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# Stability in distribution of neutral stochastic partial differential delay equations driven by $\alpha$ -stable process

Yanchao Zang\* and Junping Li

\*Correspondence:  
zycmail@126.com  
School of Mathematics and  
Statistics, Central South University,  
Changsha, Hunan 410075, P.R. China

## Abstract

We consider a class of neutral stochastic partial differential equations driven by an  $\alpha$ -stable process. We prove the existence and uniqueness of the mild solution to the equation by the Banach fixed-point theorem under some suitable assumptions. Sufficient conditions for the stability in the distribution of the mild solution are derived.

**MSC:** 39A11; 37H10

**Keywords:** neutral stochastic partial differential equation;  $\alpha$ -stable process; mild solution; stability in distribution

## 1 Introduction

The theory of stochastic partial differential equations has been widely applied in scientific fields such as physics, mechanical engineering, and economics. Especially, the study of stochastic neutral functional differential equations has received a great deal of attention in recent years. For example, Bao *et al.* [1] extended the existence and uniqueness of mild solutions to a class of more general stochastic neutral partial functional differential equations under non-Lipschitz conditions. Caraballo *et al.* [2] investigated the exponential stability and ultimate boundedness of the solutions to a class of neutral stochastic semilinear partial delay differential equations.

Also the stability in a distribution is an important notion like the stability in probability or in the moment of stochastic differential equations. Such a stability is much weaker than stability in probability and it is useful sometimes to know whether or not the probability distribution of the solution will converge to some distribution but not necessarily to zero. There is an extensive literature concerned with the stability in the distribution of stochastic differential equations. Using an excellent stopping time technique and an  $M$ -matrix trick, Yuan and Mao [3] investigated the stability in the distribution of nonlinear SDEs with Markovian switching. Yuan *et al.* [4] discussed a class of stochastic differential delay equation with Markovian switching, where the sufficient conditions of stability in the distribution were established. Tan *et al.* [5] considered weak convergence of functional SDEs with variable delays. For the case of stochastic partial differential equations, we refer to Bao *et al.* [1, 6]. Furthermore, for the nonlinear regime switching jump diffusion, we can refer to Yang and Yin [7].

Although many scholars have discussed the stability in the distribution of SDEs or functional SDEs where the noises are Brownian motion and jumps, the methods applied therein will not work if the considered noises are  $\alpha$ -stable processes. As we know, for an  $\alpha$ -stable process ( $0 < \alpha < 2$ ), it only has a  $p < \alpha$  moment. Therefore, some useful techniques involved in the above references, such as the Burkholder-Davis-Gundy inequality and Da Prato-Kwapień-Zabczyk's factorization technique [8], are not available. On the other hand, it seems that little is known about the stability in the distribution of the neutral stochastic partial differential equations driven by an  $\alpha$ -stable process, and there are few systematic works so far in which the noise source is an  $\alpha$ -stable process as well. For more studies of stochastic systems driven by stable processes, we refer to [9–11].

In this paper, we study the existence, uniqueness, and stability in the distribution of mild solutions for the following neutral stochastic differential equation with finite delay:

$$\begin{cases} d[X(t) + g(X(t - \tau))] = [AX(t) + f(X(t - \tau))] dt + dZ(t), & 0 \leq t \leq T, \\ X_0(\cdot) = \xi \in D([- \tau, 0], H), \end{cases} \quad (1)$$

where  $D([- \tau, 0], H)$  is the space of all càdlàg functions paths from  $[- \tau, 0]$  into  $H$ , a Hilbert space, equipped with the supremum norm  $\|\varphi\|_\infty = \sup_{-\tau \leq t \leq 0} \|\varphi(t)\|_H$ . And  $g, f : H \rightarrow H$  are given functions to be specified later.

The contents of the paper are as follows. In Section 2, we briefly present some basic notations and preliminaries. In Section 3 the existence and uniqueness of mild solutions are proved. In the last section, we devote to give the sufficient conditions of the stability in the distribution of the mild solution to Eq. (1).

## 2 Preliminary

Let  $(H, \langle \cdot, \cdot \rangle_H, \|\cdot\|_H)$  be a real separable Hilbert space. Denote by  $\mathcal{D} := D([- \tau, 0]; H)$  the space of all  $H$ -valued càdlàg functions defined on  $[- \tau, 0]$  equipped with the uniform norm  $\|\xi\|_\infty := \sup_{-\tau \leq \theta \leq 0} \|\xi(\theta)\|_H$ . Recall that a path  $f : [- \tau, 0] \mapsto H$  is called càdlàg if it is right-continuous having finite left-hand limits.

Let  $Z(t)$  be a cylindrical  $\alpha$ -stable process,  $\alpha \in (0, 2)$ , defined by

$$Z(t) := \sum_{m=1}^{\infty} \beta_m Z_m(t) e_m. \quad (2)$$

Here  $\{e_m\}_{m \geq 1}$  is an orthonormal basis of  $H$ ,  $\{Z_m(t)\}_{m \geq 1}$  are independent, real-valued, normalized, symmetric  $\alpha$ -stable Lévy processes defined on stochastic basis  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ , and  $\{\beta_m\}_{m \geq 1}$  is a sequence of positive numbers. Recall that a stochastic process  $\{Z_{\alpha, \beta}(t) : t \geq 0\}$  is called an  $\alpha$ -stable Lévy process if

- (i)  $Z_{\alpha, \beta}(0) = 0$  a.s.;
- (ii)  $Z_{\alpha, \beta}(t)$  has independent increments;
- (iii)  $Z_{\alpha, \beta}(t) - Z_{\alpha, \beta}(s) \sim \eta$  for any  $0 \leq s < t < \infty$ ,

where  $\eta$  stands for an  $\alpha$ -stable random variable, which is uniquely determined by its characteristic function involving four parameters:  $\alpha \in (0, 2]$ , the index of stability;  $\beta \in [-1, 1]$ , the skewness parameter;  $\sigma \in (0, \infty)$ , the scale parameter;  $\mu \in (-\infty, \infty)$ , the shift, and which has the form

$$\phi_\eta(u) = E \exp(iu\eta) = \exp\{-\sigma^\alpha |u|^\alpha (1 - i\beta \operatorname{sgn}(u)\Phi) + i\mu u\}, \quad u \in \mathbb{R},$$

where  $\Phi = \tan(\pi\alpha/2)$  for  $\alpha \neq 1$  and  $\Phi = -(2/\pi) \log |u|$  for  $\alpha = 1$ . We call  $\eta$  is strictly  $\alpha$ -stable whenever  $\mu = 0$ , and if, in addition,  $\beta = 0$ ,  $\eta$  is said to be symmetric  $\alpha$ -stable. For a real-valued normalized (standard) symmetric  $\alpha$ -stable Lévy process  $z(t)$ ,  $\alpha \in (0, 2)$ , it has the characteristic function

$$E \exp(iuz(t)) = e^{-t|u|^\alpha}, \quad u \in \mathbb{R},$$

and the Lévy measure  $\lambda_\alpha(dx) := \frac{c_\alpha}{|x|^{1+\alpha}}$ ,  $x \in \mathbb{R} - 0$ , where  $c_\alpha$  is some constant. For more details of  $\alpha$ -stable processes, we can refer to [12] and [13].

