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The chaos of the solution semigroup for some partial differential equations in weighted Banach spaces

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Abstract

In this paper we deal with the solution semigroup of some partial differential equations in a weighted Banach space on the real axis. We aim at showing the connection between complex-analytic approach and chaotic theory. With the approach of Carleman's formula and Joel H Shapiro's construction, we could construct dense systems of functions from which the chaos of the solution semigroup follows. The novelty of our paper is the usage of the complex-analytic approach in investigation on chaos of some partial differential equations. As far as we know, our manuscript is the first paper in this direction.

MSC: 35K30; 30E20

Keywords: partial differential equation; C_0-semigroup; weighted Banach spaces; chaos; Carleman's formula

1 Introduction

In [1] and [2], the author considered the following partial differential equation:

$$\frac{\partial}{\partial t}u + c(x)\frac{\partial}{\partial x}u = h(x)u, \quad t \geq 0, x \in [0,\infty), \tag{1}$$

with an initial condition

$$u(0,x) = f(x), \quad x \in [0,\infty), \tag{2}$$

where $c(x) = a$ or $c(x) = ax$ with a a positive constant and $h(x)$ is a continuous bounded function on $[0,\infty)$. In case of $f(x) \in C_0([0,\infty))$, where $C_0([0,\infty))$ consists of all complex-valued functions on $[0,\infty)$ satisfying $\lim_{x\to\infty} f(x) = 0$ with the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$, both the hypercyclicity and the chaos of the solution semigroup $\{Q(t)\}_{t\geq 0}$ of (1) and (2) in the form

$$\bigl(Q(t)\bigr)(x) = \exp\biggl(\int_x^{x+t} h(s)\,\mathrm{d}s\biggr) f(x+t) \tag{3}$$

are characterized.

It is natural to ask the following question:

Does the hypercyclicity or chaos of the solution semigroup $\{Q(t)\}_{t\geq 0}$ *of* (1) *and* (2) *still hold when* $f(x)$ *is in other Banach spaces*?

In the present paper we are concerned with the above question. Our study will be focused on the weighted Banach space C_α. Let $\alpha(x)$ be a nonnegative continuous function defined on $\mathbb{R}^+ = [0, \infty)$, henceforth called a *weight*, satisfying

$$\lim_{x\to+\infty} x^{-1}\alpha(x) = \infty. \tag{4}$$

Given a weight $\alpha(x)$, the weighted Banach space C_α consists of complex continuous functions f defined on the half real axis with $f(x)\exp(-\alpha(x))$ vanishing at infinity, and is normed by

$$\|f\|_\alpha = \sup\bigl\{\bigl|f(x)\exp\bigl(-\alpha(x)\bigr)\bigr| : x \in [0, +\infty)\bigr\}$$

for $f \in C_\alpha$.

Following [3, 4], and [5], we define the operator $T_a : C_\alpha \to C_\alpha$ of *translation by the complex number a* by

$$T_a f(x) = f(x + a) \quad (f \in C_\alpha, a \in \mathbb{C}).$$

In [4], the normal family in an open set in the complex plane which is integer translates of an entire function is characterized. The hypercyclicity of bounded translation operators on Hilbert spaces of entire functions which have slow growth is characterized in [3]. The translation operators are engaged in [5] to get the intriguing and beautiful chaotic characterizations of simple connectivity. For the reader's convenience, we shall recall some basic facts on the concept of chaos.

In the last decade it has been observed that chaotic behavior in the sense of Devaney [6] can occur in some infinite-dimensional space for a linear operator. Recall a continuous linear operator T on a topological vector space X is called *hypercyclic* if there exists a vector $x \in X$ whose orbit $\{T^n x | n = 0, 1, \ldots\}$ is dense in X. A *periodic point* for T is a vector $x \in X$ such that $T^n x = x$ for some $n \in \mathbb{N}$. T is said to be *chaotic* if it is hypercyclic and its set of periodic points is dense in X.

