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On a weakly singular quadratic integral equations of Volterra type in Banach algebras

Xiulan Yu[1], Chun Zhu[2] and JinRong Wang[2,3*]

*Correspondence:
wjr9668@126.com
[2]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China
[3]Industrial Internet of Things Engineering Research Center of the Higher Education Institutions of Guizhou Province and School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China
Full list of author information is available at the end of the article

Abstract

In this paper, we present existence and uniqueness theorems of nonnegative, asymptotically stable, and ultimately nondecreasing solutions for weakly singular quadratic integral equations of Volterra type in Banach algebras. The concept of the measure of noncompactness and a fixed point theorem due to Darbo acting in a Banach algebra are the main tools in carrying out our proof. An effective numerical example is given to illustrate our theory results.

MSC: 45G05; 47H30

Keywords: weakly singular quadratic integral equations; Banach algebras; measure of noncompactness; fixed point theorem

1 Introduction

Quadratic integral equations with nonsingular kernels are associated with epidemic models [1–5]. Recently, quadratic integral equations with singular kernels have received a lot of attention because of their useful applications in describing numerous events and problems of the real world. A simple type of quadratic integral equation involving singular kernels in Banach algebras on an unbounded interval can be written as

$$x(t) = (V_1x)(t)(V_2x)(t), \quad t \in \mathbb{R}_+ := [0, \infty), \tag{1}$$

where

$$(V_ix)(t) = m_i(t) + f_i\big(t, x(t)\big) \int_0^t (t-s)^{-\alpha_i} u_i\big(t, s, x(s)\big)\, ds,$$

where $\alpha_i \in (0, 1)$, m_i, f_i, and u_i are functions satisfying certain conditions for $i = 1, 2$. As regards (1), Banaś and Dudek [6] apply the techniques of a certain measure of noncompactness to study the existence of theories in the Banach algebra $BC(\mathbb{R}_+)$.

As pointed out in [6], one would have to explore some sophisticated tools to study quadratic integral equations involving singular kernels in Banach algebras due to such type equations have rather complicated form. More in particular, we have the important condition (m) (see Definition 2.2) related to the operator V_i, $i = 1, 2$, which plays an essential role in the application of the technique of measures of noncompactness in a certain Banach algebras setting.

It seems interesting to ask what happens if the singular $(t-s)^{-\alpha_i}$ is replaced by a certain function of, say, $(t^{\beta_i} - s^{\beta_i})^{\alpha_i - 1} s^{\gamma_i}$ where α_i, β_i, and γ_i are pre-fixed numbers for $i = 1, 2$.

In this paper we continue the work in [6] and apply the explored tools and develops the techniques to study the existence of solutions to (1) with

$$(V_i x)(t) = m_i(t) + \frac{f_i(t,x(t))}{\Gamma(\alpha_i)} \int_0^t \left(t^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} u_i\left(t,s,x(s)\right) ds, \tag{2}$$

in $BC(\mathbb{R}_+)$, where $\Gamma(\cdot)$ is the gamma function, $\alpha_i \in (0,1)$, $\beta_i > 0$, and $\gamma_i \in \mathbb{R}$ are pre-fixed numbers for $i = 1, 2$.

Let us notice that weakly singular kernels appears in the second term of (2) are corresponding to the so-called Erdélyi-Kober type fractional integrals [7] of the function h which is given by

$${}_{EK}I^{\alpha}_{0^+;\sigma,\eta} h(t) := \frac{\sigma t^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t \frac{s^{\sigma\eta+\sigma-1} h(s)\, ds}{(t^\sigma - s^\sigma)^{1-\alpha}}.$$

In the past two decades, differential and integral equations involving fractional integral operators draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. One can also find more details of such fractional equations in physics, viscoelasticity, electrochemistry, and porous media in [8–13] and in the important monographs [7, 14–18].

In the present paper, we apply a certain measure of noncompactness introduced by Banaś and Dudek [6] and develop some useful methods in Wang *et al.* [19] to derive new existence, asymptotically stable, and ultimately nondecreasing of nonnegative solutions to (1) with (2) under the restriction on the parameters $1 > \alpha_i > 0$, $\beta_i > 0$, $\gamma_i > \beta(1 - \alpha_i) - 1$. Moreover, we also give the uniqueness result by requiring g_i to satisfy the Lipschitz continuous condition and addressing $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$. As an application, we give an effective numerical example to illustrate our theoretical results.

Comparing with the corresponding existence results in Olaru [3, 4] and Brestovanská and Medveď [5], we discuss a quadratic integral equation with a class of special singular kernels by using a different method, a fixed point theorem due to Darbo acting in a Banach algebra via the technique of the measure of noncompactness. As a result, we obtain a new and interesting existence result.

Comparing with the corresponding existence result in Banaś and Dudek [6], there are at least three different points: (i) the singular kernels in our equation is more general; (ii) we not only present the existence result but also derive the uniqueness result; (iii) we give an effective numerical example and draw a curve of the unique nonnegative, asymptotically stable and ultimately nondecreasing solution to support our theoretical results.

2 Mathematical preliminary

Let E be a Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B(x,r)$ the closed ball centered at x and with radius r. The symbol B_r stands for the ball $B(\theta,r)$. If $X \subseteq E$ we use $\overline{X}$, $\operatorname{Conv} X$ to denote the closure and convex closure of X, respectively. The symbol $\operatorname{diam} X$ denote the diameter of a bounded set X and $\|X\|$ denotes the norm of X, that is, $\|X\| = \sup\{\|x\| : x \in X\}$. Moreover, we denote by $\mathfrak{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathfrak{N}_E$ its subfamily consisting of all relatively compact sets.

We collect the following definition of a measure of noncompactness.

Definition 2.1 (see [20]) A mapping $\mu : \mathfrak{M}_E \to \mathbb{R}_+$ is said to be a measure of noncompactness in E if it satisfies the following conditions:

(i) The family $\ker\mu = \{X \in \mathfrak{M}_E : \mu(X) = 0\}$ is nonempty and $\ker\mu \subset \mathfrak{N}_E$ where $\ker\mu$ denotes the kernel of the measure of noncompactness μ.
(ii) $X \subset Y \Longrightarrow \mu(X) \le \mu(Y)$.
(iii) $\mu(\overline{X}) = \mu(X)$.
(iv) $\mu(\operatorname{Conv} X) = \mu(X)$.
(v) $\mu(\lambda X + (1-\lambda)Y) \le \lambda\mu(X) + (1-\lambda)\mu(Y)$ for $\lambda \in [0,1]$.
(vi) If (X_n) is a sequence of closed sets from $\mathfrak{M}_E$ such that $X_{n+1} \subset X_n$ ($n = 1, 2, \ldots$) and if $\lim_{n\to\infty}\mu(X_n) = 0$, then the intersection $X_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.

We see that the intersection set X_∞ from (vi) belongs to $\ker\mu$. In fact, since $\mu(X_\infty) \le \mu(X_n)$ for every n, we have $\mu(X_\infty) = 0$.

We assume that the space E has the structure of Banach algebra. In such a case we write xy in order to denote the product of elements $x, y \in E$. Similarly, we denote $XY = \{xy : x \in X, y \in Y\}$.

Now, we recall a useful concept of the condition (m).

Definition 2.2 (see [21]) One says that the measure of noncompactness μ defined on the Banach algebra E satisfies condition (m):

if $\mu(XY) \le \|X\|\mu(Y) + \|Y\|\mu(X)$ for arbitrary sets $X, Y \in \mathfrak{M}_E$.

