] \in S_{i+1}$ *for some* N_i.

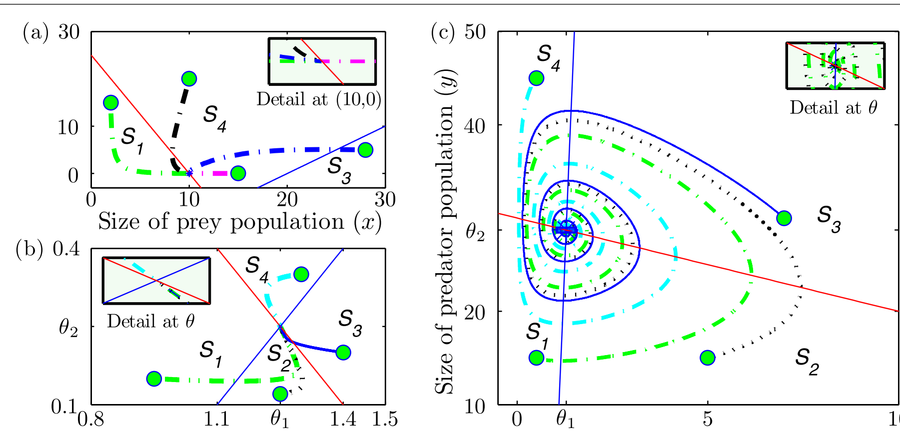

Figure 1 Two-dimensional models with $(\sigma_1, \sigma_2) = (1, -1)$. The dark circles denote initial conditions (see Examples 3.2, 3.6, and 3.7 for details). **(a)** $r_1 a_{21} - r_2 a_{11} < 0$ and the limits of the four trajectories are $(\frac{r_1}{a_{11}}, 0) = (10, 0)$. **(b)** $r_1 a_{21} - r_2 a_{11} > 0$. The solutions of equation (2.1) rotate finitely many times around the limit $\boldsymbol{\theta} = (\theta_1, \theta_2) = (1.25, 0.25)$. **(c)** $r_1 a_{21} - r_2 a_{11} > 0$. The solutions of equation (2.1) rotate infinitely many times around the limit $\boldsymbol{\theta} = (1.287, 28.713)$.

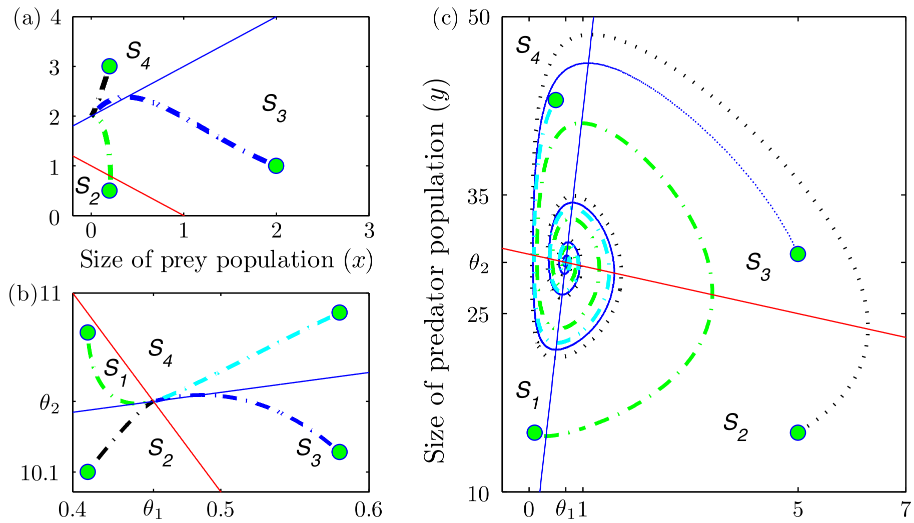

Figure 2 Two-dimensional models with $(\sigma_1, \sigma_2) = (1, 1)$. The dark circles denote initial conditions (see Example 3.10). **(a)** $r_1a_{22} - r_2a_{12} < 0$ and the limits of the four trajectories are $(0, \frac{r_2}{a_{22}}) = (0, 2)$. **(b)** $r_1a_{22} - r_2a_{12} > 0$. The solutions of equation (3.10) rotate finitely many times around the limit $\boldsymbol{\vartheta} = (0.455, 10.455)$. **(c)** $r_1a_{22} - r_2a_{12} > 0$. The solutions of equation (3.10) rotate infinitely many times around the limit $\boldsymbol{\vartheta} = (0.683, 29.317)$.

3 Dynamics of the two-dimensional predator-prey models

In this section, we first consider dynamics of the Euler difference scheme for equation (1.2) and next for equation (1.1) with $n = 2$ and $(\sigma_1, \sigma_2) = (1, 1)$. Let $\theta_1 = \frac{r_1a_{22}+r_2a_{12}}{a_{12}a_{21}+a_{11}a_{22}}$ and $\theta_2 = \frac{r_1a_{21}-r_2a_{11}}{a_{12}a_{21}+a_{11}a_{22}}$. For calculating the limits of x_k and y_k, we use the inequalities

$$\lim_{k\to\infty} x_k > 0, \qquad \lim_{k\to\infty} y_k > 0, \tag{3.1}$$

which is equivalent to $\lim_{k\to\infty}(x_k, y_k) = (\theta_1, \theta_2)$.

Theorem 3.1 *Let the assumptions of Theorem* 2.1 *hold. Suppose* $r_1a_{21} - r_2a_{11} \le 0$. *Then* (x_k, y_k) *satisfies the following dynamics with the limit* $(\frac{r_1}{a_{11}}, 0)$.

(a) *If* $(x_0, y_0) \in S_1$, *then* $(x_k, y_k) \in S_1$ *for all* k.
(b) *If* $(x_0, y_0) \in S_4$, *then* $(x_k, y_k) \in S_4$ *for all* k *or* $[\![x_{N_1}, y_{N_1}]\!] \in S_1$ *for some* N_1.
(c) *If* $(x_0, y_0) \in S_3$, *then* $[\![x_{N_2}, y_{N_2}]\!] \in S_4$ *for some* N_2.

Proof (a) Since $f^{-1}(0) = \frac{r_1}{a_{11}} \le \frac{r_2}{a_{21}} = g^{-1}(0)$, the set S_2 is empty, and then Theorem 2.4 gives $(x_k, y_k) \in S_1$ for all k. Thus equation (2.7) shows that x_k is bounded and increasing, and y_k is bounded and decreasing. Hence $\lim_{k\to\infty} x_k > 0$ and $\lim_{k\to\infty} y_k \ge 0$. Finally $\lim_{k\to\infty} y_k = 0$ since otherwise $0 < \lim_{k\to\infty} y_k = \theta_2 \le 0$, which is a contradiction. Therefore $\lim_{k\to\infty} x_k = \frac{r_1}{a_{11}}$ by using equation (2.1) with both $\lim_{k\to\infty} x_k > 0$ and $\lim_{k\to\infty} y_k = 0$.

