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Ground state periodic solutions for Duffing equations with superlinear nonlinearities

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Abstract

In this paper, we study a general second order differential equation with superlinear nonlinearity. We obtain ground state and geometrically distinct periodic solutions of this equation by a generalized Nehari manifold approach. In particular, our result extends some existing ones.

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1 Introduction and main result

In the past decades, many authors have studied the autonomous equation

$$\begin{cases} x''(t) - a(t) + g(x) = 0, & t \in \mathbb{R}, \\ x(0) = x(T), \quad x'(0) = x'(T), & T > 0, \end{cases} \tag{1.1}$$

where $a(t) \in C(\mathbb{R}, \mathbb{R})$ and $x(t) \in C^2(\mathbb{R}, \mathbb{R})$. For example, many authors have obtained the existence and multiplicity of periodic solutions by various methods, such as a generalized form of the Poincaré-Birkhoff theorem, critical point theory, phase-plane analysis combined with shooting methods or fixed point theorems of planar homeomorphisms, and continuation methods based on degree theory; see [1–7] and the references therein. Some authors [8, 9] have obtained the existence of infinitely many periodic and subharmonic solutions of (1.1) by using the Poincaré-Birkhoff theorem or Moser's twist theorem [10].

By using the coincidence degree theory of Mawhin [11], some authors [12–15] have obtained the existence of at least one positive periodic solution for the following non-autonomous equation:

$$\begin{cases} x''(t) - a(t) + g(t,x) = 0, & t \in \mathbb{R}, \\ x(0) = x(T), \quad x'(0) = x'(T), & T > 0, \end{cases} \tag{1.2}$$

where g satisfies some strong force condition near $x = 0$. If $a(t) \equiv 0$, then (1.2) becomes

$$\begin{cases} x''(t) + g(t,x) = 0, & t \in \mathbb{R}, \\ x(0) = x(T), \quad x'(0) = x'(T), & T > 0. \end{cases} \tag{1.3}$$

Torres [16] has proved (1.3) with $g(t,x) = -f(t,x)$ having one positive or negative solution.

However, in some cases in mathematical physics, the global nonnegative of $a(t)$ (*i.e.*, $a(t) \geq 0$, $\forall t \in \mathbb{R}$) is not satisfied, thus it is necessary for us to study the case that $a(t)$ is not uniformly nonnegative for all $t \in \mathbb{R}$. Therefore, we shall study the existence of infinitely many T-periodic solutions of the following general second order differential equation:

$$\begin{cases} x''(t) - a(t)x(t) + g(t,x) = 0, & t \in \mathbb{R}, \\ x(0) = x(T), \qquad x'(0) = x'(T), & T > 0. \end{cases} \tag{1.4}$$

Here, we need not assume that $g \in L^1(\mathbb{R}, \mathbb{R})$.

We are interested in the following case:

(V_1) $a(t) \in C(\mathbb{R}, \mathbb{R})$ is T-periodic and 0 belongs to a spectral gap of $\mathcal{D}$, $\mathcal{D} := -\frac{d^2}{dt^2} + a(t)$.

Let $G(t,x) := \int_0^x g(t,s)\,ds$, we assume that g satisfies the following assumptions:

(G_1) $g(t,x)$ is a Carathéodory function and it is T-periodic in t, besides, $|g(t,x)| \leq c(1 + |x|^{p-1})$ for some $c > 0$ and $p > 2$.
(G_2) $g(t,x) = o(x)$ as $|x| \to 0$ uniformly in $t \in [0,T]$.
(G_3) $\frac{G(t,x)}{x^2} \to +\infty$ as $|x| \to +\infty$ uniformly in $t \in [0,T]$.
(G_4) $x \mapsto \frac{g(t,x)}{|x|}$ is strictly increasing on $(-\infty, 0) \cup (0, +\infty)$ for all $t \in [0,T]$.

Let $E := H^1([0,T], \mathbb{R})$. By ($\mathrm{V}_1$), we have the decomposition $E = E^- \oplus E^+$, where E^+ and E^- are the positive and negative spectral subspaces of $\mathcal{D}$ in E, respectively. The corresponding functional of (1.4) is

$$\Phi(x) = \frac{1}{2}\int_0^T \left(\left|x'\right|^2 + a(t)x^2\right) dt - \int_0^T G(t,x)\,dt, \quad \forall x \in E.$$

The following set has been introduced by Pankov [17]:

$$\mathcal{M} := \left\{x \in E \setminus E^- : \left\langle \Phi'(x), x\right\rangle = 0 \text{ and } \left\langle \Phi'(x), y\right\rangle = 0, \forall y \in E^-\right\}.$$

By definition, $\mathcal{M}$ contains all nontrivial critical points of Φ.