Much of the work that has been done on hypercyclic operators depends on the following hypercyclicity criterion (see [7] and [5]).

Lemma 1.1 (The hypercyclicity criterion) *Suppose T is an operator on a Fréchet space X. Suppose further that there are dense subsets X_0 and Y_0 of X, and a mapping $S : Y_0 \to Y_0$, such that*:

(a) $T^n \to 0$ *pointwise on* X_0,
(b) $S^n \to 0$ *pointwise on* Y_0,
(c) TS *is the identity map on* Y_0.

Then T is hypercyclic on X.

In this paper we shall show that the solution semigroup of (1) which is defined in (3) is chaotic in some C_α. Our proof is based on constructing function system which is dense and periodic under the acting of the solution semigroup, which is a totally complex-analytic approach. In Section 2, we introduce some basic results from complex analysis. Our theorem on chaos of the solution semigroup of (1) will be proved in Section 3.

2 Preliminary lemmas

From now on, A denotes positive constants and it may be different at each occurrence.

Let us recall Carleman's formula, which connects the zeros of a holomorphic function with its behavior on the boundary of a circle.

With a sequence of numbers $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$, $\lambda_n \in \mathbb{C}$, we associate the function

$$N_\Lambda(R) = \sum_{|\lambda_n|\le R} \left(\frac{1}{|\lambda_n|} - \frac{|\lambda_n|}{R^2}\right)\cos\theta_n. \tag{5}$$

Carleman's formula is as follows (see [8] and [9] for more details).

Lemma 2.1 *Let $f(w)$ be a function analytic on $S = \{w = u + iv : \Re w \ge 0, |w| \le R\}$, then*

$$\begin{aligned} N_\Lambda(R) &= \frac{1}{\pi R}\int_0^\pi \log\bigl|f\bigl(Re^{i\theta}\bigr)\bigr|\cos\theta\,\mathrm{d}\theta \\ &\quad + \frac{1}{2\pi}\int_0^R \left(\frac{1}{v^2} - \frac{1}{R^2}\right)\log\bigl|f(iv)f(-iv)\bigr|\,\mathrm{d}v + d_f(R), \end{aligned}$$

where $N_\Lambda(R)$ is the function associated with the zeros of $f(w)$ in S defined by (5) *and $\{\theta_n\}$ is the corresponding sequence of arguments. Furthermore, $d_f(R)$ is a function of R depends on f, satisfying*

$$d_f(R) = O(1) \quad \text{as } R \to \infty.$$

It will be important for us to investigate the denseness of some particular systems of functions in the C_α. The basic idea of the following lemma originates from [10, 11] and [12].

Let $\alpha(x)$ be a positive continuous function on the half-axis, if for every fixed $R \ge 0$, the quantity

$$\alpha^*(R) = \sup_{x\ge 0}\bigl\{Rx - \alpha(x)\bigr\} \tag{6}$$

is finite. It is called the *Legendre transform* or the *Young dual function* for α (see [9]).

Denote by

$$M_{\alpha^*}(R) = \frac{1}{2\pi}\int_1^R \left(\frac{1}{v^2} - \frac{1}{R^2}\right)\frac{\alpha^*(4|v|)}{2}\,\mathrm{d}v. \tag{7}$$

Lemma 2.2 *Let $\alpha(x)$ be a nonnegative continuous function satisfying* (4), *let $\Lambda = \{\lambda_n = |\lambda_n|e^{i\theta_n} : n = 1, 2, \ldots\}$ be a sequence of complex numbers satisfying $\Re\lambda_n > 0$, furthermore, let $N_\Lambda(R)$ be the function associated to Λ defined by* (5), *$\alpha^*(R)$ and $M_{\alpha^*}(R)$ be defined by* (6) *and* (7) *separately. If*

$$\limsup_{R\to\infty}\left(N_\Lambda(R) - \frac{\alpha^*(R)}{R} - M_{\alpha^*}(R)\right) = \infty, \tag{8}$$

then the system $\{e^{\lambda_n x}\}$ is dense in C_α.