(b) Theorem 2.4 and (a) in this theorem show that it suffices to show $\lim_{k\to\infty}(x_k, y_k) = (\frac{r_1}{a_{11}}, 0)$ in the case $(x_k, y_k) \in S_4$ for all k, where $\lim_{k\to\infty} x_k \ge 0$ and $\lim_{k\to\infty} y_k \ge 0$ by equation (2.7). Then $\lim_{k\to\infty} y_k = 0$ since otherwise the last equation in equation (2.1) gives $\lim_{k\to\infty} x_k > 0$, and hence equation (3.1) implies $\lim_{k\to\infty} y_k = \theta_2 \le 0$, producing a contradiction to $\lim_{k\to\infty} y_k > 0$.

Note that $y_k > f(x_k)$ for all k since $(x_k, y_k) \in S_4$ for all k. Thus if $\lim_{k\to\infty} x_k = 0$, then $\lim_{k\to\infty} y_k \geq \lim_{k\to\infty} f(x_k) = \frac{r_1}{a_{12}} > 0$, which contradicts to $\lim_{k\to\infty} y_k = 0$. Hence $\lim_{k\to\infty} x_k > 0$. Consequently, $\lim_{k\to\infty} x_k = \frac{r_1}{a_{11}}$ by using equation (2.1) with $\lim_{k\to\infty} y_k = 0$.

(c) Assume, to the contrary, that $(x_k, y_k) \in S_3$ for all k. Then equation (2.7) implies that x_k and y_k have positive limits, and hence equation (3.1) gives $0 < \lim_{k\to\infty} y_k = \theta_2 \leq 0$, which is a contradiction. Therefore $[\![x_{N_2}, y_{N_2}]\!] \in S_4$ for some N_2, which gives $\lim_{k\to\infty}(x_k, y_k) = (\frac{r_1}{a_{11}}, 0)$ by (b) in this theorem. □

Example 3.2 Consider the discrete system

$$x_{k+1} = x_k + x_k(50 - 5x_k - 2y_k)\Delta t, \qquad y_{k+1} = y_k + y_k(-100 + 5x_k - 5y_k)\Delta t$$

with $\Delta t = 0.001$ and $(\chi_1, \chi_2) = (80, 90)$. Then equation (2.4) and the conditions $r_1a_{21} - r_2a_{11} \leq 0$ and $\max\{\hat{r}_1, \hat{r}_2\} < 1$ in Theorem 3.1 are satisfied. For the four initial conditions

$$(2,15) \in S_1, \qquad (28,5) \in S_3, \qquad (10,20) \in S_4, \qquad (15,0.01) \in S_4,$$

Figure 1(a) shows that the solution (x_k, y_k) satisfies Theorem 2.1 and Theorem 3.1 with the limit $(\frac{r_1}{a_{11}}, 0) = (10, 0)$. In the case $(x_0, y_0) = (15, 0.01) \in S_4$, Figure 1(a) shows that for $0 \leq i \leq 25{,}000$ and $2{,}000 \leq k \leq 25{,}000$

$$10 < 10 + 9 * 10^{-39} = x_k \leq x_{i+1} \leq x_i$$

which implies that $(x_k, y_k) \in S_4$ for all k.

In order to show the global asymptotic stability of the equilibrium point $\boldsymbol{\theta} = (\theta_1, \theta_2)$, the linearized system of equation (2.1) at $\boldsymbol{\theta}$ is used: Consider the Jacobian matrix of $T(x,y) = (F_y(x), G_x(y))$ at $\boldsymbol{\theta}$

$$DT_{\boldsymbol{\theta}} = \begin{pmatrix} 1 - \theta_1 a_{11}\Delta t & -\theta_1 a_{12}\Delta t \\ \theta_2 a_{21}\Delta t & 1 - \theta_2 a_{22}\Delta t \end{pmatrix}. \tag{3.2}$$

Letting $J = \frac{1}{\Delta t}(DT_{\boldsymbol{\theta}} - I)$ with the 2×2 identity matrix I, we have $\operatorname{tr}(J) = -\theta_1 a_{11} - \theta_2 a_{22}$, $\det(J) = \theta_1\theta_2(a_{11}a_{22} + a_{12}a_{21})$, and the eigenvalues of $DT_{\boldsymbol{\theta}}$

$$1 + \frac{1}{2}\Big\{\operatorname{tr}(J) \pm \sqrt{\operatorname{tr}^2(J) - 4\det(J)}\Big\}\Delta t. \tag{3.3}$$

The following lemma is used for showing that the equilibrium point $\boldsymbol{\theta}$ of the nonlinear system $\mathbf{x}_{k+1} = T(\mathbf{x}_k)$ is locally asymptotically stable.

Lemma 3.3 *Suppose* $r_1a_{21} - r_2a_{11} > 0$. *Let* $J = \frac{1}{\Delta t}(DT_{\boldsymbol{\theta}} - I)$. *Then all of the eigenvalues of* $DT_{\boldsymbol{\theta}}$ *have magnitude less than* 1 *if one of the following is true*.

(a) $\operatorname{tr}^2(J) \geq 4\det(J)$.

(b) $\operatorname{tr}^2(J) < 4\det(J)$ *and* $\det(J)\Delta t < -\operatorname{tr}(J)$.

In order to find conditions for when (x_n, y_n) rotates finitely or infinitely many times around $\boldsymbol{\theta}$, we need the following theorem about a hyperbolic point: A point $\mathbf{p}$ of $\mathbf{x}_{k+1} = T(\mathbf{x}_k)$ is called hyperbolic if all of the eigenvalues of $DT_{\mathbf{p}}$ have nonzero real parts.

Theorem 3.4 (Hartman-Grobman theorem for maps) *Let* $\mathbf{p}$ *be a hyperbolic fixed point of* $\mathbf{x}_{k+1} = H(\mathbf{x}_k)$, *where* H *is a continuously differentiable function defined on a neighborhood of* $\mathbf{0} \in \mathbb{R}^n$ *for* $n \geq 1$. *Then there exist neighborhoods* U *of* $\mathbf{p}$, V *of the hyperbolic fixed point* $\mathbf{0}$ *of* $\mathbf{x}_{k+1} = DH_{\mathbf{p}}(\mathbf{x}_k)$, *and a homeomorphism* $h : V \to U$ *such that* $H(h(\mathbf{x})) = h(DH_{\mathbf{p}}(\mathbf{x}))$ *for all* $\mathbf{x} \in V$.

Theorem 3.4 (see [3] for the proof) states that the nonlinear system $\mathbf{x}_{k+1} = H(\mathbf{x}_k)$ is topologically equivalent to the linearized system $\mathbf{x}_{k+1} = DH_{\mathbf{p}}(\mathbf{x}_k)$ of the nonlinear system at $\mathbf{p}$.

Theorem 3.5 *Let the assumptions of Theorem* 2.1 *hold and let* $J = \frac{1}{\Delta t}(DT_{\boldsymbol{\theta}} - I)$. *Suppose* $r_1a_{21} - r_2a_{11} > 0$ *and* $(x_0, y_0) \in \bigcup_{1\leq i\leq 4} S_i$.

(a) *If* $\mathrm{tr}^2(J) \geq 4\det(J)$, *then* (x_k, y_k) *rotates finitely many times around* $\boldsymbol{\theta}$ *in the counterclockwise direction and finally stays in one of* S_i *for* $1 \leq i \leq 4$ *with* $\lim_{k\to\infty}(x_k, y_k) = \boldsymbol{\theta}$.