Throughout the paper we impose the following assumptions:

(H1) The operator  $(A, D(A))$  is a self-adjoint compact operator on the Hilbert space  $H$  which is separable such that  $-A$  has discrete spectrum

$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_m \leq \dots \leq \lim_{m \rightarrow \infty} \lambda_m = \infty$  with corresponding eigenbasis  $\{e_m\}_{m \geq 1}$  of  $H$ . In this case  $A$  generates a compact  $C_0$ -semigroup  $S(t) = e^{tA}$ ,  $t \geq 0$ , such that  $\|e^{tA}\| \leq e^{-\lambda_1 t}$ .

(H2) There exists a positive constant  $K_1$  such that for all  $x, y \in H$

$$\|f(x) - f(y)\|_H \leq K_1 \|x - y\|_H, \quad \|f(x)\|_H \leq K_1 (1 + \|x\|_H).$$

(H3) There exist  $k \in (0, 1)$  and a positive constant  $K_2$  such that for all  $x, y \in H$  and

(i)  $\|(-A)^k g(x) - (-A)^k g(y)\|_H \leq K_2 \|x - y\|_H$ ;

(ii)  $\|(-A)^k g(x)\|_H \leq K_2 (1 + \|x\|_H)$ ;

(iii) The constants  $K_2$  and  $k$  satisfy  $10K_2^p \|(-A)^{-k}\|^p < 1$ .

(H4) There exists  $\theta \in (0, k)$  such that  $\alpha\theta \in (0, 1)$  and  $\delta := \sum_{m=1}^\infty \frac{\beta_m^\alpha}{\lambda_m^{1-\alpha\theta}} < \infty$ .

The following two lemmas will play an important role in proving our main results. So let us state them now.

**Lemma 2.1** [14] *Under (H1), for any  $k \in (0, 1]$  and  $x \in D((-A)^k)$*

$$e^{tA}(-A)^k x = (-A)^k e^{tA} x$$

and there exists  $M_k > 0$  such that for any  $t > 0$

$$\|(-A)^k e^{tA}\| \leq M_k t^{-k} e^{-\lambda t}.$$

**Lemma 2.2** [11] *Let  $T > 0$  be arbitrary. For all  $0 \leq \theta < \beta - \frac{1}{2\alpha}$  and all  $0 < p < \alpha$ , we have*

$$E \sup_{0 \leq t \leq T} \|(-A)^\theta Z_A(t)\|_H^p \leq CT^{\frac{p}{\alpha}},$$

where  $Z_A(t) := \int_0^t e^{(t-s)A} dZ(s)$ ,  $Z(s)$  is a cylindrical  $\alpha$ -stable process having the form of (2) and  $C$  depends on  $\alpha, \theta, \beta, p$ .

**Remark 2.1** In Lemma 2.2, the constant  $\beta$  satisfies  $\beta > \frac{1}{2} + \frac{1}{2\alpha}$  so that the convolutions  $Z_A(t)$  are in  $H$ . Moreover, there exist some  $C_1, C_2 > 0$  such that  $C_1 \lambda_k^{-\beta} \leq |\beta_k| \leq C_2 \lambda_k^{-\beta}$ .

### 3 Existence and uniqueness

The aim of this section is to establish the existence and uniqueness of the mild solution to Eq. (1). First, we give the following definition of mild solutions for Eq. (1).

**Definition 3.1** An  $\mathcal{F}_t$ -adapted càdlàg stochastic process  $X(t), t \in [-\tau, T]$ , is called a mild solution of Eq. (1) if it has the following properties:

- (a)  $X_0 = \xi \in D([-\tau, 0], H)$ .
- (b) For arbitrary  $t \in [0, T]$

$$\begin{aligned}
 X(t) = & e^{tA}(\xi(0) + g(\xi(-\tau))) - g(X(t - \tau)) \\
 & - \int_0^t Ae^{(t-s)A}g(X(s - \tau)) ds + \int_0^t e^{(t-s)A}f(X(s - \tau)) ds \\
 & + \int_0^t e^{(t-s)A} dZ(s), \quad \text{a.s.}
 \end{aligned} \tag{3}$$

We shall denote by  $S_T$  the Banach space of all of càdlàg  $H$ -valued processes  $X(t) \in D([-\tau, T], H)$  with initial data  $X(t) = \xi(t)$  for  $t \in [-\tau, 0]$ , and

$$\|X(t)\|_{S_T} := \left( E \sup_{-\tau \leq t \leq T} \|X(t)\|_H^p \right)^{\frac{1}{p}} < \infty.$$

We have the following result.

**Theorem 3.1** *Suppose the assumptions (H1)-(H4) hold and let  $p \in (1, \alpha), \alpha \in (1, 2)$ . Then, for any initial datum  $\xi \in L^p(\Omega, D([-\tau, 0]; H))$ , there exists a unique mild solution  $X(t)$  of Eq. (1) in  $S_T$  and there exists a constant  $C_T$ , independent of  $\xi$ , such that*

$$E \sup_{t \in [0, T]} \|X(t)\|_H^p \leq C_T(1 + E\|\xi\|_\infty^p). \tag{4}$$

*Proof* For arbitrary  $\xi \in L^p(\Omega, D([-\tau, 0]; H))$  and  $X \in S_T$ , define an operator  $\Lambda$  on  $S_T$  by that  $\Lambda(X)(t) = \xi(t), t \in [-\tau, 0]$ , and

$$\begin{aligned}
 \Lambda(X)(t) = & e^{tA}(\xi(0) + g(\xi(-\tau))) - g(X(t - \tau)) - \int_0^t Ae^{(t-s)A}g(X(s - \tau)) ds \\
 & + \int_0^t e^{(t-s)A}f(X(s - \tau)) ds + \int_0^t e^{(t-s)A} dZ(s), \quad t \in [0, T].
 \end{aligned}$$

The required assertion follows if we show that the operator  $\Lambda$  has a fixed point in the space  $S_T$  by the Banach fixed-point theorem. We divide the proof into two steps.

Step 1. We show that  $\Lambda(X)(t) \in S_T$  for  $t \in [-\tau, T]$ . It is trivial for the case  $t \in [-\tau, 0]$ . For  $0 \leq t \leq T$ , and for any fixed  $X \in S_T$ , using the trivial inequality:  $(\sum_{i=1}^n a_i)^p \leq C_r(\sum_{i=1}^n a_i^p)$ , here  $C_r = 1$  when  $p \leq 1, C_r = n^{p-1}$  when  $p > 1$ , we have

$$\begin{aligned}
 E \sup_{0 \leq t \leq T} \|\Lambda(X)(t)\|_H^p \leq & 5^{p-1}E \sup_{0 \leq t \leq T} \|e^{tA}(\xi(0) + g(\xi(-\tau)))\|_H^p \\
 & + 5^{p-1}E \sup_{0 \leq t \leq T} \|g(X(t - \tau))\|_H^p
 \end{aligned}$$

$$\begin{aligned}
 &+ 5^{p-1}E \sup_{0 \leq t \leq T} \left\| \int_0^t Ae^{(t-s)A}g(X(s-\tau)) ds \right\|_H^p \\
 &+ 5^{p-1}E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)A}f(X(s-\tau)) ds \right\|_H^p \\
 &+ 5^{p-1}E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)A} dZ(s) \right\|_H^p \\
 &=: \sum_{i=1}^5 I_i. \tag{5}
 \end{aligned}$$