In what follows we recall some measures of noncompactness in the Banach algebra $BC(\mathbb{R}_+)$ consisting of all real functions defined, continuous, and bounded on $\mathbb{R}_+$. The algebra $BC(\mathbb{R}_+)$ is endowed with the usual supremum norm $\|x\| = \sup\{|x(t)| : t \in \mathbb{R}_+\}$ for $x \in BC(\mathbb{R}_+)$. In fact, the present measures of noncompactness were considered in detail in [21].

We assume that X is an arbitrarily fixed nonempty and bounded subset of the Banach algebra $BC(\mathbb{R}_+)$, that is, $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$. Choose arbitrarily $\varepsilon > 0$ and $T > 0$. For $x \in X$ denote by $\omega^T(x,\varepsilon)$ the modulus of continuity of the function x on the interval $[t_0, T]$, *i.e.*, $\omega^T(x,\varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \le \varepsilon\}$.

Let $\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon) : x \in X\}$, $\omega_0^T(X) = \lim_{\varepsilon\to 0}\omega^T(X,\varepsilon)$ and $\omega_0^\infty(X) = \lim_{T\to\infty}\omega_0^T(X)$. Define $a(X) = \lim_{T\to\infty}\{\sup_{x\in X}\{\sup\{|x(t) - x(s)| : t, s \ge T\}\}\}$. Finally, we set $\mu_a(X) = \omega_0^\infty(X) + a(X)$.

It was shown [21] that the function μ_a is the measure of noncompactness in the algebra $BC(\mathbb{R}_+)$ as introduced in [21]. The kernel $\ker\mu_a$ of this measure contains all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and have finite limits at infinity. Moreover, all functions from the set X tend to their limits with the same rate. Further, it was also proved that the measure of noncompactness μ_a satisfies condition (m) in [21].

For our problem, we consider another measure of noncompactness. In order to define this measure, similarly as above, fix a set $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ and a number $t \in \mathbb{R}_+$. Denote by $X(t)$ the cross-section of the set X at the point t, that is, $X(t) = \{x(t) : x \in X\}$. Denote by $\operatorname{diam} X(t)$ the diameter of $X(t)$. Further, for a fixed $T > 0$ and $x \in X$ denote by $d_T(x)$ the so-called modulus of decrease of the function x on the interval $[T, \infty)$, which is defined by the formula $d_T(x) = \sup\{|x(t) - x(s)| - [x(t) - x(s)] : T \le s < t\}$. We denote $d_T(X) = \sup\{d_T(x) : x \in X\}$, $d_\infty(X) = \lim_{T\to\infty} d_T(X)$.

In a similar way one can define the modulus of increase of function x and the set X (see [21]).

Finally, let us define the set quantity μ_d in the following way:

$$\mu_d(X) = \omega_0^\infty(X) + d_\infty(X) + \lim_{t\to\infty} \sup \operatorname{diam} X(t). \tag{3}$$

Linking the facts established in [21, 22], μ_d is the measure of noncompactness in the algebra $BC(\mathbb{R}_+)$. The kernel $\ker \mu_d$ of this measure consists of all sets $X \in \mathfrak{M}_{BC(\mathbb{R}_+)}$ such that functions belonging to X are locally equicontinuous on $\mathbb{R}_+$ and the thickness of the bundle $X(t)$ formed by functions from X tends to zero at infinity. Moreover, all functions from X are ultimately nondecreasing on $\mathbb{R}_+$ (see [23]).

The measure μ_d has also an additional property.

Lemma 2.3 ([6, Theorem 6]) *The measure of noncompactness μ_d defined by* (3) *satisfies condition* (m) *on the family of all nonempty and bounded subsets X of Banach algebra $BC(\mathbb{R}_+)$ such that functions belonging to X are nonnegative on $\mathbb{R}_+$.*

The measure of noncompactness μ_d defined by (3) allows us to characterize solutions of considered operator equations in terms of the concept of asymptotic stability.

To formulate precisely that concept (see [24]) assume that Ω is a nonempty subset of the Banach algebra $BC(\mathbb{R}_+)$ and $F : \Omega \to BC(\mathbb{R}_+)$ is an operator. Consider the operator equation

$$x(t) = (Fx)(t), \quad t \in \mathbb{R}_+, \tag{4}$$

where $x \in \Omega$.

Definition 2.4 ([6, Definition 7]) One says that solutions of (4) are asymptotically stable if there exists a ball $B(x_0, r)$ in $BC(\mathbb{R}_+)$ such that $B(x_0, r) \cap \Omega \neq \emptyset$, and for each $\varepsilon > 0$ there exists $T > 0$ such that $|x(t) - y(t)| \leq \varepsilon$ for all solutions $x, y \in B(x_0, r) \cap \Omega$ of (4) and for $t \geq T$.

Now, we recall fixed point theorems for operators acting in a Banach algebra and satisfying some conditions expressed with the help of the measure of noncompactness.

Definition 2.5 ([20]) Let Ω be a nonempty subset of a Banach space E, and let $F : \Omega \to E$ be a continuous operator which transforms bounded subsets of Ω onto bounded ones. One says that F satisfies the Darbo condition with a constant k with respect to the measure of noncompactness μ if $\mu(FX) \leq k\mu(X)$ for each $X \in \mathfrak{M}_E$ such that $X \subset \Omega$. If $k < 1$, then F is called a contraction with respect to μ.

Lemma 2.6 ([6, Lemma 4]) *Let E be a Banach algebra and assume that μ is a measure of noncompactness on E satisfying condition* (m). *Assume that Ω is nonempty, bounded, closed, and convex subset of the Banach algebra E, and the operators P and T transform continuously the set Ω into E in such a way that $P(\Omega)$ and $T(\Omega)$ are bounded. Moreover, one assumes that the operator $F = P \cdot T$ transforms Ω into itself. If the operators P and T satisfy on the set Ω the Darbo condition with respect to the measure of noncompactness μ with the constants k_1 and k_2, respectively, then the operator F satisfies on Ω the Darbo condition* *with the constant* $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1$. *Particularly, if* $\|P(\Omega)\|k_2 + \|T(\Omega)\|k_1 < 1$, *then F is a contraction with respect to the measure of noncompactness* μ *and has at least one fixed point in the set* Ω.

Remark 2.7 It can be shown [20] that the set Fix F of all fixed points of the operator F on the set Ω is a member of the kernel $\ker\mu$.

In what follows we recall facts concerning the superposition operator which are drawn from [25]. In order to define this operator assume that $J \subseteq \mathbb{R}$ but $J \neq \emptyset$ and $f : \mathbb{R}_+ \times J \to \mathbb{R}$ is a given function. Denote X_J by all the functions acting from $\mathbb{R}_+$ into J. For any $x \in X_J$, a function $\mathcal{F}$ is defined by

$$(\mathcal{F}x)(t) = f\big(t, x(t)\big), \quad t \in \mathbb{R}. \tag{5}$$

Then the operator $\mathcal{F}$ defined in (5) is called the superposition operator generated by f.

The following result presents a useful property of the superposition operator which is considered in the Banach space $B(\mathbb{R}_+)$ consisting of all real functions defined and bounded on $\mathbb{R}_+$.

Lemma 2.8 (see [23]) *Assume that the following hypotheses are satisfied.*

(C1) *The function* f *is continuous on the set* $\mathbb{R}_+ \times J$.

(C2) *The function* $t \longmapsto f(t,u)$ *is ultimately nondecreasing uniformly with respect to* u *belonging to bounded subintervals of* J, *that is,*

$$\lim_{T\to\infty} \big\{\sup\big\{\big|f(t,u) - f(s,u)\big| - \big[f(t,u) - f(s,u)\big] : t > s > T, u \in J_1\big\}\big\} = 0$$

for any bounded subinterval $J_1 \subseteq J$.

(C3) *For any fixed* $t \in \mathbb{R}_+$ *the function* $u \longmapsto f(t,u)$ *is nondecreasing on* J.