(b) *If the four inequalities* $\mathrm{tr}^2(J) < 4\det(J)$, $\det(J)\Delta t < -\mathrm{tr}(J)$,

$$\Delta t \max_{1\leq i\leq 2}\bigl\{a_{i1}(\theta_1 + \chi_1) + a_{i2}(\theta_2 + \chi_2)\bigr\} < 0.5, \tag{3.4}$$

$$4\Delta t\bigl\{a_{21}(a_{11} + a_{12}) + a_{12}(a_{21} + a_{22})\bigr\} \max_{1\leq i,j\leq 2} a_{ij} < \min\{a_{21}a_{11}, a_{12}a_{22}\}, \tag{3.5}$$

are satisfied, then (x_k, y_k) *rotates infinitely many times around* $\boldsymbol{\theta}$ *in the counterclockwise direction with* $\lim_{k\to\infty}(x_k, y_k) = \boldsymbol{\theta}$.

Proof (a) Since all of the eigenvalues of $DT_{\boldsymbol{\theta}}$ in equation (3.3) are positive numbers less than 1, the fixed point $\boldsymbol{\theta}$ of $(x_{k+1}, y_{k+1}) = T(x_{k+1}, y_{k+1})$ is hyperbolic and the solutions of the linearized system of equation (2.1) at $\boldsymbol{\theta}$ rotate finitely many times around $(0, 0)$, converging to $(0, 0)$. Hence Theorem 3.4 with Theorem 2.4 gives the proof of (a) without showing $\lim_{k\to\infty}(x_k, y_k) = \boldsymbol{\theta}$.

For obtaining the limit of (x_k, y_k) consider the case $(x_k, y_k) \in S_i$ for $i = 1$ and all sufficiently large k. Then equation (2.7) yields the result that x_k is increasing with the upper bound θ_1 less than $\frac{r_1}{a_{11}}$, and hence $0 < \lim_{k\to\infty} x_k < \frac{r_1}{a_{11}}$. Consequently $\lim_{k\to\infty} y_k > 0$, since otherwise $\lim_{k\to\infty} x_k = \frac{r_1}{a_{11}}$, which is a contradiction. Therefore we obtain equation (3.1).

In the case $(x_k, y_k) \in S_i$ for $i = 2$ and all sufficiently large k, it follows from equation (2.7) that x_k and y_k are both increasing and bounded, which gives equation (3.1).

Similarly, the other two cases for $i = 3$ and 4 can be proved by using equation (3.1), equation (2.7), and the method of proof by contradiction.

(b) Since all of the eigenvalues of $DT_{\boldsymbol{\theta}}$ in equation (3.3) have positive real parts and magnitude less than 1 by $\det(J)\Delta t < -\mathrm{tr}(J)$, the fixed point $\boldsymbol{\theta}$ is hyperbolic and the solutions of the linearized system of equation (2.1) at $\boldsymbol{\theta}$ rotate infinitely many times around $(0, 0)$, converging to $(0, 0)$. Hence (b) is proved by using Theorem 3.4 and Theorem 2.4. It remains to show that $\lim_{k\to\infty}(x_k, y_k) = \boldsymbol{\theta}$. Consider a function V_k defined by

$$V_k = a_{21}(x_k - \theta_1 \ln x_k) + a_{12}(y_k - \theta_2 \ln y_k) \tag{3.6}$$

for all the solutions (x_k, y_k). Letting $\theta_{1k} = \theta_1 - x_k$, $\theta_{2k} = \theta_2 - y_k$, and $\Delta x_k = x_{k+1} - x_k$, we have $\Delta x_k = (\hat{a}_{11}\theta_{1k} + \hat{a}_{12}\theta_{2k})x_k$ and $\Delta y_k = (-\hat{a}_{21}\theta_{1k} + \hat{a}_{22}\theta_{2k})y_k$ by equation (2.1). Then the Mean

Value Theorem gives for some α, β with $0 < \alpha, \beta < 1$

$$\begin{aligned} V_{k+1} - V_k &= a_{21}\Delta x_k\left(1 - \theta_1\frac{\Delta \ln x_k}{\Delta x_k}\right) + a_{12}\Delta y_k\left(1 - \theta_2\frac{\Delta \ln y_k}{\Delta y_k}\right) \\ &= a_{21}(\hat{a}_{11}\theta_{1k} + \hat{a}_{12}\theta_{2k})\left(x_k - \theta_1\frac{x_k}{\alpha\Delta x_k + x_k}\right) \\ &\quad + a_{12}(-\hat{a}_{21}\theta_{1k} + \hat{a}_{22}\theta_{2k})\left(y_k - \theta_2\frac{y_k}{\beta\Delta y_k + y_k}\right). \end{aligned} \tag{3.7}$$

Note that

$$\begin{aligned} \frac{x_k}{\alpha\Delta x_k + x_k} &= 1 - \frac{\alpha(a_{11}\theta_{1k} + a_{12}\theta_{2k})\Delta t}{\alpha(a_{11}\theta_{1k} + a_{12}\theta_{2k})\Delta t + 1} \equiv 1 - (C_1\theta_{1k} + C_2\theta_{2k})\Delta t, \\ \frac{y_k}{\beta\Delta y_k + y_k} &= 1 - \frac{\beta(-a_{21}\theta_{1k} + a_{22}\theta_{2k})\Delta t}{\beta(-a_{21}\theta_{1k} + a_{22}\theta_{2k})\Delta t + 1} \equiv 1 - (C_3\theta_{1k} + C_4\theta_{2k})\Delta t, \end{aligned}$$

where equation (3.4) gives

$$\max_{1\le i\le 4}|C_i| < 2\max_{1\le i,j\le 4} a_{ij}. \tag{3.8}$$

Then equation (3.7) becomes

$$V_{k+1} - V_k \le -(a_{21}\hat{a}_{11} - C_5\Delta t)\theta_{1k}^2 - (a_{12}\hat{a}_{22} - C_6\Delta t)\theta_{2k}^2, \tag{3.9}$$

where $C_5 = a_{21}\{\hat{a}_{11}(|C_1| + |C_2|) + \hat{a}_{12}|C_1|\} + a_{12}\{\hat{a}_{21}(|C_3| + |C_4|) + \hat{a}_{22}|C_3|\}$, $C_6 = a_{21}\{\hat{a}_{11}|C_2| + \hat{a}_{12}(|C_1| + |C_2|)\} + a_{12}\{\hat{a}_{21}|C_4| + \hat{a}_{22}(|C_3| + |C_4|)\}$, and

$$\max\{C_5, C_6\} < 4\Delta t\{a_{21}(a_{11} + a_{12}) + a_{12}(a_{21} + a_{22})\}\max_{1\le i,j\le 2} a_{ij}$$

by equation (3.8). Hence equation (3.9) together with equation (3.5) becomes

$$V_{k+1} - V_k \le -C_7\left(\theta_{1k}^2 + \theta_{2k}^2\right)\Delta t$$

for a positive constant C_7.