From (H1), (H3), and the boundedness of  $(-A)^{-k}$  for  $k \in (0, 1)$ , we get

$$\begin{aligned}
 I_1 &= 5^{p-1}E \sup_{0 \leq t \leq T} \left\| (-A)^{-k} e^{tA} (-A)^k (\xi(0) + g(\xi(-\tau))) \right\|_H^p \\
 &\leq 10^{p-1}E \|\xi\|_\infty^p + 10^{p-1} \left\| (-A)^{-k} \right\|^p K_2^p E (1 + \|\xi\|_\infty)^p \\
 &\leq M_{k,p} (1 + E \|\xi\|_\infty^p), \tag{6}
 \end{aligned}$$

where  $M_{k,p}$  is a constant only depending on  $k$  and  $p$ . For the second term,  $I_2$ , using the assumption (H3) again, we have

$$\begin{aligned}
 I_2 &\leq 5^{p-1} \left\| (-A)^{-k} \right\|^p E \sup_{0 \leq t \leq T} \left\| (-A)^k g(X(t-\tau)) \right\|_H^p \\
 &\leq 5^{p-1} \left\| (-A)^{-k} \right\|^p K_2^p E \sup_{0 \leq t \leq T} (1 + \|X(t-\tau)\|_H)^p \\
 &\leq 10^{p-1} \left\| (-A)^{-k} \right\|^p K_2^p \left( 1 + E \sup_{-\tau \leq t \leq T} \|X(t)\|_H^p \right). \tag{7}
 \end{aligned}$$

We consider the third term,  $I_3$ : apply Lemma 2.1, use the assumption (H3) and Hölder’s inequality, and we derive

$$\begin{aligned}
 I_3 &= 5^{p-1}E \sup_{0 \leq t \leq T} \left\| \int_0^t (-A)^{1-k} e^{(t-s)A} (-A)^k g(X(s-\tau)) ds \right\|_H^p \\
 &\leq 5^{p-1} M_{1-k}^p K_2^p E \sup_{0 \leq t \leq T} \left( \int_0^t (t-s)^{-(1-k)} e^{-\lambda(t-s)} (1 + \|X(s-\tau)\|_H) ds \right)^p \\
 &\leq 10^{p-1} T M_{1-k}^p K_2^p \lambda^{(1-pk)} \left| \Gamma \left( \frac{pk-1}{p-1} \right) \right|^{p-1} \left( 1 + E \sup_{-\tau \leq t \leq T} \|X(t)\|_H^p \right). \tag{8}
 \end{aligned}$$

Similarly, for the fourth term,  $I_4$ , using Hölder’s inequality, we obtain

$$\begin{aligned}
 I_4 &\leq 5^{p-1} K_1^p E \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\lambda(t-s)} (1 + \|X(s-\tau)\|_H) ds \right)^p \\
 &\leq 5^{p-1} K_1^p \sup_{0 \leq t \leq T} \left( \int_0^t e^{-\lambda(t-s)} ds \right)^{p-1} E \sup_{0 \leq t \leq T} \int_0^t (1 + \|X(s-\tau)\|_H)^p ds \\
 &\leq 10^{p-1} T K_1^p \lambda^{(1-p)} \left( 1 + E \sup_{-\tau \leq t \leq T} \|X(t)\|_H^p \right). \tag{9}
 \end{aligned}$$

Noting that  $I_5 \leq 5^{p-1} \|(-A)^{-\theta}\|^p E \sup_{0 \leq t \leq T} \|(-A)^\theta Z_A(t)\|_H^p$ , and by Lemma 2.2, we immediately get

$$I_5 \leq 5^{p-1} \|(-A)^{-\theta}\|^p CT^{\frac{p}{\alpha}}. \tag{10}$$

Thus, we derive from (5) to (10) that, for some constants  $c_1, c_2, c_3$ ,

$$E \sup_{0 \leq t \leq T} \|\Lambda(X)(t)\|_H^p \leq c_1 + c_2 E \|\xi\|_\infty^p + c_3 E \sup_{0 \leq t \leq T} \|X(t)\|_H^p. \tag{11}$$

Hence,  $\Lambda(X)(t) \subset S_T$ .

Step 2. We shall show that the mapping  $\Lambda$  is contractive. Let  $X, Y \in S_T$ . For any fixed  $t \in [0, T]$ , we have

$$\begin{aligned} E \sup_{0 \leq t \leq T} \|\Lambda(X)(t) - \Lambda(Y)(t)\|_H^p &\leq 3^{p-1} E \sup_{0 \leq t \leq T} \|g(X(t-\tau)) - g(Y(t-\tau))\|_H^p \\ &\quad + 3^{p-1} E \sup_{0 \leq t \leq T} \left\| \int_0^t A e^{(t-s)A} [g(X(s-\tau)) - g(Y(s-\tau))] ds \right\|_H^p \\ &\quad + 3^{p-1} E \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)A} [f(X(s-\tau)) - f(Y(s-\tau))] ds \right\|_H^p. \end{aligned}$$

By the assumptions (H2), (H3), and Hölder’s inequality, we obtain

$$\begin{aligned} E \sup_{0 \leq t \leq T} \|\Lambda(X)(t) - \Lambda(Y)(t)\|_H^p &\leq 3^{p-1} K_2^p \|(-A)^{-k}\|^p E \sup_{0 \leq t \leq T} \|X(t) - Y(t)\|_H^p \\ &\quad + 3^{p-1} M_{1-k}^p K_2^p T \lambda^{(1-pk)} \left| \Gamma\left(\frac{pk-1}{p-1}\right) \right|^{p-1} E \sup_{0 \leq t \leq T} \|X(t) - Y(t)\|_H^p \\ &\quad + 3^{p-1} K_1^p T \lambda^{(1-p)} E \sup_{0 \leq t \leq T} \|X(t) - Y(t)\|_H^p \\ &\leq 3^{p-1} (K_2^p \|(-A)^{-k}\|^p + CT) E \sup_{0 \leq t \leq T} \|X(t) - Y(t)\|_H^p, \end{aligned}$$

where  $X(t) = Y(t)$  on  $[-\tau, 0]$ , and  $C > 0$  is a bounded constant. Hence, by the condition  $10K_2^p \|(-A)^{-k}\|^p < 1$ , choosing sufficiently small  $T_1$  such that  $3^{p-1} (K_2^p \|(-A)^{-k}\|^p + CT_1) < 1$ , we can conclude that  $\Lambda$  is contractive. Therefore, by the contraction principle, we have a mild solution of Eq. (1) on  $[0, T_1]$ . Moreover with such a  $T_1$  we get from (11) for the solution of Eq. (1)

$$E \sup_{0 \leq t \leq T_1} \|X(t)\|_H^p \leq \frac{1}{1 - c_3} [c_1 + c_2 E \|\xi\|_\infty^p],$$

which is inequality (4). The solution can be extended to the entire interval  $[0, T]$  in finite steps by repeating the above procedure on intervals  $[0, T_1], [T_1, 2T_1], \dots$  with  $T_1$  such that  $c_3(T_1) < 1$ . This completes the proof.  $\square$

#### 4 Asymptotic stability in the distribution

In this section, we shall derive some sufficient conditions on the stability in the distribution for the process  $Y(t) = X_t$  on  $t \geq 0$ . We need to introduce some more notations. For  $t \geq 0$ , let  $X(t; \xi)$  be the solution of Eq. (1) with initial datum  $X_0 = \xi \in D([- \tau, 0], H)$ . Correspondingly,  $X_t(\xi) = \{X(t + \theta; \xi) : -\tau \leq \theta \leq 0\}$  on  $t \geq 0$ . As usual,  $\{X_t\}_{t \geq 0}$  is called the segment process of  $\{X(t)\}_{t \geq -\tau}$ . Denote by  $p(\xi, t, d\zeta)$  the transition probability of the process  $Y(t) = X_t(\xi)$ , then  $Y(t)$  is a time homogeneous Markov process according to Mohammed [15].