(C4) *The function* $u \longmapsto f(t,u)$ *satisfies a Lipschitz condition*; *that is, there exists a constant* $k > 0$ *such that*

$$\big|f(t,u) - f(t,v)\big| \le k|u - v|$$

for all $t \in \mathbb{R}_+$ *and all* $u, v \in J$.

Then the inequality

$$d_\infty(Fx) \le k d_\infty(x)$$

holds for any function $x \in B(\mathbb{R}_+)$, *where k is the Lipschitz constant from assumption* (iv).

The following basic equality will be used in the sequel.

Lemma 2.9 (see [26]) *Let* α, β, γ, *and p be constants such that* $\alpha > 0$, $p(\gamma - 1) + 1 > 0$, *and* $p(\beta - 1) + 1 > 0$. *Then*

$$\int_0^t \big(t^\alpha - s^\alpha\big)^{p(\beta-1)} s^{p(\gamma-1)}\,ds = \frac{t^\theta}{\alpha}\mathbb{B}\left(\frac{p(\gamma - 1) + 1}{\alpha}, p(\beta - 1) + 1\right), \quad t \in \mathbb{R}_+,$$

where

$$\mathbb{B}(\xi,\eta)=\int_0^1 s^{\xi-1}(1-s)^{\eta-1}\,ds \quad \bigl(Re(\xi)>0, Re(\eta)>0\bigr)$$

is the well-known Beta function and $\theta=p[\alpha(\beta-1)+\gamma-1]+1$.

3 Main results

In order to derive the existence theorem of nonnegative, asymptotically stable and ultimately nondecreasing solution, we consider (1) with (2) under the following assumptions:

(H1) The function m_i, $i=1,2$ is nonnegative, bounded, continuous, and ultimately nondecreasing.

(H2) The function $f_i:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies the conditions (C1)-(C3) of Lemma 2.8 for $i=1,2$.

(H3) The functions f_i, $i=1,2$ satisfy the Lipschitz condition with respect to the second variable; that is, there exists a constant k_i such that

$$\bigl|f_i(t,x)-f_i(t,y)\bigr|\le k_i|x-y|,\quad t\in\mathbb{R}_+, i=1,2.$$

(H4) $u_i:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}$ is a continuous function such that $u_i:\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$, $i=1,2$.

(H5) There exists a continuous and nondecreasing function $G_i:\mathbb{R}_+\to\mathbb{R}_+$ and a bounded and continuous function $g_i:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ such that $u_i(t,s,x)=g_i(t,s)G_i(|x|)$ for $t,s\in\mathbb{R}_+$, $x\in\mathbb{R}$, and $i=1,2$.

(H6) The function $t\to\int_0^t(t^{\beta_i}-s^{\beta_i})^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds$ is bounded on $\mathbb{R}_+$ and

$$\lim_{t\to\infty}\int_0^t\bigl(t^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds=0,\quad i=1,2.$$

(H7) The function g_i, $i=1,2$, satisfies

$$\lim_{T\to\infty}\biggl\{\sup\biggl\{\int_0^{t_2}\bigl\{\bigl|\bigl(t_2^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t_2,s)-\bigl(t_1^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t_1,s)\bigr| - \bigl[\bigl(t_2^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t_2,s)-\bigl(t_1^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t_1,s)\bigr]\bigr\}\,ds:t_2>t_1\ge T\biggr\}\biggr\}=0.$$

Remark 3.1 For some certain $k,\nu,a,b,c,d\in\mathbb{R}$, we can choose $g_i(t,s)=ke^{\nu(t+s)}$ or $g_i(t,s)=\frac{1}{(at+bs+c)^d}$ to guarantee the assumptions (H6) and (H7) hold. For more details, one can see the example in Section 4.

For brevity, we set

$$\overline{F}_i=\sup\bigl\{\bigl|f_i(t,0)\bigr|:t\in\mathbb{R}_+\bigr\},\qquad \overline{g}_i=\sup\bigl\{g_i(t,s):t,s\in\mathbb{R}_+\bigr\},$$

$$\overline{G}_i=\sup\biggl\{\frac{1}{\Gamma(\alpha_i)}\int_0^t\bigl(t^{\beta_i}-s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds:t\in\mathbb{R}_+\biggr\},$$

$$\overline{F}=\max\{\overline{F}_1,\overline{F}_2\},\qquad k=\max\{k_1,k_2\},\qquad m=\max\bigl\{\|m_1\|,\|m_2\|\bigr\}.$$

(H8) There exists a solution $r_0 > 0$ of the following inequality:

$$\left[m + k\overline{G}_1 rG_1(r) + \overline{FG}_1 G_1(r)\right] \times \left[m + k\overline{G}_2 rG_2(r) + \overline{FG}_2 G_2(r)\right] \leq r \tag{6}$$

such that

$$\begin{aligned} L &:= mk\bigl(\overline{G}_1 G_1(r_0) + \overline{G}_2 G_2(r_0)\bigr) + 2k\overline{FG}_1 G_1(r_0)\overline{G}_2 G_2(r_0) \\ &\quad + 2k^2 r_0 \overline{G}_1 G_1(r_0)\overline{G}_2 G_2(r_0) < 1. \end{aligned} \tag{7}$$

Consider the operators on the space $BC(\mathbb{R}_+)$ by defining

$$(F_i x)(t) = f_i\bigl(t, x(t)\bigr),$$

and

$$(U_i x)(t) = \frac{1}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i - 1} s^{\gamma_i} u_i\bigl(t, s, x(s)\bigr)\, ds, \quad i = 1, 2.$$

As a result, we obtain

$$(V_i x)(t) = m_i(t) + (F_i x)(t)(U_i x)(t), \quad t \in \mathbb{R}_+, i = 1, 2. \tag{8}$$

In order to achieve our goal, we present the following results.

Lemma 3.2 *The operators U_i and F_i, $i = 1, 2$ transform continuously the set $\Omega \subseteq BC(\mathbb{R}_+)$ into $BC(\mathbb{R}_+)$ in such a way that $U_i(\Omega)$ and $F_i(\Omega)$ are positive and bounded.*

Proof For any $x \in \Omega$, keeping in mind of our assumptions (H1)-(H5) one can derive the fact that $U_i x$ is nonnegative on $\mathbb{R}_+$, $i = 1, 2$. Moreover, for $t \in \mathbb{R}_+$, linking (2) with the imposed assumptions, we derive

$$\begin{aligned} (V_i x)(t) &\leq m_i(t) + \frac{k_i x(t) + f_i(t, 0)}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i - 1} s^{\gamma_i} u_i\bigl(t, s, x(s)\bigr)\, ds \\ &\leq m_i(t) + \frac{[k_i x(t) + f_i(t, 0)] G_i(\|x\|)}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i - 1} s^{\gamma_i} g_i(t, s)\, ds \\ &\leq \|m_i\| + k_i \overline{G}_i \|x\| G_i\bigl(\|x\|\bigr) + \overline{FG}_i G_i\bigl(\|x\|\bigr), \quad i = 1, 2, \end{aligned} \tag{9}$$

which shows that $V_i x$, $i = 1, 2$, is bounded on $\mathbb{R}_+$.

Next, keeping in mind of the properties of the superposition operator in [27] and (H2) we find that $F_i x$ is continuous and bounded on $\mathbb{R}_+$, $i = 1, 2$.

To show that $V_i x$ is continuous on $\mathbb{R}_+$, $i = 1, 2$, it is sufficient to show that $U_i x$ is continuous on $\mathbb{R}_+$, $i = 1, 2$.