Now assume, to the contrary, that (x_k, y_k) does not converge to $\boldsymbol{\theta}$. Since $\boldsymbol{\theta}$ is locally asymptotically stable by the linearization method (see [4]), the assumption implies that $\theta_{1k}^2 + \theta_{2k}^2$ has a positive lower bound. Then there exists a positive constant $\mathcal{C}$ such that for all k with $k \ge 0$

$$V_{k+1} - V_k \le -\mathcal{C} \quad \text{and hence} \quad V_k \le V_0 - k\mathcal{C}.$$

This is a contradiction, since $\lim_{k\to\infty}(V_0 - k\mathcal{C}) = -\infty$ and V_k is bounded by Theorem 2.1. □

Example 3.6 Consider the discrete system

$$x_{k+1} = x_k + x_k(2.75 - 2x_k - y_k)\Delta t, \qquad y_{k+1} = y_k + y_k(-1 + x_k - y_k)\Delta t$$

and the four initial conditions

$$(0.9, 0.15) \in S_1, \qquad (1.25, 0.12) \in S_2, \qquad (1.4, 0.2) \in S_3, \qquad (1.3, 0.35) \in S_4$$

with $\Delta t = 0.001$ and $(\chi_1, \chi_2) = (100, 100)$. Then $r_1 a_{21} - r_2 a_{11} > 0$, $\max\{\hat{r}_1, \hat{r}_2\} < 1$, and $\boldsymbol{\theta} = (1.25, 0.25)$. Since $\operatorname{tr}(J) = -2.75$ and $\det(J) = 0.938$, the condition $\operatorname{tr}^2(J) \geq 4\det(J)$ in Theorem 3.5(a) is satisfied. Figure 1(b) shows the dynamics in Theorem 3.5(a) with the limit $\boldsymbol{\theta}$.

Example 3.7 Consider the discrete system

$$x_{k+1} = x_k + x_k(30 - x_k - y_k)\Delta t, \qquad y_{k+1} = y_k + y_k(-1 + x_k - 0.01 y_k)\Delta t,$$

and the four initial conditions (x_0, y_0)

$$(0.5, 15) \in S_1, \qquad (5, 15) \in S_2, \qquad (7, 30) \in S_3, \qquad (0.5, 0.45) \in S_4$$

with $\Delta t = 0.0001$ and $(\chi_1, \chi_2) = (100, 100)$. Then $r_1 a_{21} - r_2 a_{11} > 0$, $\max\{\hat{r}_1, \hat{r}_2\} < 1$, and $\boldsymbol{\theta} = (1.287, 28.713)$. Since $\operatorname{tr}(J) = -1.574$ and $\det(J) = 37.327$, the conditions in Theorem 3.5(b) are satisfied. Figure 1(c) shows the dynamics in Theorem 3.5(b) with the limit $\boldsymbol{\theta}$.

In the remainder of this section we consider the Euler difference scheme for equation (1.1) with $n = 2$ and $(\sigma_1, \sigma_2) = (1, 1)$

$$x_{k+1} = x_k(1 + \hat{r}_1 - \hat{a}_{11} x_k - \hat{a}_{12} y_k), \qquad y_{k+1} = y_k(1 + \hat{r}_2 + \hat{a}_{21} x_k - \hat{a}_{22} y_k), \tag{3.10}$$

which has the three nonzero equilibrium points

$$\left(\frac{r_1}{a_{11}}, 0\right), \qquad \left(0, \frac{r_2}{a_{22}}\right), \qquad \boldsymbol{\vartheta} = \left(\frac{r_1 a_{22} - r_2 a_{12}}{a_{12} a_{21} + a_{11} a_{22}}, \frac{r_1 a_{21} + r_2 a_{11}}{a_{12} a_{21} + a_{11} a_{22}}\right).$$

Replace r_2 in Section 2 with $-r_2$. For example, $g(x) = \frac{-r_2 + a_{21} x}{a_{22}}$ is replaced with $g(x) = \frac{r_2 + a_{21} x}{a_{22}}$. Then Theorem 2.1, equation (2.7), and Theorem 2.4 remain true. In the case $r_1 a_{22} - r_2 a_{12} \leq 0$, the set S_1 is empty, and hence we can prove the following theorem, which corresponds to Theorem 3.1.

Theorem 3.8 *Let the assumptions of Theorem* 2.1 *hold with r_2 replaced by $-r_2$ and let $g(x) = \frac{-r_2 + a_{21} x}{a_{22}}$. Suppose $r_1 a_{22} - r_2 a_{12} \leq 0$. Then the solution (x_k, y_k) of equation* (3.10) *satisfies the following dynamics with the limit $(0, \frac{r_2}{a_{22}})$.*

(a) *If $(x_0, y_0) \in S_4$, then $(x_k, y_k) \in S_4$ for all k.*
(b) *If $(x_0, y_0) \in S_3$, then $(x_k, y_k) \in S_3$ for all k or $[\![x_{N_1}, y_{N_1}]\!] \in S_4$ for some N_1.*
(c) *If $(x_0, y_0) \in S_2$, then $[\![x_{N_2}, y_{N_2}]\!] \in S_3$ for some N_2.*

Remark 3.9 The Jacobian matrix of equation (3.10) at $\boldsymbol{\vartheta}$ equals $DT_{\boldsymbol{\vartheta}}$ as defined in equation (3.2), and then Theorem 3.5 remains true if $r_1 a_{21} - r_2 a_{11} > 0$, r_2, and $\boldsymbol{\theta}$ in Theorem 3.5 are replaced with $r_1 a_{22} - r_2 a_{12} > 0$, $-r_2$, and $\boldsymbol{\vartheta}$, respectively. Therefore only the two equilibrium points $(0, \frac{r_2}{a_{22}})$ and $\boldsymbol{\vartheta}$ of equation (3.10) are globally asymptotically stable.

Example 3.10 Let $E = (r_1, a_{11}, a_{12}, r_2, a_{21}, a_{22})$. Consider the Euler difference scheme for equation (3.10) with $\Delta t = 0.001$.

(a) $E = (1,1,1,2,1,1)$ with the three initial conditions $(0.2, 0.5) \in S_2$, $(2,1) \in S_3$, and $(0.2, 3) \in S_4$. Then the conditions in Theorem 3.8 are satisfied. Figure 2(a) shows that the three trajectories converge to $(0, \frac{r_2}{a_{22}}) = (0, 2)$ as in Theorem 3.8. Replacing the values $r_1 = 1$ in E with $r_1 = 0.0000001$ and letting $(x_0, y_0) = (0.0001, 1.5)$, we have $y_i \le y_{i+1} \le y_k = 2 - 0.75 * 10^{-36} < 2$ for $0 \le i \le 125{,}000$ and $50{,}000 \le k \le 125{,}000$, which imply $(x_k, y_k) \in S_3$ for all k.

(b) $E = (1.5, 1, 0.1, 1, 0.1, 0.1)$ with the four initial conditions $(0.41, 10.8) \in S_1$, $(0.41, 10.1) \in S_2$, $(0.58, 10.2) \in S_3$, and $(0.58, 10.9) \in S_4$. Then $\boldsymbol{\vartheta} = (0.455, 10.455)$ and $r_1 a_{22} - r_2 a_{12} = 0.05 > 0$. Since $\operatorname{tr}(J) = -1.5$ and $\det(J) = 0.523$, the condition $\operatorname{tr}^2(J) \ge 4\det(J)$ in Theorem 3.5(a) is satisfied. Figure 2(b) shows that solutions rotate finitely many times around the limit $\boldsymbol{\vartheta}$.