Now, we introduce the concept of stability in the distribution and prepare some useful lemmas as follows.

**Definition 4.1** [16] The process  $Y(t)$  is said to be asymptotically stable in the distribution if there exists a probability measure  $\pi(\cdot)$  on  $D([- \tau, 0]; H)$  such that the transition probability function  $p(\xi, t, d\zeta)$  of  $Y(t)$  converges weakly to  $\pi(d\zeta)$  as  $t \rightarrow \infty$  for every  $\xi \in D([- \tau, 0]; H)$ . Equation (1) is said to be asymptotically stable in the distribution if the solution process  $Y(t) = X_t(\xi)$  is asymptotically stable in the distribution.

By Yuan *et al.* [4], to show the stability in the distribution of Eq. (1), it is sufficient to verify that for some  $p > 0$  and bounded subset  $U \subset \mathcal{D}$ ,

- (N1)  $\sup_{t \geq 0} \sup_{\xi \in U} E \|X_t(\xi)\|_\infty^p < \infty$ ;
- (N2)  $\lim_{t \rightarrow \infty} \sup_{\xi, \eta \in U} E \|X_t(\xi) - X_t(\eta)\|_\infty^p = 0$ , for every  $x, y, z \in X$ .

In what follows, we need to prepare some useful lemmas to demonstrate that (N1) and (N2) hold under some imposed conditions.

**Lemma 4.1** Let  $y : [- \tau, +\infty) \rightarrow [0, +\infty)$  be Borel measurable. If  $y(t)$  is a solution of the delay integral inequality, then

$$y(t) \leq \begin{cases} \|\phi\|_\infty e^{-rt} + b_1 \|y_t\|_\infty + b_2 \int_0^t e^{-r(t-s)} \|y_s\|_\infty ds + J, & t \geq 0, \\ \phi(t), & t \in [- \tau, 0], \end{cases}$$

where  $\phi(t) \in D([- \tau, 0], [0, +\infty))$ ,  $r > 0$ ,  $b_1, b_2$ , and  $J$  are nonnegative constants. If  $\|\phi\|_\infty \leq K$  for some constant  $K > 0$  and

$$b_1 + \frac{b_2}{r} := \rho < 1,$$

then there are constants  $\gamma \in (0, r)$  and  $N \geq K$  such that

$$y(t) \leq Ne^{-\gamma t} + \frac{J}{1 - \rho}, \quad \forall t \geq 0,$$

where  $\gamma$  and  $N$  satisfy

$$\rho_\gamma := b_1 e^{\gamma \tau} + \frac{b_2 e^{\gamma \tau}}{r - \gamma} < 1 \quad \text{and} \quad N \geq \frac{K}{1 - \rho_\gamma},$$

or  $b_2 \neq 0$  and

$$\rho_\gamma \leq 1 \quad \text{and} \quad N \geq \frac{(r - \gamma)[K - \frac{b_2 J}{r(1 - \rho)}]}{b_2 e^{\gamma \tau}}.$$

*Proof* The proof is similar to Lemma 3.1 of [17], so we omit it here. □

**Lemma 4.2** *Assume that (H1)-(H4) hold, and that the inequality*

$$\rho := 10 \left( \|(-A)^{-k}\|^p K_2^p + M_{1-k}^p K_2^p \lambda^{-pk} \left| \Gamma \left( \frac{pk-1}{p-1} \right) \right|^{p-1} + \lambda^{-p} K_1^p \right) < 1 \tag{12}$$

*holds for  $k \in (0, 1]$ ,  $p \in (1, \alpha)$ , where  $\Gamma(\cdot)$  is the gamma function. Then*

$$\sup_{0 \leq t < \infty} E \|X_t(\xi)\|_\infty^p < \infty \tag{13}$$

*for any  $\xi \in D([-\tau, 0], H)$ .*

*Proof* First, we shall show  $\sup_{-\tau \leq t < \infty} E \|X(t)\|_H^p < \infty$ . For  $p \in (1, \alpha)$ , we obtain from (3) that

$$\begin{aligned} E \|X(t)\|_H^p &\leq 5E \|e^{tA}[\xi(0) + g(\xi(-\tau))]\|_H^p \\ &\quad + 5E \|g(X(t-\tau))\|_H^p + 5E \left\| \int_0^t Ae^{(t-s)A} g(X(s-\tau)) ds \right\|_H^p \\ &\quad + 5E \left\| \int_0^t e^{(t-s)A} f(X(s-\tau)) ds \right\|_H^p + 5E \left\| \int_0^t e^{(t-s)A} dZ(s) \right\|_H^p \\ &=: \sum_{i=1}^5 J_i, \end{aligned} \tag{14}$$

where  $J_i$  ( $i = 1, 2, \dots, 5$ ) stands for the  $i$ th term behind the first inequality. It follows from Lemma 2.1 and the assumption (H3) that

$$\begin{aligned} J_1 &= 5E \|e^{tA}[\xi(0) + g(\xi(-\tau))]\|_H^p \\ &\leq 5e^{-\lambda pt} E [\|\xi\|_\infty + \|(-A)^{-k}\| \cdot \|(-A)^k g(\xi(-\tau))\|_H]^p \\ &\leq 5e^{-\lambda pt} E [\|\xi\|_\infty + K_2 \|(-A)^{-k}\| (1 + \|\xi\|_\infty)]^p \\ &\leq M^* e^{-\lambda pt} (1 + E \|\xi\|_\infty^p), \end{aligned} \tag{15}$$

where  $M^* \geq 1$  is an appropriate constant only depend on  $k$  and  $p$ . For the second term  $J_2$ ,

$$\begin{aligned} J_2 &= 5E \|g(X(t-\tau))\|_H^p \leq 5E [\|(-A)^{-k}\| \cdot \|(-A)^k g(X(t-\tau))\|_H]^p \\ &\leq 5 \|(-A)^{-k}\|^p K_2^p E (1 + \|X(t-\tau)\|_H)^p \\ &\leq 10 \|(-A)^{-k}\|^p K_2^p \left( 1 + \sup_{-\tau \leq \theta \leq 0} E \|X(t+\theta)\|_H^p \right). \end{aligned} \tag{16}$$

Similarly, by using Lemma 2.1, (H3), and Hölder's inequality, we obtain

$$\begin{aligned} J_3 &= 5E \left\| \int_0^t Ae^{(t-s)A} g(X(s-\tau)) ds \right\|_H^p \\ &\leq 5E \left( \int_0^t \|(-A)^{1-k} e^{(t-s)A} (-A)^k g(X(s-\tau))\|_H ds \right)^p \end{aligned}$$



$$\begin{aligned} &\leq 5M_{1-k}^p K_2^p \left[ \int_0^t \frac{e^{-\lambda(t-s)}}{(t-s)^{\frac{p(1-k)}{p-1}}} ds \right]^{p-1} \int_0^t e^{-\lambda(t-s)} E(1 + \|X(s-\tau)\|_H)^p ds \\ &\leq 10M_{1-k}^p K_2^p \lambda^{1-pk} \left| \Gamma\left(\frac{pk-1}{p-1}\right) \right|^{p-1} \int_0^t e^{-\lambda(t-s)} \left(1 + \sup_{-\tau \leq \theta \leq 0} E\|X(s+\theta)\|_H^p\right) ds. \end{aligned} \tag{17}$$