Fix $T > 0$ and $\varepsilon > 0$, we choose any $t_1, t_2 \in [0, T]$ such that $|t_2 - t_1| \leq \varepsilon$. Without loss of generality we assume that $t_1 < t_2$. Since $\gamma_i > \beta_i(1 - \alpha_i) - 1$ and $\alpha_i > 0$ for $i = 1, 2$, we can take $\zeta_i > 1$ such that $\zeta_i \gamma_i > \zeta_i \beta_i(1 - \alpha_i) - 1$ and $\zeta_i(\alpha_i - 1) + 1 > 0$, $i = 1, 2$. Set $\zeta_i^* := \frac{\zeta_i}{\zeta_i - 1}$ $(i = 1, 2)$. By

Lemma 2.9 and the Hölder inequality, we have

$$
\begin{aligned}
&\big|(U_ix)(t_2)-(U_ix)(t_1)\big| \\
&\quad\le \bigg|\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_2,s,x(s)\big)\,ds \\
&\qquad -\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds\bigg| \\
&\qquad +\bigg|\frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds \\
&\qquad -\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds\bigg| \\
&\qquad +\bigg|\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds \\
&\qquad -\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}\big(t_1^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds\bigg| \\
&\quad\le \frac{1}{\Gamma(\alpha_i)}\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\big|u_i\big(t_2,s,x(s)\big)-u_i\big(t_1,s,x(s)\big)\big|\,ds \\
&\qquad +\frac{1}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t_1,s,x(s)\big)\,ds \\
&\qquad +\frac{1}{\Gamma(\alpha_i)}\int_0^{t_1}u_i\big(t_1,s,x(s)\big)s^{\gamma_i}\big[\big(t_1^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}-\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}\big]\,ds \\
&\quad\le \frac{\omega_{\|x\|}^T(u_i,\varepsilon)}{\Gamma(\alpha_i)}\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\,ds+\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\,ds \\
&\qquad +\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\int_0^{t_1}\big[\big(t_1^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}-\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}\big]s^{\gamma_i}\,ds \\
&\quad\le \frac{\omega_{\|x\|}^T(u_i,\varepsilon)}{\beta_i\Gamma(\alpha_i)}T^{\beta_i(\alpha_i-1)+\gamma_i+1}\mathbb{B}\bigg(\frac{\gamma_i+1}{\beta_i},\alpha_i\bigg)+\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\sqrt[\frac{1}{\xi^*}]{t_2-t_1} \\
&\qquad \times\sqrt[\zeta]{\int_{t_1}^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\zeta(\alpha_i-1)}s^{\zeta\gamma_i}\,ds}+\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\bigg[\int_0^{t_1}\big(t_1^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\,ds \\
&\qquad -\int_0^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\,ds\bigg]+\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\int_{t_1}^{t_2}\big(t_2^{\beta_i}-s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\,ds \\
&\quad\le \frac{\omega_{\|x\|}^T(u_i,\varepsilon)}{\beta_i\Gamma(\alpha_i)}T^{\beta_i(\alpha_i-1)+\gamma_i+1}\mathbb{B}\bigg(\frac{\gamma_i+1}{\beta_i},\alpha_i\bigg) \\
&\qquad +\frac{2G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\sqrt[\frac{1}{\xi^*}]{\varepsilon}\sqrt[\zeta]{\frac{T^{\xi\beta_i(\alpha_i-1)+\xi\gamma_i+1}\mathbb{B}(\frac{\xi\gamma_i+1}{\beta_i},\alpha_i)}{\beta_i}} \\
&\qquad +\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\big[t_2^{\beta_i(\alpha_i-1)+\gamma_i+1}-t_1^{\beta_i(\alpha_i-1)+\gamma_i+1}\big]\mathbb{B}\bigg(\frac{\gamma_i+1}{\beta_i},\alpha_i\bigg) \\
&\quad\le \bigg[\frac{\omega_{\|x\|}^T(u_i,\varepsilon)}{\beta_i\Gamma(\alpha_i)}T^{\beta_i(\alpha_i-1)+\gamma_i+1}+\frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\omega^T(h,\varepsilon)\bigg]\mathbb{B}\bigg(\frac{\gamma_i+1}{\beta_i},\alpha_i\bigg) \\
&\qquad +\frac{2G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\sqrt[\frac{1}{\xi^*}]{\varepsilon}\sqrt[\zeta]{\frac{T^{\xi\beta_i(\alpha_i-1)+\xi\gamma_i+1}\mathbb{B}(\frac{\xi\gamma_i+1}{\beta_i},\alpha_i)}{\beta_i}},
\end{aligned}
\tag{10}
$$

where

$$\omega^T_{\|x\|}(u_i,\varepsilon) = \sup\bigl\{\bigl|u_i(t_2,s,x) - u_i(t_1,s,x)\bigr| : t_1,t_2,s \in [0,T],\\ |t_2 - t_1| \le \varepsilon, x \in [-d,d]\bigr\}, \tag{11}$$

and $h(t) := t^{\beta_i(\alpha_i-1)+\gamma_i+1}$ is continuous on $[0,T]$.

The estimate (10) implies that U_i is uniformly continuous on $[0,T] \times [0,T] \times [-\|x\|,\|x\|]$. Thus, we known that U_ix, $i = 1,2$ is continuous on $\mathbb{R}_+$, which guarantees that V_i, $i = 1,2$, transforms the set Ω into Ω due to (10). The proof is done. □

Lemma 3.3 *There exists a* $r_0 > 0$ *such that* $W = V_1V_2$ *transforms* Ω_{r_0} *into* Ω_{r_0} *where*

$$\Omega_{r_0} = \bigl\{x \in BC(\mathbb{R}_+) : 0 \le x(t) \le r_0, t \in \mathbb{R}_+\bigr\}, \tag{12}$$

with

$$\bigl\|V_i(\Omega_{r_0})\bigr\| \le m + k_i\overline{G}r_0G_i(r_0) + \overline{FG}G_i(r_0), \quad i = 1,2. \tag{13}$$

Proof Note that (9) and (6), there exists a $r_0 > 0$ such that $W = V_1V_2$ transforms Ω_{r_0} into Ω_{r_0} and (13) holds. □

Lemma 3.4 *The operator* $W = V_1V_2$ *is a contraction with respect to the measure of noncompactness* μ_d *with the constant L given in* (7).

Proof Choose a nonempty subset X of the set Ω_{r_0} and choose any fixed $T > 0$ and $\varepsilon > 0$. Then, for $x \in X$ and for $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| \le \varepsilon$ and $t_2 \ge t_1$, we have

$$\bigl|(V_ix)(t_2) - (V_ix)(t_1)\bigr| \le \omega^T(m_i,\varepsilon) + \bigl|(F_ix)(t_2) - (F_ix)(t_1)\bigr|\bigl|(U_ix)(t_2)\bigr|\\ + \bigl|(F_ix)(t_1)\bigr|\bigl|(U_ix)(t_2) - (U_ix)(t_1)\bigr|. \tag{14}$$

In a similar way we obtain

$$\bigl|(F_ix)(t_2) - (F_ix)(t_1)\bigr| \le k_i\bigl|x(t_2) - x(t_1)\bigr| + \bigl|f_i\bigl(t_2,x(t_1)\bigr) - f_i\bigl(t_1,x(t_1)\bigr)\bigr|\\ \le k_i\omega^T(x,\varepsilon) + \omega^T_{\|x\|}(f_i,\varepsilon), \tag{15}$$

where

$$\omega^T_d(f_i,\varepsilon) = \sup\bigl\{\bigl|f_i(t_2,x) - f_i(t_1,x)\bigr| : t_1,t_2 \in [0,T], |t_2 - t_1| \le \varepsilon, x \in [-d,d]\bigr\}.$$

Moreover,

$$\bigl|(U_ix)(t_2)\bigr| \le \frac{G_i(\|x\|)}{\Gamma(\alpha_i)}\int_0^{t_2}\bigl(t_2^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds \le G_i\bigl(\|x\|\bigr)\overline{G}_i,$$

$$\bigl|(F_ix)(t_1)\bigr| \le k_i\bigl|x(t_1)\bigr| + \bigl|f_i(t_1,0)\bigr| \le k_ir_0 + \overline{F}_i,$$

for any $t_1, t_2 \in \mathbb{R}_+$.