(c) $E = (30, 1, 1, 0.1, 2, 0.05)$ with the four initial conditions $(0.1, 15) \in S_1$, $(5, 15) \in S_2$, $(5, 30) \in S_3$, and $(0.5, 43) \in S_4$. Then $\boldsymbol{\vartheta} = (0.683, 29.317)$ and $r_1 a_{22} - r_2 a_{12} = 1.4 > 0$. Since $\operatorname{tr}(J) = -2.149$ and $\det(J) = 41.044$, the conditions in Theorem 3.5(b) are satisfied. Figure 2(c) shows that the spiral trajectories rotate infinitely many times around the limit $\boldsymbol{\vartheta}$.

4 *n*-Dimensional predator-prey models

In this section, we consider the Euler difference scheme for equation (1.1)

$$x_{k+1}^i = F_{\mathbf{x}_k^i}\left(x_k^i\right) \quad \text{for } 1 \le k, 1 \le i \le n, \tag{4.1}$$

where $\mathbf{x}_k^i$ denotes $x_k^1, \dots, x_k^{i-1}, x_k^{i+1}, \dots, x_k^n$ and

$$F_{\mathbf{x}_k^i}\left(x_k^i\right) = x_k^i\left(1 + \sigma_i \hat{r}_i + \sum_{1 \le j \le i-1} \hat{a}_{ij} x_k^j - \sum_{i \le j \le n} \hat{a}_{ij} x_k^j\right) \tag{4.2}$$

with $\sigma_1 = 1$ and $\sigma_i \in \{-1, 1\}$ for $2 \le i \le n$.

Let $\boldsymbol{\zeta}^i$ denote $\zeta_1, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_n$. Note that if $\zeta_1, \dots, \zeta_n$ are positive constants such that for $1 \le i \le n$

$$U_i\left(\boldsymbol{\zeta}^i\right) = \frac{1 + \sigma_i \hat{r}_i + \sum_{1 \le j \le i-1} \hat{a}_{ij} \zeta_j - \sum_{i+1 \le j \le n} \hat{a}_{ij} \zeta_j}{2\hat{a}_{ii}} > 0,$$

then

$$F_{\boldsymbol{\zeta}^i}(\zeta_i) \text{ is increasing on } 0 \le \zeta_i < U_i\left(\boldsymbol{\zeta}^i\right). \tag{4.3}$$

Assume that there exist positive constants χ_i such that for $1 \le i \le n$

$$\chi_i < \frac{1 + \sigma_i \hat{r}_i - \sum_{i+1 \le j \le n} \hat{a}_{ij} \chi_j}{2\hat{a}_{ii}}, \tag{4.4}$$

$$\frac{\sigma_i r_i + \sum_{1 \le j \le i-1} a_{ij} \chi_j}{a_{ii}} < \chi_i, \tag{4.5}$$

$$\sigma_i \hat{r}_i + \sum_{1 \le j \le i-1} \hat{a}_{ij} \chi_j < 1, \tag{4.6}$$

which is the generalization of equation (2.4).

Theorem 4.1 *Let* Δt *and* χ_i *satisfy equations* (4.4)-(4.6). *If* $(x_0^1, \dots, x_0^n) \in \prod_{1 \le i \le n}(0, \chi_i)$, *then* $(x_k^1, \dots, x_k^n) \in \prod_{1 \le i \le n}(0, \chi_i)$ *for all* k.

Proof It follows from equation (4.4) and the definition of U_i that for $1 \le i \le n$

$$0 < x_0^i < \chi_i < \frac{1 + \sigma_i \hat{r}_i - \sum_{i+1 \le j \le n} \hat{a}_{ij} \chi_j}{2\hat{a}_{ii}} < U_i\bigl(\mathbf{x}_0^i\bigr).$$

Then equation (4.3) with $\boldsymbol{\zeta}^i = \mathbf{x}_0^i$ gives

$$x_1^i = F_{\mathbf{x}_0^i}\bigl(x_0^i\bigr) > F_{\mathbf{x}_0^i}(0) = 0.$$

If $\sigma_i r_i + \sum_{1 \le j \le i-1} a_{ij} x_0^j - \sum_{i \le j \le n} a_{ij} x_0^j \le 0$ for $1 \le i \le n$, then

$$x_1^i \le x_0^i < \chi_i,$$

and otherwise, we have $0 < x_0^i < f_i(\mathbf{x}_0^i)$ with $f_i(\mathbf{x}_0^i) = \frac{\sigma_i r_i + \sum_{1 \le j \le i-1} a_{ij} x_0^j - \sum_{i+1 \le j \le n} a_{ij} x_0^j}{a_{ii}}$.

Since $0 < f_i(\mathbf{x}_0^i) < U_i(\mathbf{x}_0^i)$ by equation (4.6), it follows from equation (4.3) with $\boldsymbol{\zeta}^i = \mathbf{x}_0^i$ that

$$x_1^i = F_{\mathbf{x}_0^i}\bigl(x_0^i\bigr) < F_{\mathbf{x}_0^i}\bigl(f_i\bigl(\mathbf{x}_0^i\bigr)\bigr) = f_i\bigl(\mathbf{x}_0^i\bigr) < \frac{\sigma_i r_i + \sum_{1 \le j \le i-1} a_{ij} \chi_j}{a_{ii}} < \chi_i,$$

where the last inequality is obtained by equation (4.5). Therefore we find that if $(x_0^1, \dots, x_0^n) \in \prod_{1 \le i \le n}(0, \chi_i)$, then

$$\bigl(x_1^1, \dots, x_1^n\bigr) \in \prod_{1 \le i \le n} (0, \chi_i),$$

and hence mathematical induction completes the proof. □

From now on, we consider the global asymptotic stability of the equilibrium point of equation (4.1) whose components are all positive. Let $\sigma_{ij} = 1$ if $1 \le i < j \le n$, and otherwise, $\sigma_{ij} = -1$. Assume that

$$\begin{aligned} &\text{the inverse matrix of the } n \times n \text{ matrix } (\sigma_{ij} a_{ij}) \text{ exists and} \\ &(\sigma_{ij} a_{ij})^{-1} \begin{pmatrix} \sigma_1 r_1 & \sigma_2 r_2 & \cdots & \sigma_n r_n \end{pmatrix}^T \text{ is a positive matrix,} \end{aligned} \tag{4.7}$$

where $(\sigma_1 r_1\ \sigma_2 r_2\ \cdots\ \sigma_n r_n)^T$ is the transpose of $1 \times n$ matrix $(\sigma_1 r_1\ \sigma_2 r_2\ \cdots\ \sigma_n r_n)$.

Let

$$\boldsymbol{\theta}^T = (\sigma_{ij} a_{ij})^{-1} \begin{pmatrix} \sigma_1 r_1 & \sigma_2 r_2 & \cdots & \sigma_n r_n \end{pmatrix}^T,$$

which gives for $1 \leq i \leq n$

$$\sigma_i r_i + \sum_{1\leq j\leq i-1} a_{ij}\theta_j - \sum_{i\leq j\leq n} a_{ij}\theta_j = 0. \tag{4.8}$$

Then equation (4.7) is the generalization of the condition $r_1 a_{21} - r_2 a_{11} > 0$ in both Lemma 3.3 and Theorem 3.5.