By (H2) and Hölder’s inequality, we obtain

$$\begin{aligned} J_4 &= 5E \left\| \int_0^t e^{(t-s)A} f(X(s-\tau)) ds \right\|_H^p \\ &\leq 5E \left( \int_0^t e^{-\lambda(t-s)} K_1 (1 + \|X(s-\tau)\|_H) ds \right)^p \\ &\leq 10\lambda^{1-p} K_1^p \int_0^t e^{-\lambda(t-s)} \left(1 + \sup_{-\tau \leq \theta \leq 0} E\|X(s+\theta)\|_H^p\right) ds. \end{aligned} \tag{18}$$

For the estimation of  $J_5$ , by the argument of [10, Theorem 4.4] and applying (H4), we have

$$J_5 = 5E \left\| \int_0^t e^{(t-s)A} dZ(s) \right\|_H^p \leq 5\tilde{c}_p \left( \sum_{m \geq 1} \beta_m^\alpha \frac{1 - e^{-\alpha\lambda_m t}}{\alpha\lambda_m} \right)^{p/\alpha} \leq 5\tilde{c}_p (\delta / (\alpha\lambda_1^{\alpha\theta}))^{p/\alpha}, \tag{19}$$

where  $\tilde{c}_p$  depends only on  $p$ . Thus, substituting (15)-(19) into (14), we get

$$\begin{aligned} E\|X(t)\|_H^p &\leq M^* e^{-\lambda pt} E\|\xi\|_\infty^p + b_1 \sup_{-\tau \leq \theta \leq 0} E\|X(t+\theta)\|_H^p \\ &\quad + b_2 \int_0^t e^{-\lambda(t-s)} \sup_{-\tau \leq \theta \leq 0} E\|X(s+\theta)\|_H^p ds + J, \end{aligned} \tag{20}$$

where  $b_1 = 10\|(-A)^{-k}\|_H^p K_2^p$ ,  $b_2 = 10(M_{1-k}^p K_2^p \lambda^{1-pk} |\Gamma(\frac{pk-1}{p-1})|^{p-1} + \lambda^{1-p} K_1^p)$ , and

$$\begin{aligned} J &= M^* e^{-\lambda pt} + 10\|(-A)^{-k}\|_H^p K_2^p + 10M_{1-k}^p K_2^p \lambda^{-pk} \left| \Gamma\left(\frac{pk-1}{p-1}\right) \right|^{p-1} \\ &\quad + 10\lambda^{-p} K_1^p + 5\tilde{c}_p (\delta / (\alpha\lambda_1^{\alpha\theta}))^{p/\alpha}. \end{aligned}$$

In the light of (12),  $\rho = b_1 + \frac{b_2}{\lambda} < 1$  holds. Furthermore, there exist  $\bar{K} \geq 0, \bar{N} > 0, \gamma \in (0, \lambda)$ , such that

$$M^* E\|\xi\|_\infty^p \leq \bar{K}, \quad \rho_\gamma = b_1 e^{\gamma\tau} + \frac{b_2 e^{\gamma\tau}}{\lambda - \gamma} \leq 1 \quad \text{and} \quad \frac{(\lambda - \gamma)[\bar{K} - \frac{b_2 J}{\lambda(1-\rho)}]}{b_2 e^{\gamma\tau}} \leq \bar{N}.$$

Consequently, combining (12), (20) with Lemma 4.1, we arrive at

$$\sup_{-\tau \leq t < \infty} E\|X(t)\|_H^p \leq \bar{N} e^{\gamma\tau} + \frac{J}{(1-\rho)} < \infty. \tag{21}$$

Next, we use the following trivial inequality: for any  $a, b, c, d, e \geq 0$  and  $\epsilon > 0$ ,

$$(a + b + c + d + e)^2 \leq (1 + \epsilon)a^2 + 4\left(1 + \frac{1}{\epsilon}\right)(b^2 + c^2 + d^2 + e^2).$$

For any integer  $n \geq 1$ , similar to the above computations, one has

$$\begin{aligned}
 E\|X_{n\tau}(\xi)\|_{\infty}^p &\leq (1 + \epsilon)^{\frac{p}{2}} E\|g(X_{(n-1)\tau}(\xi))\|_{\infty}^p \\
 &\quad + 2^p \left(1 + \frac{1}{\epsilon}\right)^{\frac{p}{2}} \left\{ E \sup_{-\tau \leq \theta \leq 0} \|e^{(\theta+\tau)A} [X((n-1)\tau) + g(X((n-2)\tau))]\|_H^p \right. \\
 &\quad + E \sup_{-\tau \leq \theta \leq 0} \left\| \int_{(n-1)\tau}^{n\tau+\theta} A e^{(n\tau+\theta-s)A} g(X(s-\tau)) ds \right\|_H^p \\
 &\quad + E \sup_{-\tau \leq \theta \leq 0} \left\| \int_{(n-1)\tau}^{n\tau+\theta} e^{(n\tau+\theta-s)A} f(X(s-\tau)) ds \right\|_H^p \\
 &\quad \left. + E \sup_{-\tau \leq \theta \leq 0} \left\| \int_{(n-1)\tau}^{n\tau+\theta} e^{(n\tau+\theta-s)A} dZ(s) \right\|_H^p \right\} \\
 &\leq (1 + \epsilon)^p K_2^p \|(-A)^{-k}\|^p E\|X_{(n-1)\tau}(\xi)\|_{\infty}^p \\
 &\quad + C \left( 1 + \sup_{(n-2)\tau \leq t \leq (n-1)\tau} E\|X(t, \xi)\|_H^p + E \sup_{0 \leq t \leq \tau} \left\| \int_0^t e^{(t-s)A} d\tilde{Z}(s) \right\|_H^p \right),
 \end{aligned}$$

where  $\tilde{Z}(s) := Z(s + (n-1)\tau) - Z((n-1)\tau)$  is also a cylindrical stable process,  $C$  is a constant not depending on  $n$  and its value is not important and may change from one line to another. Hence, according to (21) and Lemma 2.2,

$$E\|X_{n\tau}(\xi)\|_{\infty}^p \leq (1 + \epsilon)^p K_2^p \|(-A)^{-k}\|^p E\|X_{(n-1)\tau}(\xi)\|_{\infty}^p + C$$

holds. In view of (iii) of (H3) and (21), there exist constants  $\vartheta$  and  $M$  such that, for sufficient small  $\epsilon$ ,  $(1 + \epsilon)^p K_2^p \|(-A)^{-k}\|^p < \vartheta < 1$ , and for any integer  $n \geq 1$ ,

$$\begin{aligned}
 E\|X_{n\tau}(\xi)\|_{\infty}^p &\leq \vartheta E\|X_{(n-1)\tau}(\xi)\|_{\infty}^p + M \\
 &\leq \vartheta^n \|\xi\|_{\infty}^p + M \{1 + \vartheta + \dots + \vartheta^{n-1}\} \\
 &\leq \|\xi\|_{\infty}^p + \frac{M}{1 - \vartheta}.
 \end{aligned} \tag{22}$$

Observe that for any  $t \geq 0$ , there exists an  $n \geq 0$  such that  $t \in [n\tau, (n+1)\tau)$  and

$$E\|X_t(\xi)\|_{\infty}^p \leq E\|X_{(n+1)\tau}(\xi)\|_{\infty}^p + E\|X_{n\tau}(\xi)\|_{\infty}^p.$$

Then the desired assertion (13) follows immediately from (22). □

**Lemma 4.3** *Let the conditions of (H1)-(H4) and (12) hold. Then for any bounded subset  $K$  of  $D([-\tau, 0], H)$ ,*

$$\lim_{t \rightarrow \infty} E\|X_t(\xi) - X_t(\eta)\|_{\infty}^p = 0 \tag{23}$$

uniformly in  $\xi, \eta \in K$ .