Further, linking (14), (15) with the above fact, we arrive at

$$\begin{aligned}
&\big|(V_i x)(t_2) - (V_i x)(t_1)\big| \\
&\quad \le \omega^T(m_i,\varepsilon) + \big[k_i\omega^T(x,\varepsilon) + \omega^T_{\|x\|}(f_i,\varepsilon)\big]G_i\big(\|x\|\big)\overline{G}_i + [k_i r_0 + \overline{F}_i] \\
&\qquad \times \Biggl\{\biggl[\frac{\omega^T_{\|x\|}(u_i,\varepsilon)}{\beta_i\Gamma(\alpha_i)}T^{\beta_i(\alpha_i-1)+\gamma_i+1} + \frac{G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\omega^T(h,\varepsilon)\biggr]\mathbb{B}\biggl(\frac{\gamma_i+1}{\beta_i},\alpha_i\biggr) \\
&\qquad + \frac{2G_i(\|x\|)\overline{g}_i}{\Gamma(\alpha_i)}\sqrt[\frac{1}{\xi^*}]{\varepsilon}\sqrt[\zeta]{\frac{T^{\xi\beta_i(\alpha_i-1)+\xi\gamma_i+1}\mathbb{B}(\frac{\xi\gamma_i+1}{\beta_i},\alpha_i)}{\beta_i}}\Biggr\}.
\end{aligned} \tag{16}$$

Clearly, $\omega^T_{\|x\|}(f_i,\varepsilon)$ and $\omega^T_{\|x\|}(u_i,\varepsilon)$ tend to zero as $\varepsilon \to 0$ since the functions f_i and u_i are uniformly continuous on the set $[0,T]\times[-\|x\|,\|x\|]$ and $[0,T]\times[0,T]\times[-\|x\|,\|x\|]$, respectively. Hence we have

$$\omega_0^T(V_i X) \le f_i\overline{G}_i G_i(r_0)\omega_0^T(X) \tag{17}$$

and

$$\omega_0^\infty(V_i X) \le f_i\overline{G}_i G_i(r_0)\omega_0^\infty(X). \tag{18}$$

Next, choose any $x,y \in X$ and $t \in \mathbb{R}_+$. Keeping in mind our assumptions, we obtain

$$\begin{aligned}
&\big|(V_i x)(t) - (V_i y)(t)\big| \\
&\quad \le \frac{|f_i(t,x(t)) - f_i(t,y(t))|}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}u_i\big(t,s,x(s)\big)\,ds \\
&\qquad + \frac{f_i(t,y(t))}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}\big|u_i\big(t,s,x(s)\big) - u_i\big(t,s,y(s)\big)\big|\,ds \\
&\quad \le \frac{k_i|x(t) - y(t)|G_i(\|x\|)}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds \\
&\qquad + \frac{k_i y(t) + f_i(t,0)}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\big[G_i\big(x(s)\big) - G_i\big(y(s)\big)\big]\,ds \\
&\quad \le \frac{k_i|x(t) - y(t)|G_i(r_0)}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds \\
&\qquad + \frac{(k_i r_0 + \overline{F}_i)2G_i(r_0)}{\Gamma(\alpha_i)}\int_0^t \big(t^{\beta_i} - s^{\beta_i}\big)^{\alpha_i-1}s^{\gamma_i}g_i(t,s)\,ds.
\end{aligned} \tag{19}$$

Now using (H6), we derive

$$\lim_{t\to\infty}\sup\operatorname{diam}(V_i X)(t) = 0. \tag{20}$$

In what follows, we prove that V_i is continuous on Ω_{r_0}.

Fix $\varepsilon > 0$ and take $x,y \in \Omega_{r_0}$ such that $\|x-y\| \le \varepsilon$. In view of (20) we know that we may find a number $T > 0$ such that for any $t \ge T$ we get

$$\big|(V_i x)(t) - (V_i y)(t)\big| \le \varepsilon.$$

On the other hand, for any $t \in [0, T]$, we have

$$\begin{aligned}
&\bigl|(V_ix)(t) - (V_iy)(t)\bigr| \\
&\quad \le \frac{|f_i(t,x(t)) - f_i(t,y(t))|}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} u_i\bigl(t,s,x(s)\bigr)\,ds \\
&\qquad + \frac{f_i(t,y(t))}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} \bigl|u_i\bigl(t,s,x(s)\bigr) - u_i\bigl(t,s,y(s)\bigr)\bigr|\,ds \\
&\quad \le \frac{k_i|x(t) - y(t)|G_i(\|x\|)}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t,s)\,ds \\
&\qquad + \frac{[k_iy(t) + f_i(t,0)]\omega_{r_0}^T(u_i,\varepsilon)}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i}\,ds \\
&\quad \le \frac{k_i\varepsilon\overline{G}_iG_i(r_0)}{\Gamma(\alpha_i)} + \frac{k_ir_0 + \overline{F}}{\beta_i\Gamma(\alpha_i)} T^{\beta_i(\alpha_i-1)+\gamma_i+1}\mathbb{B}\biggl(\frac{\gamma_i+1}{\beta_i},\alpha_i\biggr)\omega_{r_0}^T(u_i,\varepsilon),
\end{aligned} \tag{21}$$

where we denoted

$$\omega_{r_0}^T(u_i,\varepsilon) = \sup\bigl\{\bigl|u_i(t,s,x) - u_i(t,s,y)\bigr| : t,s \in [0,T], x,y \in [-r_0,r_0], |x-y| \le \varepsilon\bigr\}.$$

By using the uniform continuity of the function u_i on $[0,T] \times [0,T] \times [r_0,r_0]$, we derive

$$\omega_{r_0}^T(u_i,\varepsilon) \quad \text{as } \varepsilon \to 0.$$

This shows that we can find $T > 0$ such that $\omega_{r_0}^T(u_i,\varepsilon)$ is sufficiently small for $t \ge T$ and $i = 1,2$.

Now, we take any fixed $T > 0$ and choose t_1, t_1 such that $t_2 > t_1 \ge T$. Then, for any $x \in X$, we obtain

$$\begin{aligned}
&\bigl|(V_ix)(t_2) - (V_ix)(t_1)\bigr| - \bigl[(V_ix)(t_2) - (V_ix)(t_1)\bigr] \\
&\quad \le d_T(m_i) + d_T(F_ix)(V_ix)(t_2) \\
&\qquad + (F_ix)(t_1)\bigl\{\bigl|(U_ix)(t_2) - (U_ix)(t_1)\bigr| - \bigl[(U_ix)(t_2) - (U_ix)(t_1)\bigr]\bigr\}.
\end{aligned} \tag{22}$$