First, we consider ground state T-periodic solutions of (1.4), that is, solutions corresponding to the least energy of the action functional of (1.4):

Theorem 1.1 *If* (V_1) *and* (G_1)-(G_4) *hold, then* (1.4) *has at least one ground state T-periodic solution.*

Remark 1.1 By the Schauder fixed point theorem, Esmailzadeh and Nakhaie-Jazar [18] obtained a periodic solution for the Mathieu-Duffing type equation

$$\begin{cases} x'' + (a_0 + b_0 \cos t)x + c_0 x^3 = 0, & t \in \mathbb{R}, \\ x(0) = x(T), \qquad x'(0) = x'(T), & T = 2\Omega. \end{cases} \tag{1.5}$$

But it does not exclude the trivial solution, which always exists. Obviously, the nonlinearity $g(t,x) = c_0 x^3$ with $c_0 > 0$ satisfies (G_1)-(G_4), thus our Theorem 1.1 implies (1.5) admits a ground state 2Ω-periodic solution if 0 belongs to a spectral gap of $-\frac{d^2}{dt^2} - (a_0 + b_0 \cos t)$, that is, $\min_{t \in [0,2\Omega]}(-a_0 - b_0 \cos t) < 0$.

Remark 1.2 Torres [16] proved (1.4) with $a(t) \equiv 0$ and $g(t,x) = -f(t,x)$ has one positive or negative solution, where $f \in L^1(\mathbb{R},\mathbb{R})$ is a Carathéodory function and T-periodic in t. But we consider a more general equation (1.4) than the equation in [16], and we obtain a ground state T-periodic solution of (1.4) by a Nehari manifold approach. Therefore, our result extends the result in [16].

Now, we consider the multiplicity of solutions of (1.4). For $k \in \mathbb{Z}$ and $x \in E$, let $x(\cdot - k) \in E$ be defined by $x(\cdot - k) := x(t - kT)$. We note that if x is a solution of (1.4), then so are all elements of the orbit of x under the action of $\mathbb{Z}$, $\mathcal{O}(x) := \{x(\cdot - k) : k \in \mathbb{Z}\}$. Two solutions x_1 and x_2 are said to be geometrically distinct if $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$ are disjoint.

Theorem 1.2 *If* (V_1), (G_1)-(G_4), *and $g(t,x)$ is odd in x hold, then* (1.4) *admits infinitely many pairs of geometrically distinct T-periodic solutions.*

Example 1.1 As simple applications of Theorems 1.1 and 1.2, we consider the following examples:
Ex1. $g(t,x) = c(t)x|x|^{p-2}$;
Ex2. $g(t,x) = c(t)x\ln(1 + |x|)$,
where $p > 2$, $x \in \mathbb{R}$ and $c(t) > 0$ with T-periodic in t. It is not hard to check that the above functions all satisfy assumptions (G_1)-(G_4).

Remark 1.3 Our method is based on the generalized Nehari manifold [19]. In fact, there are many papers where the method of the generalized Nehari manifold has been used, see [20–22] and so on.

The rest of our paper is organized as follows. In Section 2, we establish the variational framework associated with (1.4), and we also give some preliminary lemmas, which are useful in the proofs of our results, and then we give the detailed proofs of our Theorems 1.1 and 1.2.

2 Variational frameworks and preliminary lemmas

Throughout this paper we denote by $\|\cdot\|_{L^q}$ the usual $L^q([0,T],\mathbb{R})$ norm and C for generic constants.