*Proof* Since the argument is similar to Lemma 4.2, we only sketch the main proof to point out the difference with that of Lemma 4.2 here. First, we shall prove that

$$\lim_{t \rightarrow \infty} E\|X(t, \xi) - X(t, \eta)\|_H^p = 0$$

uniformly in  $\xi, \eta \in K$ . Following a similar argument to derive (20), we can get

$$\begin{aligned}
 E\|X(t, \xi) - X(t, \eta)\|_H^p &= 4E\|e^{tA}[\xi(0) + g(\xi(-\tau)) - \eta(0) - g(\eta(-\tau))]\|_H^p \\
 &\quad + 4E\|g(X(t - \tau, \xi)) - g(X(t - \tau, \eta))\|_H^p \\
 &\quad + 4E\left\|\int_0^t Ae^{(t-s)A}[g(X(s - \tau, \xi)) - g(X(s - \tau, \eta))] ds\right\|_H^p \\
 &\quad + 4E\left\|\int_0^t e^{(t-s)A}[f(X(s - \tau, \xi)) - f(X(s - \tau, \eta))] ds\right\|_H^p \\
 &\leq \tilde{b}_1 e^{-\lambda pt} + \tilde{b}_2 \sup_{-\tau \leq \theta \leq 0} E\|X(t + \theta, \xi) - X(t + \theta, \eta)\|_H^p \\
 &\quad + \tilde{b}_3 \int_0^t e^{-\lambda(t-s)} \sup_{-\tau \leq \theta \leq 0} E\|X(s + \theta, \xi) - X(s + \theta, \eta)\|_H^p ds, \quad (24)
 \end{aligned}$$

where  $\tilde{b}_1 = 8E\|\xi - \eta\|_\infty^p(1 + K_2^p\|(-A)^{-k}\|^p)$ ,  $\tilde{b}_2 = 4K_2^p\|(-A)^{-k}\|^p$  and

$$\tilde{b}_3 = 4\left(M_{1-k}^p K_2^p \lambda^{(1-pk)} \left|\Gamma\left(\frac{pk-1}{p-1}\right)\right|^{p-1} + \lambda^{1-p} K_1^p\right).$$

As a result, noting the condition (12) and using [18, Lemma 3.1], we derive from (24) that

$$E\|X(t, \xi) - X(t, \eta)\|_H^p \leq M_2 e^{-\mu t}, \quad t \geq -\tau,$$

where  $\mu$  is a positive root of the equation  $(\tilde{b}_2 + (\tilde{b}_3/\lambda - \mu)) = 1$  and  $M_2 = \max\{\tilde{b}_1(\lambda - \mu)/\tilde{b}_3 e^{\mu\tau}, \tilde{b}_1\}$ . Hence,

$$\lim_{t \rightarrow \infty} E\|X(t, \xi) - X(t, \eta)\|_H^p = 0. \quad (25)$$

Now, for  $t \geq 2\tau$  and  $-\tau \leq \theta \leq 0$ , according to the fundamental inequality and assumptions (H1)-(H4), we arrive at

$$\begin{aligned}
 &E\|X_t(\xi) - X_t(\eta) + g(X_{t-\tau}(\xi)) - g(X_{t-\tau}(\eta))\|_\infty^p \\
 &\leq 3E \sup_{-\tau \leq \theta \leq 0} \|e^{(\theta+\tau)A}[X(t - \tau, \xi) - X(t - \tau, \eta) + g(X(t - 2\tau, \xi)) - g(X(t - 2\tau, \eta))]\|_H^p \\
 &\quad + 3E \sup_{-\tau \leq \theta \leq 0} \left\|\int_{t-\tau}^{t+\theta} Ae^{(t+\theta-s)A}[g(X(s - \tau, \xi)) - g(X(s - \tau, \eta))] ds\right\|_H^p \\
 &\quad + 3E \sup_{-\tau \leq \theta \leq 0} \left\|\int_{t-\tau}^{t+\theta} e^{(t+\theta-s)A}[f(X(s - \tau, \xi)) - f(X(s - \tau, \eta))] ds\right\|_H^p \\
 &\leq 6E\|X(t - \tau, \xi) - X(t - \tau, \eta)\|_H^p + 6K_2^p\|(-A)^{-k}\|^p E\|X(t - 2\tau, \xi) - X(t - 2\tau, \eta)\|_H^p \\
 &\quad + 3M_{1-k}^p K_2^p e^{\lambda p\tau} \lambda^{1-pk} \left|\Gamma\left(\frac{pk-1}{p-1}\right)\right|^{p-1} \int_{t-\tau}^t e^{-\lambda(t-s)} E\|X(s - \tau, \xi) - X(s - \tau, \eta)\|_H^p ds \\
 &\quad + 3K_1^p e^{\lambda p\tau} \lambda^{1-p} \int_{t-\tau}^t e^{-\lambda(t-s)} E\|X(s - \tau, \xi) - X(s - \tau, \eta)\|_H^p ds. \quad (26)
 \end{aligned}$$

This, together with (25), yields

$$\lim_{t \rightarrow \infty} E \|X_t(\xi) - X_t(\eta) + g(X_{t-\tau}(\xi)) - g(X_{t-\tau}(\eta))\|_\infty^p = 0. \tag{27}$$

Finally, we claim that the desired equation (23) must hold. If Eq. (23) is not true, then there exist positive constants  $L$  and  $T$ , where, when  $t \geq T$ ,

$$E \|X_t(\xi) - X_t(\eta)\|_\infty^p \geq L.$$

On the other hand, when  $t \rightarrow \infty$ ,

$$\begin{aligned} & E^{\frac{1}{p}} \|X_t(\xi) - X_t(\eta) + g(X_{t-\tau}(\xi)) - g(X_{t-\tau}(\eta))\|_\infty^p \\ & \geq E^{\frac{1}{p}} \|X_t(\xi) - X_t(\eta)\|_\infty^p - K_2 \|(-A)^{-k}\| E^{\frac{1}{p}} \|X_{t-\tau}(\xi) - X_{t-\tau}(\eta)\|_\infty^p \\ & \geq (1 - K_2 \|(-A)^{-k}\|)L. \end{aligned}$$

This obviously contradicts (27). Therefore, (23) holds. □

In what follows we aim to prove the stability in the distribution of Eq. (1). Denote by  $\mathcal{P}(D([-\tau, 0]; H))$  the space of all probability measures on  $D([-\tau, 0]; H)$ . For  $P_1, P_2 \in \mathcal{P}(D([-\tau, 0]; H))$ , define

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{f \in \mathbb{L}} \left| \int_H f(\xi) P_1(d\xi) - \int_H f(\eta) P_2(d\eta) \right|,$$

where, for any  $\xi, \eta \in D([-\tau, 0]; H)$ ,

$$\mathbb{L} = \{f : D([-\tau, 0]; H) \rightarrow R : |f(\xi) - f(\eta)| \leq \|\xi - \eta\| \text{ and } |f(\cdot)| \leq 1\}.$$

**Lemma 4.4** *Let assumptions (H1)-(H4), (13), and (23) hold. Then, for any initial data  $\xi \in D([-\tau, 0]; H)$ ,  $\{p(\xi, t, \cdot) : t \geq 0\}$  is Cauchy in the space  $\mathcal{P}(D([-\tau, 0]; H))$  with the metric  $d_{\mathbb{L}}$ .*