Proof We use the Hahn-Banach theorem. Suppose ϕ is a continuous linear functional on C_α that annihilates each exponential function $\{e^{\lambda_n x}\}$ for $\lambda_n \in \Lambda$. By Hahn-Banach we will be done if we can show that $\phi = 0$ on C_α.

The Riesz representation theorem provides a complex measure μ satisfying

$$\|\mu\| = \int_0^{+\infty} e^{\alpha(x)} \big|\mathrm{d}\mu(x)\big| = \|\phi\|$$

such that

$$\phi(f) = \int_0^{+\infty} f(x)\,\mathrm{d}\mu(x)$$

for $f \in C_\alpha$. In particular,

$$\phi\big(e^{\lambda_n x}\big) = \int_0^{+\infty} e^{\lambda_n x}\,\mathrm{d}\mu(x)$$

for each $\lambda_n \in \Lambda$. Now the last equation shows that the function defined on the complex plane by

$$\Phi(w) = \phi\big(e^{wx}\big) = \int_0^{+\infty} e^{wx}\,\mathrm{d}\mu(x) \quad (w = u + iv) \in \mathbb{C}$$

is holomorphic in the closed right half plane $\mathbb{C}_+ = \{w : \Re w \geq 0\}$, satisfying

$$\Phi(\lambda_n) = 0$$

for each $\lambda_n \in \Lambda$.

By the definition of μ, we have

$$\big|\Phi(w)\big| = \left|\int_0^{+\infty} e^{wx}\,\mathrm{d}\mu(x)\right| \leq \|\mu\| \cdot e^{\alpha^*(|w|)}.$$

Thanks to Lemma 2.1, we will see that (8) verifies Lemma 2.2. Consider $\Phi(w)$ in the closed half circle $S = \{w : \Re w \geq 0, |w| \leq R\}$. Without loss of generality, we can suppose that $|\lambda_n| > 1$. Application of Carleman's formula in Lemma 2.1 yields

$$N_\Lambda(R) \leq \frac{\alpha^*(R)}{R} + M_{\alpha^*}(R) + d_f(R).$$

Recall the function $d_f(R)$ defined in Lemma 2.1 remains bounded as $R \to \infty$. This forces

$$\limsup_{R\to\infty} \left(N_\Lambda(R) - \frac{\alpha^*(R)}{R} - M_{\alpha^*}(R)\right) < \infty,$$

which gives $\Phi(w) \equiv 0$, proving Lemma 2.2. □

For a function $f(z)$ regular in the right half plane $\Re z \geq 0$, the indicator function of $f(z)$ is defined by (see [8])

$$h(\theta) = \limsup_{r\to\infty} r^{-1} \log\big|f\big(re^{i\theta}\big)\big|, \quad |\theta| \leq \frac{\pi}{2}.$$

We also need the following uniqueness theorem on holomorphic functions of exponential type growth on the right half plane characterized by the indicator functions (see [8]).

Lemma 2.3 (Carleson's theorem) *Suppose that $f(z)$ is regular and exponential type in $\Re z \geq 0$ and $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) < 2\pi$; then $f(z) \equiv 0$ if and only if $f(k) = 0$, $k = 1, 2, \ldots$.*

3 The chaotic theorem

With the useful criteria for the density in Lemma 2.2 and Lemma 2.3 in hand, we are able to prove Theorem 3.1, which is the main result of this paper.

Theorem 3.1 *Let $Q(t)$ be the solution semigroup of* (1) *defined in* (3) *with $f(x) \in C_\alpha$, where $x \in [0, +\infty)$ and $h(x)$ is a bounded continuous function. If*

$$\exp\left(\int_x^{x+t} h(s)\,\mathrm{d}s\right) = \exp\left(\int_x^{x+t+a} h(s)\,\mathrm{d}s\right) \tag{9}$$

holds for some positive constant a, then the discrete semigroup $\{Q(na)\}_{n=1}^{\infty}$ is chaotic in C_α.