Note that

$$\begin{aligned}
&\bigl|(U_ix)(t_2) - (U_ix)(t_1)\bigr| - \bigl[(U_ix)(t_2) - (U_ix)(t_1)\bigr] \\
&\quad \le \frac{1}{\Gamma(\alpha_i)}\biggl|\int_0^{t_2} \bigl(t_2^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t_2,s)G_i\bigl(x(s)\bigr)\,ds \\
&\qquad - \int_0^{t_2} \bigl(t_1^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t_1,s)G_i\bigl(x(s)\bigr)\,ds\biggr| \\
&\qquad + \frac{1}{\Gamma(\alpha_i)}\biggl|\int_{t_1}^{t_2} \bigl(t_1^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t_1,s)G_i\bigl(x(s)\bigr)\,ds\biggr| \\
&\qquad - \frac{1}{\Gamma(\alpha_i)}\biggl[\int_0^{t_2} \bigl(t_2^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t_2,s)G_i\bigl(x(s)\bigr)\,ds \\
&\qquad - \int_0^{t_2} \bigl(t_1^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t_1,s)G_i\bigl(x(s)\bigr)\,ds\biggr]
\end{aligned}$$

$$
\begin{aligned}
&- \frac{1}{\Gamma(\alpha_i)} \int_{t_1}^{t_2} \left(t_1^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_1, s) G_i\left(x(s)\right) ds \\
&\le \frac{1}{\Gamma(\alpha_i)} \left\{ \int_0^{t_2} \left| \left(t_2^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_2, s) G_i\left(x(s)\right) \right. \right. \\
&\quad \left. - \left(t_1^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_1, s) G_i\left(x(s)\right) \right| ds \\
&\quad - \int_0^{t_2} \left[\left(t_2^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_2, s) G_i\left(x(s)\right) \right. \\
&\quad \left. \left. - \left(t_1^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_1, s) G_i\left(x(s)\right) \right] ds \right\} \\
&\le \frac{G_i(\|x\|)}{\Gamma(\alpha_i)} \int_0^{t_2} \left\{ \left| \left(t_2^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_2, s) - \left(t_1^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_1, s) \right| \right. \\
&\quad \left. - \left[\left(t_2^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_2, s) - \left(t_1^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t_1, s) \right] \right\} ds. \qquad (23)
\end{aligned}
$$

By (H1), (H5), and (H7) and (22), we obtain

$$d_\infty(V_i x) \le d_\infty(F_i x) G_i(r_0) \overline{G}_i, \quad i = 1, 2.$$

Hence, in view of Lemma 2.8, we derive

$$d_\infty(V_i x) \le k_i \overline{G}_i G_i(r_0) d_\infty(x), \quad i = 1, 2.$$

Linking (23), (15), (18), and (22), we obtain

$$\mu_d(V_i X) \le k_i \overline{G}_i G_i(r_0) \mu_d(X), \quad i = 1, 2.$$

It comes from (13) and (7) we obtain

$$L := \left\| V_1(\Omega_{r_0}) \right\| k_2 + \left\| V_2(\Omega_{r_0}) \right\| k_1 < 1,$$

which implies that W is a contraction with respect to the measure of noncompactness μ_d with the constant $0 < L < 1$. This completes the proof. □

Now we are ready to state the main result in this paper.

Theorem 3.5 *Let the assumptions* (H1)-(H8) *be satisfied. Then* (1) *with* (2) *has at least one solution $x = x(\cdot)$ in the space $BC(\mathbb{R}_+)$. Moreover, this solution is nonnegative, asymptotically stable, and ultimately nondecreasing.*

Proof By Lemmas 3.2-3.4, one can see that all the assumptions in Lemma 2.6 are satisfied. Thus, one can infer that the operator W has at least one fixed point $x \in \Omega_{r_0}$. Due to Remark 2.7 we know that x is nonnegative on $\mathbb{R}_+$, asymptotically stable, and ultimately nondecreasing. The proof is done. □

To end this section, we establish some sufficient conditions to derive the uniqueness of solution.

Theorem 3.6 *Let the assumptions of Theorem* 3.5 *be satisfied. There exists a positive constant* h_i *such that*

$$\left|u(t,s,x) - u(t,s,y)\right| \le h_i|x-y|, \quad i = 1,2 \tag{24}$$

for any $t \in \mathbb{R}_+$ *and all* $x, y \in \Omega_{r_0}$, *where* Ω_{r_0} *is defined in* (12).

Then for

$$\alpha_1 = \alpha_2 \in (0,1), \qquad \beta_1 = \beta_2 \in (0,+\infty), \qquad \gamma_1 = \gamma_2 \in \bigl(\beta_1(1-\alpha_1) - 1, +\infty\bigr),$$

(1) *with* (2) *has a unique nonnegative, asymptotically stable, and ultimately nondecreasing solution.*

Proof Suppose that y be another nonnegative and nondecreasing solution of (1). Then y satisfies the following integral equation:

$$y(t) = (V_1y)(t)(V_2y)(t), \quad t \in \mathbb{R}_+.$$

Note that

$$\begin{aligned}\left|x(t) - y(t)\right| &\le \left|(V_1x)(t)(V_2x)(t) - (V_1y)(t)(V_2x)(t)\right| \\ &\quad + \left|(V_1y)(t)(V_2x)(t) - (V_1y)(t)(V_2y)(t)\right| \\ &\le \left|(V_2x)(t)\right|\left|(V_1x)(t) - (V_1y)(t)\right| + \left|(V_1y)(t)\right|\left|(V_2x)(t) - (V_2y)(t)\right|, \end{aligned} \tag{25}$$

where

$$\begin{aligned}&\left|(V_ix)(t) - (V_iy)(t)\right| \\ &\quad \le \frac{|f_i(t,x(t)) - f_i(t,y(t))|}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} u_i\bigl(t,s,x(s)\bigr)\,ds \\ &\qquad + \frac{f_i(t,y(t))}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} \left|u_i\bigl(t,s,x(s)\bigr) - u_i\bigl(t,s,y(s)\bigr)\right|\,ds \\ &\quad \le \frac{k_i|x(t) - y(t)|G_i(\|x\|)}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} g_i(t,s)\,ds \\ &\qquad + \frac{h_i[k_iy(t) + f_i(t,0)]}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} \left|x(s) - y(s)\right|\,ds \\ &\quad \le \frac{k_i\overline{G}_iG_i(r_0)}{\Gamma(\alpha_i)}\left|x(t) - y(t)\right| \\ &\qquad + \frac{h_i(k_ir_0 + \overline{F})}{\Gamma(\alpha_i)} \int_0^t \bigl(t^{\beta_i} - s^{\beta_i}\bigr)^{\alpha_i-1} s^{\gamma_i} \left|x(s) - y(s)\right|\,ds, \quad i = 1,2. \end{aligned} \tag{26}$$

Using (26) and (9) in (25), we obtain

$$\begin{aligned}\left|x(t) - y(t)\right| &\le L\left|x(t) - y(t)\right| \\ &\quad + \frac{2h(kr_0 + \overline{F})[m + k\overline{G}r_0G + \overline{FG}G]}{\Gamma(\alpha_1)} \int_0^t \bigl(t^{\beta_1} - s^{\beta_1}\bigr)^{\alpha_1-1} s^{\gamma_1} \left|x(s) - y(s)\right|\,ds\end{aligned}$$

for any $t \in \mathbb{R}_+$ and all $x, y \in \Omega_{r_0}$, where L is given in (7), $\overline{G} = \max\{\overline{G}_1, \overline{G}_2\}$, $G = \max\{G_1(r_0), G_2(r_0)\}$, and $h = \max\{h_1, h_2\}$.