In order to use the linearized system of equation (4.1) at $\boldsymbol{\theta}$, consider the Jacobian matrix of $T(x_k^1,\dots,x_k^n) = (F_{\mathbf{x}_k^1}(x_k^1),\dots,F_{\mathbf{x}_k^n}(x_k^n))$ at $\boldsymbol{\theta}$

$$DT_{\boldsymbol{\theta}} = \begin{pmatrix} 1-\theta_1 a_{11}\Delta t & -\theta_1 a_{12}\Delta t & \cdots & -\theta_1 a_{1n}\Delta t \\ \theta_2 a_{21}\Delta t & 1-\theta_2 a_{22}\Delta t & \cdots & -\theta_2 a_{2n}\Delta t \\ \vdots & \vdots & \cdots & \vdots \\ \theta_n a_{n1}\Delta t & \theta_n a_{n2}\Delta t & \cdots & 1-\theta_n a_{nn}\Delta t \end{pmatrix}.$$

Since the global stability of $\boldsymbol{\theta}$ implies that $\boldsymbol{\theta}$ is locally stable, we need to assume the condition for the local stability of $\boldsymbol{\theta}$:

All of the eigenvalues of $DT_{\boldsymbol{\theta}}$ have magnitude less than 1. (4.9)

Let $J = \frac{1}{\Delta t}(DT_{\boldsymbol{\theta}} - I)$ for the $n \times n$ identity matrix I and let $P(\lambda)$ be the characteristic polynomial of the matrix J. Note that $1 + \lambda_J \Delta t$ are the eigenvalues of $DT_{\boldsymbol{\theta}}$ for the roots λ_J of $P(\lambda)$. Thus equation (4.9) is equivalent to the condition

all of the roots of $P(\lambda)$ are negative or have negative real parts, (4.10)

and $\left|\mathrm{Re}(\lambda_J)\right| > \dfrac{\Delta t}{2}\|\lambda_J\|^2$ if $\mathrm{Re}(\lambda_J) < 0$ and $\mathrm{Im}(\lambda_J) \neq 0$, (4.11)

where $\|\lambda_J\|^2 = \mathrm{Re}^2(\lambda_J) + \mathrm{Im}^2(\lambda_J)$, $\mathrm{Re}(\lambda_J)$ and $\mathrm{Im}(\lambda_J)$ are the real and imaginary parts of λ_J, respectively. Hence equation (4.10) together with equation (4.11) is the generalization of Lemma 3.3.

Remark 4.2 The condition equation (4.9) with $1 + \lambda_J \Delta t \neq 0$ implies that θ of equation (4.1) is asymptotically stable by using Theorem 3.4 with the following two facts: first, θ is a hyperbolic point since all of the eigenvalues of $DT_{\boldsymbol{\theta}}$ are of the form $1 + \lambda_J \Delta t$, which is nonzero. Second, equation (4.9) is equivalent to the fact that the linearized system of equation (4.1) is asymptotically stable (see Theorem 3.3.20 in [5]).

Remark 4.3 Using Routh-Hurwitz criteria, we can find conditions for both r_i and a_{ij} to satisfy equation (4.10). Let $P(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ with real constants a_i for $1 \leq i \leq n$. Define the n Hurwitz matrices H_i for $1 \leq i \leq n$

$$H_1 = (a_1), \qquad H_2 = \begin{pmatrix} a_1 & 1 \\ a_3 & a_2 \end{pmatrix}, \qquad \dots, \qquad H_n = \begin{pmatrix} a_1 & 1 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{pmatrix}.$$

Then equation (4.10) is equivalent to $\det(H_i) > 0$ for $1 \leq i \leq n$. Thus Routh-Hurwitz criteria for $n = 2$ and 3 are

$$\begin{aligned} n = 2:&\quad a_1 > 0 \text{ and } a_2 > 0, \\ n = 3:&\quad a_1 > 0, a_3 > 0, \text{and } a_1 a_2 > a_3. \end{aligned} \tag{4.12}$$

If we assume equation (4.9) with $1 + \lambda_J \Delta t \neq 0$, then $\boldsymbol{\theta}$ is asymptotically stable (see Remark 4.2). Hence the following lemma can be proved.

Lemma 4.4 *Let r_i, a_{ij}, Δt satisfy equations* (4.4)-(4.7), (4.10), *and* (4.11) *with* $1 + \lambda_J \Delta t \neq 0$. *If* $\lim_{k\to\infty}(x_k^1, \dots, x_k^n) \neq \boldsymbol{\theta}$, *then there exists a positive constant C such that*

$$\max_{1\leq i\leq n} \left(\theta_i - x_k^i\right)^2 > C > 0 \quad \textit{for all } k.$$

We need to find constants α_i that play the same role as a_{12} and a_{21} in equations (3.6), (3.7), and (3.9): If $x = a_{21}$ and $y = a_{12}$, then

$$\begin{aligned} &-x\left(a_{11}\theta_{1n}^2 + a_{12}\theta_{1n}\theta_{2n}\right) - y\left(-a_{21}\theta_{1n}\theta_{2n} + a_{22}\theta_{2n}^2\right) \\ &\quad = -\begin{pmatrix}\theta_{1n} & \theta_{2n}\end{pmatrix} \begin{pmatrix} xa_{11} & \frac{1}{2}(xa_{12} - ya_{21}) \\ \frac{1}{2}(xa_{12} - ya_{21}) & ya_{22} \end{pmatrix} \begin{pmatrix}\theta_{1n} & \theta_{2n}\end{pmatrix}^T \leq 0. \end{aligned}$$

Hence we choose positive constants α_i such that the matrices

$$A_{ij} = \begin{pmatrix} \alpha_i a_{ii} & \frac{n-1}{2}(\alpha_i a_{ij} - \alpha_j a_{ji}) \\ \frac{n-1}{2}(\alpha_i a_{ij} - \alpha_j a_{ji}) & \alpha_j a_{jj} \end{pmatrix} \quad \text{for } 1 \leq i < j \leq n \tag{4.13}$$

are positive definite. Note that all of the eigenvalues of A_{ij} are positive if and only if

$$\alpha_i a_{ii} \alpha_j a_{jj} > \left(\frac{n-1}{2}\right)^2 (\alpha_i a_{ij} - \alpha_j a_{ji})^2,$$

which is equivalent to

$$(n-1)^2 a_{ji}^2 \left(\frac{\alpha_j}{\alpha_i}\right)^2 - 2\left\{(n-1)^2 a_{ij} a_{ji} + 2a_{ii}a_{jj}\right\}\frac{\alpha_j}{\alpha_i} + (n-1)^2 a_{ij}^2 < 0.$$

Hence consider the quadratic equation in the variable t

$$(n-1)^2 a_{ji}^2 t^2 - 2\left\{(n-1)^2 a_{ij} a_{ji} + 2a_{ii}a_{jj}\right\} t + (n-1)^2 a_{ij}^2 = 0, \tag{4.14}$$

which has two different solutions denoted by m_{ij} and M_{ij} with $m_{ij} < M_{ij}$ for $1 \leq i < j \leq n$. Therefore in order that the eigenvalues of A_{ij} are positive, assume that there exist positive constants α_i such that for $1 \leq i < j \leq n$

$$m_{ij} < \frac{\alpha_j}{\alpha_i} < M_{ij}. \tag{4.15}$$

Lemma 4.5 *Suppose that there exist positive constants* α_i $(1 \leq i \leq n)$ *satisfying equation* (4.15). *Let* λ_{ij} *be the minimum of the two positive eigenvalues of* A_{ij} *for* $1 \leq i < j \leq n$ *in* *equation* (4.13). *Then for all* $(x, y) \in \mathbb{R}^2$

$$\begin{pmatrix} x & y \end{pmatrix} A_{ij} \begin{pmatrix} x & y \end{pmatrix}^T \geq \lambda_{ij}\left(x^2 + y^2\right). \tag{4.16}$$