*Proof* For any fixed  $\xi \in D([-\tau, 0]; H)$ , we need to show that for any  $\epsilon > 0$ , there is a  $T_1 > 0$  such that

$$d_{\mathbb{L}}(p(\xi, t + s, \cdot), p(\xi, t, \cdot)) \leq \epsilon \quad \forall t \geq T_1, s > 0. \tag{28}$$

That is,

$$\sup_{f \in \mathbb{L}} |Ef(X_{t+s}(\xi)) - Ef(X_t(\xi))| \leq \epsilon$$

for any  $t \geq T_1$  and  $s > 0$ . Now, for any  $f \in \mathbb{L}$  and  $t, s > 0$ , we compute

$$\begin{aligned} & |Ef(X_{t+s}(\xi)) - Ef(X_t(\xi))| \\ & = |E[E(f(X_{t+s}(\xi)) | \mathcal{F}_s)] - Ef(X_t(\xi))| \\ & = \left| \int_H Ef(X_t(\zeta)) p(\xi, s, d\zeta) - Ef(X_t(\xi)) \right| \end{aligned}$$

$$\begin{aligned} &\leq \int_H |Ef(X_t(\zeta)) - Ef(X_t(\xi))| p(\xi, s, d\zeta) \\ &\leq 2p(\xi, s, \overline{H}_R) + \int_{H_R} |Ef(X_t(\zeta)) - Ef(X_t(\xi))| p(\xi, s, d\zeta), \end{aligned} \tag{29}$$

where  $H_R = \{\xi \in D([-\tau, 0], H) : \|\xi\| \leq R\}$  and  $\overline{H}_R = D([-\tau, 0], H) \setminus H_R$ . By the argument (13), there exists a positive number  $R$  sufficiently large for which

$$p(\xi, s, \overline{H}_R) < \frac{\epsilon}{4}, \quad \forall s > 0. \tag{30}$$

On the other hand, by Lemma 4.3, there exists a  $T_3 > 0$  such that

$$\sup_{f \in \mathbb{L}} |Ef(X_t(\zeta)) - Ef(X_t(\xi))| \leq \frac{\epsilon}{2}, \quad t \geq T_3. \tag{31}$$

Substituting (30), (31) into (29) yields

$$|Ef(X_{t+s}(\xi)) - Ef(X_t(\xi))| \leq \epsilon, \quad t \geq T_3, s > 0.$$

Since  $f$  is arbitrary, the desired inequality (28) must hold. □

Based on the results above, we can now state our main result.

**Theorem 4.1** *Let the assumptions (H1)-(H4) and (12) hold; then the process  $Y(t) = X_t$  is stable in the distribution.*

*Proof* By Definition 4.1, it suffices to prove that there is a probability measure  $\pi(\cdot) \in \mathcal{P}(D([-\tau, 0], H))$  such that for any  $\xi \in D([-\tau, 0], H)$ , the transition probabilities  $\{p(\xi, t, \cdot) : t \geq 0\}$  converge weakly to  $\pi(\cdot)$ . According to the well-known fact that the weak convergence of probability measure is a metric concept (see [19]), we therefore need to show that, for any  $\xi \in D([-\tau, 0], H)$ ,

$$\lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(\xi, t, \cdot), \pi(\cdot)) = 0.$$

By Lemma 4.4,  $\{p(\eta, t, \cdot) : t \geq 0\}$  is Cauchy in the metric space  $\mathcal{P}(D([-\tau, 0], H))$  for any fixed  $\eta \in D([-\tau, 0], H)$ . Therefore, there exists a probability measure  $\pi(\cdot)$  such that

$$\lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(\eta, t, \cdot), \pi(\cdot)) = 0.$$

Furthermore, for any  $\xi \in D([-\tau, 0], H)$ , Lemma 4.3 shows that

$$\lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(\xi, t, \cdot), p(t, \eta, \cdot)) = 0.$$

Hence

$$\begin{aligned} \lim_{t \rightarrow \infty} d_{\mathbb{L}}(p(\xi, t, \cdot), \pi(\cdot)) &\leq \lim_{t \rightarrow \infty} [d_{\mathbb{L}}(p(\xi, t, \cdot), p(t, \eta, \cdot)) + d_{\mathbb{L}}(p(t, \eta, \cdot), \pi(\cdot))] \\ &= 0 \end{aligned}$$

as required. □

To demonstrate the applications of Theorem 4.1, we give an illustrative example, motivated by [20, Example 4.1].

**Example 4.1** Let  $\phi : \mathbb{R} \mapsto \mathbb{R}$  be Lipschitzian, *i.e.*, there exists  $L > 0$  such that  $|\phi(x) - \phi(y)| \leq L|x - y|$ ,  $x, y \in \mathbb{R}$ . Assume further that  $\varphi : [-\tau, 0] \times [0, \pi] \times [0, \pi] \mapsto \mathbb{R}$  is measurable such that  $\varphi(\cdot, \cdot, 0) = \varphi(\cdot, \cdot, \pi) = 0$  and

$$N := \int_0^\pi \int_0^\pi \left( \frac{\partial}{\partial x} \varphi(-\tau, \zeta, x) \right)^2 d\zeta dx < \infty. \tag{32}$$

Consider the following stochastic neutral partial functional differential equation:

$$\begin{aligned} d \left[ u(t, x) + \int_0^\pi \varphi(-\tau, \zeta, x) u(t - \tau, \zeta) d\zeta \right] \\ = \left[ \frac{\partial^2}{\partial x^2} u(t, x) + \phi(u(t - \tau, x)) \right] dt + dZ(t, x), \end{aligned} \tag{33}$$

with the Dirichlet boundary condition

$$u(t, 0) = u(t, \pi) = 0, \quad t \in [0, T],$$

and the initial condition

$$u(\theta, x) = \psi(\theta, x), \quad \theta \in [-\tau, 0], x \in (0, \pi).$$

Let  $H = L^2[0, \pi]$ ;  $A$  is given by

$$\begin{cases} A := \frac{\partial^2}{\partial x^2}, \\ D(A) = H^2(0, \pi) \cap H_0^1(0, \pi), \end{cases}$$

where  $H^k(0, \pi)$ ,  $k = 1, 2$ , represent Sobolev spaces, and  $H_0^1(0, \pi)$  is the subspace of  $H^1(0, \pi)$  of all functions vanishing at 0 and  $\pi$ . Then we get

$$Au = \sum_{n=1}^\infty -n^2 \langle u, e_n \rangle_H e_n,$$

where  $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$ ,  $n = 1, 2, \dots$ ,  $x \in [0, \pi]$ . Let  $Z(t, x) := \sum_{n=1}^\infty n^\beta Z_n(t) e_n(x)$ , where  $\alpha\beta \in (0, 1)$  and  $\{Z_n(t)\}_{n \geq 1}$  is an independent, real-valued, normalized, symmetric  $\alpha$ -stable process sequence. It is trivial to see that  $\delta = \sum_{n=1}^\infty \frac{1}{n^{2(1-\alpha\theta) - \alpha\beta}}$ . Due to  $\alpha\beta \in (0, 1)$ , we have  $1 < 2 - \alpha\beta < 2$ . Hence there exists  $\theta \in (0, 1)$  such that  $1 < 2(1 - \alpha\theta) - \alpha\beta < 2$ , and therefore  $\delta = \sum_{n=1}^\infty \frac{1}{n^{2(1-\alpha\theta) - \alpha\beta}} < \infty$ . In other words, the assumption (H4) holds for such a case.