Note that (7), we have

$$\begin{aligned}
&(1-L)\left|x(t)-y(t)\right| \\
&\quad\le \frac{2h(kr_0+\overline{F})[m+k\overline{G}r_0G+\overline{FG}G]}{\Gamma(\alpha_1)} \int_0^t \left(t^{\beta_1}-s^{\beta_1}\right)^{\alpha_1-1} s^{\gamma_1}\left|x(s)-y(s)\right| ds \\
&\quad\le \frac{2h(kr_0+\overline{F})[m+k\overline{G}r_0G+\overline{FG}G]}{\Gamma(\alpha_1)} \sqrt[\zeta]{\int_0^t \left(t^{\beta_1}-s^{\beta_1}\right)^{\zeta(\alpha_1-1)} s^{\zeta\gamma_1}\, ds} \\
&\qquad\times \sqrt[\zeta^*]{\int_0^t \left|x(s)-y(s)\right|^{\zeta^*} ds} \\
&\quad\le \frac{2h(kr_0+\overline{F})[m+k\overline{G}r_0G+\overline{FG}G]}{\Gamma(\alpha_1)} \\
&\qquad\times \sqrt[\zeta]{\frac{t^{\zeta\beta_1(\alpha_1-1)+\zeta\gamma_1+1}}{\beta_1}\mathbb{B}\left(\frac{\zeta\gamma_1+1}{\beta_1}, \zeta(\alpha_1-1)+1\right)} \sqrt[\zeta^*]{\int_0^t \left|x(s)-y(s)\right|^{\zeta^*} ds},
\end{aligned}$$

where ζ and ζ^* are defined in the proof of Lemma 3.2.

In view of (7), we can rewrite the above inequality to

$$z(t) \le \hat{c}(t)\int_0^t z(s)\, ds \le \left(1+\hat{c}(t)\right)\int_0^t z(s)\, ds, \tag{27}$$

where $z(t) := |x(t)-y(t)|^{\zeta^*}$ and

$$\begin{aligned}
\hat{c}(t) :&= \frac{2^{\zeta^*}h^{\zeta^*}(kr_0+\overline{F})^{\zeta^*}[m+k\overline{G}r_0G+\overline{FG}G]^{\zeta^*}}{(1-L)^{\zeta^*}\Gamma(\alpha_1)^{\zeta^*}} \\
&\quad\times \left(\frac{t^{\zeta\beta_1(\alpha_1-1)+\zeta\gamma_1+1}}{\beta_1}\mathbb{B}\left(\frac{\zeta\gamma_1+1}{\beta_1}, \zeta(\alpha_1-1)+1\right)\right)^{\frac{\zeta^*}{\zeta}}.
\end{aligned}$$

From (27), we get

$$\frac{z(t)}{1+\hat{c}(t)} \le \int_0^t \left(1+\hat{c}(s)\right)\left[\frac{z(s)}{1+\hat{c}(s)}\right] ds,$$

and the Gronwall inequality implies $\frac{z(t)}{1+\hat{c}(t)} = 0$, so $z(t) = 0$. This completes the proof. □

4 An example

Motivated by Example 11 in [6], we treat a numerical example to illustrate the main results. Consider the following quadratic fractional integral equation:

$$x(t) = (V_1x)(t)(V_2x)(t), \tag{28}$$

where

$$(V_1x)(t) = \frac{2t}{5t+1} + \frac{\frac{2}{3}\arctan(t^2+x(t))}{\Gamma(\frac{2}{3})}\int_0^t \left(t^{\frac{1}{2}}-s^{\frac{1}{2}}\right)^{-\frac{1}{3}} se^{-t-s}x^2(t)\, ds, \tag{29}$$

$$(V_2x)(t) = \frac{1-e^{-2t}}{3} + \frac{\frac{1}{2}\ln(x(t)+1)}{\Gamma(\frac{2}{3})}\int_0^t \frac{x^4(t)(t^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t+s+2)^3}\,ds, \tag{30}$$

for $t \in \mathbb{R}_+$.

Clearly, $\alpha_1 = \alpha_2 = \frac{2}{3}$, $\beta_1 = \beta_2 = \frac{1}{2}$, $\gamma_1 = \gamma_2 = 1$, and the functions in (1) have the form $m_1 = \frac{2t}{5t+1}$, $m_2 = \frac{1-e^{-2t}}{3}$, $f_1(t,x) = \frac{2}{3}\arctan(t^2 + x(t))$, $f_2(t,x) = \frac{1}{2}\ln(x(t)+1)$, $u_1(t,s,x) = e^{-t-s}x^2$, $u_2(t,s,x) = \frac{x^4}{(t+s+2)^3}$.

In what follows, we check that the above functions will satisfy all the assumptions of Theorem 3.5.

Step 1, the function m_i, $i = 1,2$ is nonnegative, bounded, and continuous on $\mathbb{R}_+$. Since m_1 and m_2 are increasing on $\mathbb{R}_+$, they must be ultimately nondecreasing on $\mathbb{R}_+$. Meanwhile, $\|m_1\| = \frac{2}{5}$ and $\|m_2\| = \frac{1}{3}$. Thus, (H1) holds.

Step 2, f_i, $i = 1,2$ transform continuously the set $\mathbb{R}_+ \times \mathbb{R}_+$ into $\mathbb{R}_+$. Moreover, f_1 is nondecreasing with respect to both variables and satisfies the Lipschitz condition with the constant $k_1 = \frac{2}{3}$. Similarly, the function $f_2 = f_2(t,x)$ is increasing with respect to x and satisfies the Lipschitz condition with the constant $k_2 = \frac{1}{2}$. Also, $\overline{F}_1 = \frac{\pi}{3}$, $\overline{F}_2 = 0$. Thus, f_1 and f_2 satisfy (H2) and (H3).

Step 3, $u_i(t,s,x)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$ and transforms $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$. Meanwhile, $u_i(t,s,x) = g_i(t,s)G_i(|x|)$, $i = 1,2$, where $g_1(t,s) = e^{-t-s}$, $G_1(x) = x^2$, $g_2(t,s) = \frac{1}{(t+s+2)^3}$, and $G_2(x) = x^4$. It is easily seen that (H4) and (H5) are satisfied for u_1 and u_2.

Step 4, one has

$$\int_0^t \left(t^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}} s e^{-t-s}\,ds \le e^{-t}\int_0^t \left(t^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}} s\,ds = 2\mathbb{B}\left(4, \frac{2}{3}\right)e^{-t}t^{\frac{11}{6}},$$

and

$$\int_0^t \frac{(t^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t+s+2)^3}\,ds \le \frac{1}{(t+2)^3}\int_0^t \left(t^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}} s\,ds = \frac{2\mathbb{B}(4,\frac{2}{3})t^{\frac{11}{6}}}{(t+2)^3}.$$

Thus,

$$\lim_{t\to\infty}\int_0^t \left(t^{\beta_i} - s^{\beta_i}\right)^{\alpha_i - 1} s^{\gamma_i} g_i(t,s)\,ds = 0, \quad i = 1,2.$$

So we find that (H6) is satisfied.

Step 5, for $0 < T \le t_1 < t_2$, we get

$$\begin{aligned}
&\int_0^{t_2}\bigl\{\bigl|\left(t_2^{\beta_1} - s^{\beta_1}\right)^{\alpha_1-1}s^{\gamma_1}g_1(t_2,s) - \left(t_1^{\beta_1} - s^{\beta_1}\right)^{\alpha_1-1}s^{\gamma_1}g_1(t_1,s)\bigr| \\
&\qquad - \bigl[\left(t_2^{\beta_1} - s^{\beta_1}\right)^{\alpha_1-1}s^{\gamma_1}g_1(t_2,s) - \left(t_1^{\beta_1} - s^{\beta_1}\right)^{\alpha_1-1}s^{\gamma_1}g_1(t_1,s)\bigr]\bigr\}\,ds \\
&\quad = \int_0^{t_2}\bigl\{\bigl|\left(t_2^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_2-s} - \left(t_1^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_1-s}\bigr| \\
&\qquad - \bigl[\left(t_2^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_2-s} - \left(t_1^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_1-s}\bigr]\bigr\}\,ds \\
&\quad = 2\int_0^{t_2}\bigl[\left(t_1^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_1-s} - \left(t_2^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}se^{-t_2-s}\bigr]\,ds \\
&\quad \le 2e^{-t_1}\int_0^{t_2}\left(t_1^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}s\,ds - 2e^{-2t_2}\int_0^{t_2}\left(t_2^{\frac{1}{2}} - s^{\frac{1}{2}}\right)^{-\frac{1}{3}}s\,ds
\end{aligned}$$