Assume that

$$\Delta t \max_{1\leq i\leq n} \sum_{1\leq j\leq n} a_{ij}(\theta_j + \chi_j) < \frac{1}{2}, \tag{4.17}$$

$$\begin{aligned} &2\Delta t \sum_{1\leq s\leq n} \alpha_s\theta_s \left\{ \left(\max_{1\leq i\leq n} a_{si}\right) n + \left(\sum_{1\leq j\leq n} a_{sj}\right) \right\} \max_{1\leq i,j\leq n} a_{ij} \\ &\quad < \min_{1\leq i\leq n} \sum_{i+1\leq j\leq n} \frac{\lambda_{ij}}{n-1}, \end{aligned} \tag{4.18}$$

which are the generalization of equations (3.4) and (3.5), respectively.

Theorem 4.6 *Let the assumptions in Lemma* 4.4 *hold and let equations* (4.15), (4.17), *and* (4.18) *be satisfied. If* $(x_0^1, \ldots, x_0^n) \in \prod_{1\leq i\leq n}(0, \chi_i)$, *then* $\lim_{k\to\infty}(x_k^1, \ldots, x_k^n) = \boldsymbol{\theta}$.

Proof Consider a function

$$V_k = \sum_{1\leq i\leq n} \alpha_i\left(x_k^i - \theta_i \ln x_k^i\right)$$

for all the solutions $x_k^1, \ldots, x_k^n$ of equation (4.1). The Mean Value Theorem with $\theta_{ik} = \theta_i - x_k^i$ and equation (4.8) shows that there exist constants $c_i \in (0, 1)$ such that

$$\begin{aligned} V_{k+1} - V_k &= \sum_{1\leq i\leq n} \alpha_i\left(x_{k+1}^i - x_k^i\right)\left(1 - \theta_i \frac{\ln x_{k+1}^i - \ln x_k^i}{x_{k+1}^i - x_k^i}\right) \\ &= \sum_{1\leq i\leq n} \alpha_i \Theta_{ik}\left(x_k^i - \theta_i \frac{x_k^i}{c_i(x_{k+1}^i - x_k^i) + x_k^i}\right) \end{aligned} \tag{4.19}$$

with

$$\Theta_{ik} = -\sum_{1\leq j\leq i-1} \hat{a}_{ij}\theta_{jk} + \hat{a}_{ii}\theta_{ik} + \sum_{i+1\leq j\leq n} \hat{a}_{ij}\theta_{jk}. \tag{4.20}$$

Note that

$$\frac{x_k^i}{c_i(x_{k+1}^i - x_k^i) + x_k^i} = \frac{1}{c_i\Theta_{ik}\Delta t + 1} \equiv 1 - \sum_{1\leq \ell\leq n} c_{i\ell}\theta_{\ell k}\Delta t, \tag{4.21}$$

where $c_{i\ell} = -\frac{c_i a_{i\ell}}{c_i\Theta_{ik}\Delta t + 1}$ for $1 \leq \ell \leq i-1$, $c_{i\ell} = \frac{c_i a_{i\ell}}{c_i\Theta_{ik}\Delta t + 1}$ for $i \leq \ell \leq n$, and

$$|c_{i\ell}| \leq 2\max\{a_{ij} | 1 \leq i, j \leq n\} \tag{4.22}$$

by equation (4.17). Then equation (4.19) with (4.21) becomes

$$V_{k+1} - V_k = -\sum_{1\leq i\leq n} \alpha_i\Theta_{ik}\theta_{ik} + \Delta t\left(\sum_{1\leq i\leq n} \alpha_i\Theta_{ik} \sum_{1\leq \ell\leq n} \theta_i c_{i\ell}\theta_{\ell k}\right). \tag{4.23}$$

It follows from equations (4.20) and (4.13) that

$$\begin{aligned}\sum_{1\le i\le n}\alpha_i\Theta_{ik}\theta_{ik} &= -\sum_{1\le i\le n}\sum_{1\le j<i}\alpha_i\hat{a}_{ij}\theta_{ik}\theta_{jk}+\sum_{1\le i\le n}\alpha_i\hat{a}_{ii}\theta_{ik}^2+\sum_{1\le i\le n}\sum_{i<j\le n}\alpha_i\hat{a}_{ij}\theta_{ik}\theta_{jk}\\ &= \sum_{1\le i<j\le n}(\alpha_i\hat{a}_{ij}-\alpha_j\hat{a}_{ji})\theta_{ik}\theta_{jk}+\sum_{1\le i<j\le n}\frac{1}{n-1}\big(\alpha_i\hat{a}_{ii}\theta_{ik}^2+\alpha_j\hat{a}_{jj}\theta_{jk}^2\big)\\ &= \frac{\Delta t}{n-1}\sum_{1\le i<j\le n}\big(\theta_{ik}\quad\theta_{jk}\big)A_{ij}\big(\theta_{ik}\quad\theta_{jk}\big)^T \end{aligned}\tag{4.24}$$

and

$$\sum_{1\le i\le n}\alpha_i\Theta_{ik}\sum_{1\le\ell\le n}\theta_i c_{i\ell}\theta_{\ell k}\le\Delta t\sum_{1\le i\le n}C_i\theta_{ik}^2,\tag{4.25}$$

where

$$C_i=\sum_{1\le s\le n}\alpha_s\theta_s\left\{a_{si}\left(\sum_{1\le\ell\le n}|c_{s\ell}|\right)+\left(\sum_{1\le j\le n}a_{sj}\right)|c_{si}|\right\}.\tag{4.26}$$

Hence substituting equations (4.24), (4.25), and Lemma 4.5 into equation (4.23) yields

$$V_{k+1}-V_k\le-\Delta t\sum_{1\le i\le n}\left\{\left(\frac{1}{n-1}\sum_{i<j\le n}\lambda_{ij}\right)-C_i\Delta t\right\}\theta_{ik}^2.\tag{4.27}$$

Using equations (4.26), (4.22), and (4.18), we obtain $\frac{1}{n-1}\sum_{i+1\le j\le n}\lambda_{ij}-C_i\Delta t>0$, and then equation (4.27) becomes

$$V_{k+1}-V_k\le-\mathcal{C}_1\Delta t\sum_{1\le i\le n}\theta_{ik}^2\tag{4.28}$$

for some positive constant $\mathcal{C}_1$. Now assume, to the contrary, that $(x_k^1,\dots,x_k^n)$ does not converge to $\boldsymbol{\theta}$ as k goes to infinity. Then combining equation (4.28) with Lemma 4.4, we see that there exists a positive constant $\mathcal{C}_2$ such that

$$V_{k+1}-V_k\le-\mathcal{C}_2\quad\text{for all } k\ge 0,$$

and hence

$$V_k\le V_0-k\mathcal{C}_2\quad\text{for all } k\ge 0.$$

This is a contradiction, since $\lim_{k\to\infty}(V_0-k\mathcal{C}_2)=-\infty$ and V_k is lower bounded for all k by using the boundedness of the solutions x_k^i for $1\le i\le n$. □

Example 4.7 Consider the three-dimensional scheme

$$\begin{aligned} x_{k+1}^1&=x_k^1+x_k^1\big(2-x_k^1-2x_k^2-2x_k^3\big)\Delta t,\\ x_{k+1}^2&=x_k^2+x_k^2\big(-2+3x_k^1-2x_k^2-2x_k^3\big)\Delta t,\\ x_{k+1}^3&=x_k^3+x_k^3\big(-2+2x_k^1+x_k^2-x_k^3\big)\Delta t, \end{aligned}\tag{4.29}$$

which is equation (4.1) with $n = 3$, $\sigma_1 = 1$, $\sigma_2 = \sigma_3 = -1$, and $\Delta t = 0.0001$. The initial condition (x_0^1, x_0^2, x_0^3) is $(1.5, 1.5, 1.0)$.