Proof It is obvious that the discrete semigroup $\{Q(na)\}_{n=1}^{\infty}$ is very close to the translation operators T_a. We shall follow the proof for T_a in [5]. We will proceed with the proof in two steps.

Step 1: *The solution semigroup Q_a is hypercyclic in C_α.*

Our business is to find the dense subspaces X_0 and Y_0 and the inverting operator S required by the hypercyclicity criterion in Lemma 1.1.

Let us define

$$F_- = \left\{e^{\lambda_j x} : \Re\lambda_j < 0 \text{ and } \lambda_j \in \Lambda, \text{where } \Lambda \text{ satisfies } (8), j = 1, 2, \ldots\right\}$$

and

$$F_+ = \left\{e^{\lambda_j x} : \Re\lambda_j > 0 \text{ and } \lambda_j \in \Lambda, \text{where } \Lambda \text{ satisfies } (8), j = 1, 2, \ldots\right\}.$$

By the density Lemma 2.2, it is obvious that F_+ is dense in C_α. The case of F_- can be done in a similar fashion, that is, applying Carleman's formula in Lemma 2.1 to the closed half circle on the left half plane $S_- = \{w : \Re w \leq 0, |w| \leq R\}$. This argument works as well for Lemma 2.1 and Lemma 2.2. Thus, we have obtained the dense subspaces of C_α.

Let us verify the hypercyclicity criterion for the dense subset $F_+, F_- \subset C_\alpha$:

(a) For every $e^{\lambda_j x} \in F_-$, $Q_a^n e^{\lambda_j x} = \exp(\int_x^{x+a} h(s)\,\mathrm{d}s) e^{\lambda_j x + na\lambda_j}$. Note that $e^{na\lambda_j x}$ can be written as $e^{na\lambda_j x} = e^{\lambda_j x}(\cos(na\lambda_j) + i\sin(na\lambda_j))$, therefore, $\lim_{n\to\infty} Q_a^n e^{\lambda_j x} = 0$.

(b) Let S_a be the operator translation by $-a$, *i.e.* $S_a = Q_a^{-1}$. For every $e^{\lambda_j x} \in F_+$, $S_a^n e^{\lambda_j x} = \exp(\int_{x-a}^{x} h(s)\,\mathrm{d}s) e^{\lambda_j t - na\lambda_j}$. Note that $e^{-na\lambda_j x}$ can be written as $e^{-na\lambda_j} = e^{-na\Re\lambda_j}(\cos(-na\Im\lambda_j x) + i\sin(-na\Im\lambda_j x))$, therefore, $\lim_{n\to\infty} S_a^n e^{\lambda_j x} = 0$.

(c) Finally, we have $Q_a S_a = I$ on F_+ where I is the identity operator.

Step 2: *C_α admits a dense periodic points subset.*

Since the obvious periodic points $e^{\lambda x}$ for $a\lambda = 2\pi iq$, where q is a (real) rational number, no longer span a dense subspace of C_α, it requires a bit more work.

Denote by $E = \{\frac{1}{(x+an)^k}, k = 1, 2, \ldots\}$. Since $\alpha(x)$ satisfies (4), we know that E is a subset of C_α. We are going to show that E is also dense in C_α. We proceed with the proof in Lemma 2.2 word by word. Denote by $\Phi(w)$ the function induced by the bounded linear functional ϕ, then

$$\Phi(w) = \phi\left(\frac{1}{(x+na)^w}\right) = \int_0^{+\infty} \exp\left(\int_x^{x+a} h(s)\,\mathrm{d}s\right) \frac{1}{(x+na)^w}\,\mathrm{d}\mu(x).$$

Denote by $w = u + iv = re^{i\theta}$, then for fixed n, we have

$$\left|\Phi(w)\right| \le A \sup_{\theta\in[0,2\pi)} e^{-r\cos\theta},$$

where A is some positive constant depend on n. Thus $\Phi(w)$ is regular and exponential type in the closed right half plane $\mathbb{C}_+ = \{w : \Re w \ge 0\}$, satisfying $h(\frac{\pi}{2}) + h(-\frac{\pi}{2}) < 2\pi$. By Lemma 2.3, we can deduce $\Phi(w) \equiv 0$ from $\Phi(k) = 0$ for all $k \in \mathbb{N}$. Thus E is dense in C_α.