$$
\begin{aligned}
&\le 2e^{-t_1}\int_0^{t_1}\big(t_1^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s\,ds+2e^{-t_1}\int_{t_1}^{t_2}\big(t_1^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s\,ds-4\mathbb{B}\left(4,\frac{2}{3}\right)e^{-2t_2}t_2^{\frac{11}{6}}\\
&\le 4\mathbb{B}\left(4,\frac{2}{3}\right)\big[e^{-t_1}t_1^{\frac{11}{6}}-e^{-2t_2}t_2^{\frac{11}{6}}\big]+4e^{-t_1}\int_{t_1}^{t_2}\big(t_1^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s^{\frac{3}{2}}\,ds^{\frac{1}{2}}\\
&\le 4\mathbb{B}\left(4,\frac{2}{3}\right)\big[e^{-t_1}t_1^{\frac{11}{6}}-e^{-2t_2}t_2^{\frac{11}{6}}\big]-12e^{-t_1}t_2^{\frac{3}{2}}\big(t_1^{\frac{1}{2}}-t_2^{\frac{1}{2}}\big)^{\frac{2}{3}}\\
&\le 4\mathbb{B}\left(4,\frac{2}{3}\right)\big[e^{-t_1}t_1^{\frac{11}{6}}-e^{-2t_2}t_2^{\frac{11}{6}}\big]\to 0\quad \text{as } t_2>t_1\to\infty.
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
&\int_0^{t_2}\big\{\big|\big(t_2^{\beta_2}-s^{\beta_2}\big)^{\alpha_2-1}s^{\gamma_2}g_2(t_2,s)-\big(t_1^{\beta_2}-s^{\beta_2}\big)^{\alpha_2-1}s^{\gamma_2}g_2(t_1,s)\big|\\
&\quad-\big[\big(t_2^{\beta_2}-s^{\beta_2}\big)^{\alpha_2-1}s^{\gamma_2}g_2(t_2,s)-\big(t_1^{\beta_2}-s^{\beta_2}\big)^{\alpha_2-1}s^{\gamma_2}g_2(t_1,s)\big]\big\}\,ds\\
&=\int_0^{t_2}\left\{\left|\frac{(t_2^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_2+s+2)^3}-\frac{(t_1^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_1+s+2)^3}\right|\right.\\
&\quad\left.-\left[\frac{(t_2^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_2+s+2)^3}-\frac{(t_1^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_1+s+2)^3}\right]\right\}ds\\
&\le 2\int_0^{t_2}\left[\frac{(t_1^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_1+s+2)^3}-\frac{(t_2^{\frac{1}{2}}-s^{\frac{1}{2}})^{-\frac{1}{3}}s}{(t_2+s+2)^3}\right]ds\\
&\le\frac{2}{(t_1+2)^3}\int_0^{t_2}\big(t_1^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s\,ds-\frac{2}{(2t_2+2)^3}\int_0^{t_2}\big(t_2^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s\,ds\\
&=\frac{4\mathbb{B}(1,\frac{2}{3})t_1^{\frac{11}{6}}}{(t_1+2)^3}-\frac{4\mathbb{B}(1,\frac{2}{3})t_2^{\frac{11}{6}}}{(2t_2+2)^3}+\frac{2}{(t_1+2)^3}\int_{t_1}^{t_2}\big(t_1^{\frac{1}{2}}-s^{\frac{1}{2}}\big)^{-\frac{1}{3}}s\,ds\\
&=\frac{4\mathbb{B}(1,\frac{2}{3})t_1^{\frac{11}{6}}}{(t_1+2)^3}-\frac{4\mathbb{B}(1,\frac{2}{3})t_2^{\frac{11}{6}}}{(2t_2+2)^3}+\frac{12t_2^{\frac{3}{2}}(t_1^{\frac{1}{2}}-t_2^{\frac{1}{2}})^{\frac{2}{3}}}{(t_1+2)^3}\to 0\quad\text{as } t_2>t_1\to\infty.
\end{aligned}
$$

Thus, we see that (H7) is satisfied.

Step 6, using the above facts we find $m=\max\{\|m_1\|,\|m_2\|\}=\frac{2}{5}$, $k=\max\{k_1,k_2\}=\frac{2}{3}$, $\overline{F}=\max\{\overline{F}_1,\overline{F}_2\}=\frac{\pi}{3}$ and $\overline{G}_1=0.3962$, $\overline{G}_2=0.0489$. Thus, the first inequality in (H8) reduces to

$$
\left(\frac{2}{5}+\frac{2}{3}\overline{G}_1r^3+\frac{\pi}{3}\overline{G}_1r^2\right)\left(\frac{2}{5}+\frac{2}{3}\overline{G}_2r^5+\frac{\pi}{3}\overline{G}_2r^4\right)\le r.
$$

One can verify that $r_0=1$ is a solution of the above inequality such that it satisfies also the second inequality in (H8)

$$
\frac{2}{5}\times\frac{2}{3}\big(\overline{G}_1r_0^2+\overline{G}_2r_0^4\big)+\frac{4\pi}{9}\overline{G}_1\overline{G}_2r_0^6+\left(\frac{2}{3}\right)^2\overline{G}_1\overline{G}_2r_0^7=0.1544<1.
$$

As a result, all the assumptions in Theorem 3.5 are satisfied. Moreover, (24) in Theorem 3.6 is also satisfied. Thus, (28) with V_1, V_2 in (29) and (30) has a unique solution $x\in\Omega_1$ where $\Omega_1=\{x\in BC(\mathbb{R}_+):0\le x(t)\le 1 \text{ for } t\in\mathbb{R}_+\}$, which is asymptotically stable and ultimately nondecreasing.

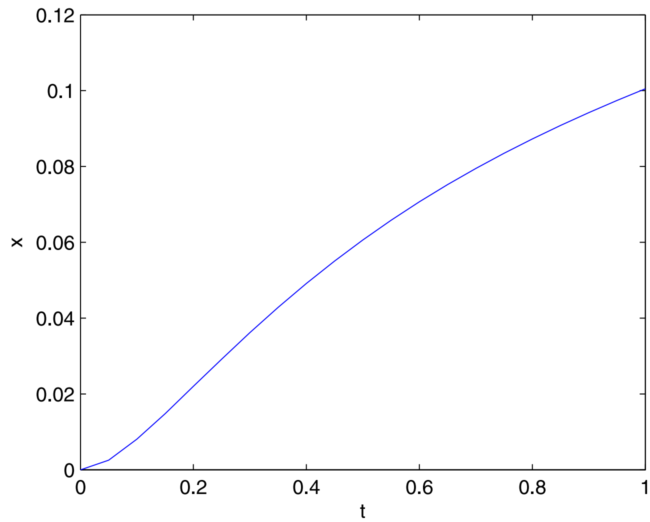

Figure 1 The unique asymptotically stable and ultimately nondecreasing solution.

The unique asymptotically stable and ultimately nondecreasing solution of (28) with (29) and (30) is displayed in Figure 1.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
This work was carried out in collaboration between all authors. JRW raises these interesting problems in this research. JRW, ZC and XLY proved the theorems, interpreted the results and wrote the article. All authors defined the research theme, read and approved the manuscript.

Author details
[1]College of Applied Mathematics, Shanxi University of Finance and Economics, Taiyuan, Shanxi 030031, P.R. China. [2]Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, P.R. China. [3]Industrial Internet of Things Engineering Research Center of the Higher Education Institutions of Guizhou Province and School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, P.R. China.

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