For  $t \in [0, T]$  and  $x \in [0, \pi]$ , let

$$X(t)(x) := u(t, x), \quad g(X(t - \tau))(x) := \int_0^\pi \varphi(-\tau, \zeta, x) u(t - \tau, \zeta) d\zeta$$

and

$$f(X(t - \tau))(x) := \phi(u(t - \tau, x)).$$

Then Eq. (33) can be rewritten in the form (1). Observe that  $A$  generates a strongly continuous semigroup  $\{e^{tA}\}_{t \in [0, T]}$ , which is compact, analytic and self-adjoint, and

$$e^{tA}u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, e_n \rangle_H e_n, \quad u \in H.$$

Thus (H1) holds and  $\|e^{tA}\| \leq e^{-t}$ . Furthermore note that

$$\begin{aligned} (-A)^{-\frac{1}{2}} \zeta &= \sum_{n=1}^{\infty} \frac{1}{n} \langle \zeta, e_n \rangle e_n, \quad \zeta \in H, \\ (-A)^{\frac{1}{2}} \zeta &= \sum_{n=1}^{\infty} n \langle \zeta, e_n \rangle e_n, \quad \zeta \in D((-A)^{\frac{1}{2}}) \end{aligned} \tag{34}$$

which in particular yields  $\|(-A)^{-\frac{1}{2}}\| = 1$ . As a result, using  $\varphi(\cdot, \cdot, 0) = \varphi(\cdot, \cdot, \pi) = 0$ , together with (32), (34), and Hölder's inequality, we get

$$\begin{aligned} &\|(-A)^{\frac{1}{2}}(g(X(t-\tau)) - g(Y(t-\tau)))\|_H^2 \\ &= \left\| \sum_{n=1}^{\infty} n \langle g(X(t-\tau)) - g(Y(t-\tau)), e_n \rangle_H e_n \right\|_H^2 \\ &= \sum_{n=1}^{\infty} \left( n \int_0^\pi (g(X(t-\tau))(x) - g(Y(t-\tau))(x)) e_n(x) dx \right)^2 \\ &= \sum_{n=1}^{\infty} \left( n \int_0^\pi \int_0^\pi \varphi(-\tau, \zeta, x) C(t-\tau, \zeta) d\zeta e_n(x) dx \right)^2 \\ &= \sum_{n=1}^{\infty} \left( \int_0^\pi \int_0^\pi \frac{\partial}{\partial x} \varphi(-\tau, \zeta, x) C(t-\tau, \zeta) d\zeta \tilde{e}_n(x) dx \right)^2 \\ &= \sum_{n=1}^{\infty} \left\langle \int_0^\pi \frac{\partial}{\partial x} \varphi(-\tau, \zeta, \cdot) C(t-\tau, \zeta) d\zeta, \tilde{e}_n \right\rangle_H^2 \\ &= \left\| \int_0^\pi \frac{\partial}{\partial x} \varphi(-\tau, \zeta, \cdot) C(t-\tau, \zeta) d\zeta \right\|_H^2 \\ &= \int_0^\pi \left( \int_0^\pi \frac{\partial}{\partial x} \varphi(-\tau, \zeta, x) C(t-\tau, \zeta) d\zeta \right)^2 dx \\ &\leq \int_0^\pi \int_0^\pi \left( \frac{\partial}{\partial x} \varphi(-\tau, \zeta, x) \right)^2 d\zeta dx \cdot \int_0^\pi C^2(t-\tau, \zeta) d\zeta \\ &\leq N \|X(t-\tau) - Y(t-\tau)\|_H^2, \end{aligned}$$

where  $C(t) = X(t) - Y(t)$  and  $\tilde{e}_n := \sqrt{\frac{2}{\pi}} \cos nx$ , which is also a complete orthonormal system of  $H$ . Hence, (H3) holds. Consequently, by Theorem 3.1, there exists a unique mild solution to Eq. (33). In addition, Let  $p = \frac{3}{2}$ ,  $k = \frac{1}{2}$ ,  $\lambda = 1$ ,  $K_1 = L$ ,  $K_2 = \sqrt{N}$ ; if  $10(N^{\frac{3}{4}} + M^{\frac{3}{2}} N^{\frac{3}{4}} |\Gamma(-\frac{1}{2})|^{\frac{1}{2}} + L^{\frac{3}{2}}) < 1$ , then by Theorem 4.1, Eq. (33) is stable in the distribution.

#### Competing interests

The authors declare that they have no competing interests.

#### Authors' contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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#### References

1. Bao, J, Hou, Z: Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients. *Comput. Math. Appl.* **59**, 207-214 (2010)
2. Caraballo, T, Real, J, Taniguchi, T: The exponential stability of neutral stochastic delay partial differential equations. *Discrete Contin. Dyn. Syst., Ser. A* **18**, 195-313 (2007)
3. Yuan, C, Mao, X: Asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Stoch. Process. Appl.* **103**, 277-291 (2003)
4. Yuan, C, Zou, J, Mao, X: Stability in distribution of stochastic differential delay equations with Markovian switching. *Syst. Control Lett.* **50**, 195-207 (2003)
5. Tan, L, Jin, W, Hou, Z: Weak convergence of functional stochastic differential equations with variable delays. *Stat. Probab. Lett.* **83**, 2592-2599 (2013)
6. Bao, J, Truman, A, Yuan, C: Stability in distribution of mild solutions to stochastic partial differential delay equations with jumps. *Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci.* **465**, 2111-2134 (2009)
7. Yang, Z, Yin, G: Stability of nonlinear regime switching jump diffusion. *Nonlinear Anal.* **75**, 3854-3873 (2012)
8. Da Prato, G, Zabczyk, J: *Stochastic Equations in Infinite Dimensions*. Encyclopedia of Mathematics and Its Applications, vol. 45. Cambridge University Press, Cambridge (1992)
9. Dong, Z, Xu, L, Zhang, X: Invariant measures of stochastic 2D Navier-Stokes equation driven by  $\alpha$ -stable processes. *Electron. Commun. Probab.* **16**, 678-688 (2011)
10. Priola, E, Zabczyk, J: Structural properties of semilinear SPDEs driven by cylindrical stable processes. *Probab. Theory Relat. Fields* **149**, 97-137 (2011)
11. Xu, L: Ergodicity of the stochastic real Ginzburg-Landau equation driven by  $\alpha$ -stable noises. *Stoch. Process. Appl.* **123**, 3710-3736 (2013)
12. Applebaum, D: *Lévy Processes and Stochastic Calculus*, 2nd edn. Cambridge University Press, Cambridge (2009)
13. Sato, K: *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press, Cambridge (1999)
14. Pazy, A: *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, New York (1983)
15. Mohammed, S-EA: *Stochastic Functional Differential Equations*. Pitman, Boston (1984)
16. Bao, J, Hou, Z, Yuan, C: Stability in distribution of neutral stochastic differential delay equations with Markovian switching. *Stat. Probab. Lett.* **79**, 1663-1673 (2009)
17. Long, S, Teng, L, Xu, D: Global attracting set and stability of stochastic neutral partial functional differential equations with impulses. *Stat. Probab. Lett.* **82**, 1699-1709 (2012)
18. Chen, H: Integral inequality and exponential stability for neutral stochastic partial differential equations with delays. *J. Inequal. Appl.* **2009**, Article ID 297478 (2009)
19. Ikeda, N, Watanabe, S: *Stochastic Differential Equations and Diffusion Process*. North-Holland, Amsterdam (1981)
20. Hernandez, E, Henriquez, HR: Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* **221**, 452-475 (1998)

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