(a) Let $\chi_1 = \frac{r_1}{a_{11}} + 1 = 3$, $\chi_2 = \frac{a_{21}\chi_1}{a_{22}} = 4.5$, and $\chi_3 = \frac{a_{31}\chi_1 + a_{32}\chi_2}{a_{33}} = 10.5$. Then equations (4.4), (4.5), and (4.6) are satisfied.

(b) The condition (4.7) is satisfied and $\boldsymbol{\theta} = (1, 0.25, 0.25)$.

(c) The conditions (4.10), (4.11), and (4.12) are satisfied, since $P(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = \lambda^3 + 1.75\lambda^2 + 3.5\lambda + 1$ with the three roots $-0.710 + 1.590i$, -0.330, and $-0.710 - 1.509i$.

(d) The values $\alpha_1 = \alpha_2 = \alpha_3 = 1$ satisfy equation (4.15), since the three equations $a_{ji}^2 t^2 - (2a_{ij}a_{ji} + a_{ii}a_{jj})t + a_{ij}^2 = 0$ $(1 \le i < j \le 3)$ have solutions $\{0.378, 1.178\}$, $\{0.610, 1.640\}$, and $\{0.764, 5.236\}$, respectively.

(e) The inequalities (4.17) and (4.18) are satisfied, since $\lambda_{12} = 0.382$, $\lambda_{13} = 1$, and $\lambda_{23} = 0.382$.

Hence the conditions in Theorem 4.6 are satisfied. Therefore the solutions of equation (4.29) are positive and the equilibrium point $\boldsymbol{\theta}$ is globally asymptotically stable, which are demonstrated in Figure 3(a).

Example 4.8 Let $E = (r_1, a_{11}, a_{12}, a_{13}, r_2, a_{21}, a_{22}, a_{23}, r_3, a_{31}, a_{32}, a_{33})$. Consider the following three difference schemes for equation (1.1) with $n = 3$, $\Delta t = 0.0001$, and $\sigma_1 = 1$.

(a) $\sigma_2 = -1$, $\sigma_3 = 1$, and $E = (3.5, 1, 0.5, 0.5, 0.5, 3, 0.5, 0.5, 1, 1, 0.5, 1)$ with the initial condition $(4, 3, 2)$ (see Figure 3(b)).

(b) $\sigma_2 = 1$, $\sigma_3 = -1$, and $E = (3.5, 1, 0.5, 0.5, 4, 1, 1, 1, 2, 1, 2, 1)$ with the initial condition $(3, 1, 1)$ (see Figure 3(c)).

(c) $\sigma_2 = 1$, $\sigma_3 = 1$, and $E = (3.5, 1, 0.5, 0.5, 4, 11, 1, 1, 1, 0.5, 1)$ with the initial condition $(1.5, 1, 1)$ (see Figure 3(d)).

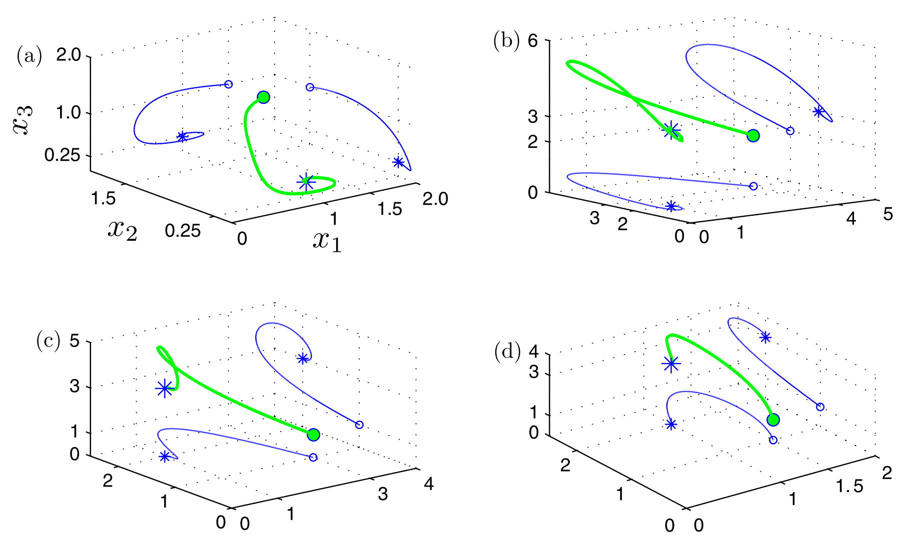

Figure 3 Three-dimensional models with $\sigma_1 = 1$ and $\sigma_2, \sigma_3 \in \{-1, 1\}$. The four thicker curves denote the trajectories of the solutions of equation (4.1) with $n = 3$ and the other curves are the projections of the trajectories in two planes among x_1x_2, x_1x_3, and x_2x_3 planes. The four larger stars denote the limits $\boldsymbol{\theta}$ (see Examples 4.7 and 4.8). **(a)** $(\sigma_2, \sigma_3) = (-1, -1)$ and $\boldsymbol{\theta} = (1, 0.25, 0.25)$. **(b)** $(\sigma_2, \sigma_3) = (-1, 1)$ and $\boldsymbol{\theta} = (1, 2, 3)$. **(c)** $(\sigma_2, \sigma_3) = (1, -1)$ and $\boldsymbol{\theta} = (1, 2, 3)$. **(d)** $(\sigma_2, \sigma_3) = (1, 1)$ and $\boldsymbol{\theta} = (1, 2, 3)$.

Set $\chi_1 = 15$, $\chi_2 = 25$, $\chi_3 = 65$, $\boldsymbol{\theta} = (1,2,3)$, and $(\alpha_1,\alpha_2,\alpha_3) = (1,0.99,0.792)$ for the three difference schemes. The points $(\lambda_{12},\lambda_{13},\lambda_{23})$ in (a), (b), and (c) are $(0.403,0.467,0.058)$, $(0.505,0.586,0.289)$, and $(0.505,0.586,0.289)$, respectively. Hence the conditions in Theorem 4.6 are satisfied. Consequently, the solutions of the three difference schemes are positive and the equilibrium point $\boldsymbol{\theta}$ is globally asymptotically stable, which are demonstrated in Figure 3(b), (c), and (d).

Remark 4.9 In the n-dimensional cases, we only consider the equilibrium point which components are all positive. Thus a future study is to investigate dynamics on the other equilibrium points with some zero components.

Competing interests
The author declares that he has no competing interests.

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