Now we proceed to construct dense periodic subset of C_α with the help of E. We claim that for each point λ of the unit circle, the series

$$\sum_{n=0}^{+\infty} \frac{\lambda^n}{(x+na)^k} \tag{10}$$

converges in the norm of C_α to an eigenvector f of Q_λ corresponding to the eigenvalue λ. Since $k \ge 2$, for fixed $x \in \mathbb{R}^+$, we have

$$(x+na)^{-k} = O\left(|n|^{-k}\right) \quad \text{as } n \to +\infty.$$

Thus the absolute series

$$\sum_{n=0}^{+\infty} \int_0^{+\infty} \exp\left(\int_x^{x+a} h(s)\,\mathrm{d}s\right) \frac{\lambda^n}{(x+na)^k}\,\mathrm{d}\mu(x)$$

converges.

Let E_Σ denote the collection of all these eigenvectors where λ is a root of unity, it is obvious that E_Σ is a set of periodic points of Q_a. As aforementioned, the set

$$E_0 = \left\{e^{\lambda x} : a\lambda \text{ is a root of unity}\right\}$$

is another collection of periodic points for Q_a. Thus the linear span of $E_0 \cup E_\Sigma$ also consists entirely of periodic points. To prove Theorem 3.1, it remains to show that $E_0 \cup E_\Sigma$ is dense in C_α. We will see that it reduces to show that E_Σ is dense in H_α.

Recall that we have proved the density of the systems E. So it is just a job for the Hahn-Banach theorem to show that the closure of the span of E_Σ contains E. Suppose ϕ is a nontrivial bounded linear functional that annihilates every function in E_Σ. By the Hahn-Banach theorem it is enough to prove that ϕ also annihilates every function in E_Σ. That is, we are assuming that $\phi(f) = 0$ for all $|\lambda| = 1$ and we want to prove $\phi(\exp(\int_x^{x+a} h(s)\,\mathrm{d}s)\frac{1}{(x+na)^k}) = 0$ for $n = 0, 1, 2, \ldots$.

Denote by $a_n(x) = \exp(\int_x^{x+a} h(s)\,\mathrm{d}s)\frac{1}{(x+na)^k}$ and write

$$b_n = \phi(a_n)$$

for which we will prove are zero. Since ϕ is continuous and the series in (10) converges to f in the norm of C_α, it follows that

$$\sum_{n=0}^{+\infty} b_n\lambda^n = \phi(f) = 0 \quad \bigl(\text{all } |\lambda| = 1\bigr). \tag{11}$$

We will show that the left-hand side of (11) has square-summable coefficients. Thus we can deduce all of the coefficients b_n must be zero by the uniqueness theorem from the L^2 theory of Fourier series.

By the definition of the continuity of a linear functional, there exists some positive constant A such that

$$\bigl|\phi(f)\bigr| \le A\|f\| \quad (f \in C_\alpha).$$

In particularly, (4) yields

$$|b_n| = \bigl|\phi(a_n)\bigr| \le A\|a_n\| \le \frac{A_1}{|n|^{2k}}$$

for all $n = 0,1,2,\ldots$, where $A_1 = A\int_0^{+\infty} x^2\,\mathrm{d}\mu(x)$. Since $k > 1$, we have $\sum_{n=0}^{+\infty} |a_n|^2 < \infty$, which show that the right-hand side of (11) belongs to L^2 of the unit circle. Thus by (11) all the coefficients a_n must be zero, proving Theorem 3.1. □

Competing interests
The author declares that they have no competing interests.

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