Let $E := H^1([0,T],\mathbb{R})$ under the usual norm and the corresponding inner product defined by

$$\|x\|_E = \left(\int_0^T \left(|x|^2 + \left|x'\right|^2\right) dt\right)^{1/2}.$$

Thus E is a Hilbert space. We will seek solutions of (1.4) as critical points of the functional Φ associated with (1.4) and given by

$$\Phi(x) = \frac{1}{2}\int_0^T \left(\left|x'\right|^2 + a(t)|x|^2\right) dt - \int_0^T G(t,x)\, dt, \quad \forall x \in E.$$

Let $I(x) = \int_0^T G(t,x)\, dt$, then $\Phi, I \in C^1(E,\mathbb{R})$ and the derivatives are given by

$$\left\langle I'(x), y\right\rangle = \int_0^T g(t,x)y\, dt, \qquad \left\langle \Phi'(x), y\right\rangle = \int_0^T \left(x'y' + a(t)xy\right) dt - \left\langle I'(x), y\right\rangle, \quad \forall x, y \in E,$$

which implies that (1.4) is the corresponding Euler-Lagrange equation for Φ. Therefore, we have reduced the problem of finding a nontrivial solution of (1.4) to that of seeking a nonzero critical point of the functional Φ on E.

In what follows, we always assume that (V_1) and (G_1)-(G_4) are satisfied. Obviously, (G_1) and (G_2) imply that for each $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that

$$\left|g(t,x)\right| \leq \varepsilon|x| + C_\varepsilon|x|^{p-1} \quad \text{for all } (t,x) \in [0,T] \times \mathbb{R}. \tag{2.1}$$

By (V_1), we have the decomposition $E = E^- \oplus E^+$, where E^+ and E^- are the positive and negative spectral subspaces of $\mathcal{D}$ in E, respectively. Let

$$Q(x,x) := \int_0^T \left(\left|x'\right|^2 + a(t)|x|^2\right) dt. \tag{2.2}$$

Obviously, the quadratic part of Φ, $Q(x)$ is positive on E^+ and negative on E^-. Moreover, we may define an new inner product $(\cdot,\cdot)$ on E with corresponding norm $\|\cdot\|$ such that

$$\int_0^T \left(\left|x'\right|^2 + a(t)|x|^2\right) dt = \pm\|x\|^2, \quad \forall x \in E^\pm. \tag{2.3}$$

Therefore, Φ can be rewritten as

$$\Phi(x) = \frac{1}{2}\left\|x^+\right\|^2 - \frac{1}{2}\left\|x^-\right\|^2 - \int_0^T G(t,x)\,dt.$$

Let $\mathbb{R}^+ = [0,\infty)$. We define for $x \in E \setminus E^-$ the following subspaces of E:

$$E(x) := E^- \oplus \mathbb{R}x = E^- \oplus \mathbb{R}x^+ \tag{2.4}$$

and the convex subset

$$\hat{E}(x) := E^- \oplus \mathbb{R}^+x = E^- \oplus \mathbb{R}^+x^+. \tag{2.5}$$

2.1 Proof of Theorem 1.1

Lemma 2.1 $\frac{1}{2}g(t,x)x > G(t,x) > 0$ *for all* $x \in \mathbb{R} \setminus \{0\}$.

Proof This follows immediately from (G_2) and (G_4). □

Lemma 2.2 ([19]) *Let* $x, y, s \in \mathbb{R}$ *be numbers with* $s \geq -1$ *and* $q := sx + y \neq 0$. *Then*

$$g(t,x)\left[s\left(\frac{s}{2}+1\right)x + (s+1)y\right] + G(t,x) - G(t,q+x) < 0.$$

Lemma 2.3 *If* $x \in \mathcal{M}$, *then* $\Phi(x+q) < \Phi(x)$ *for any* $q \in H := \{sx + y : s \geq -1, y \in E^-\}$, $q \neq 0$. *Hence* x *is the unique global maximum of* $\Phi|_{\hat{E}(x)}$.

Proof We rewrite Φ by

$$\Phi(x) = \frac{1}{2}Q\left(x^+,x^+\right) + \frac{1}{2}Q\left(x^-,x^-\right) - \int_0^T G(t,x)\,dt.$$

Since $\Phi'(x) = 0$, we have

$$\begin{aligned}
0 &= \left\langle \Phi'(x), s\left(\frac{s}{2}+1\right)x + (s+1)y \right\rangle \\
&= s\left(\frac{s}{2}+1\right)Q\left(x^+,x^+\right) + s\left(\frac{s}{2}+1\right)Q\left(x^-,x^-\right) + (s+1)Q\left(x^-,y\right) \\
&\quad - \int_0^T g(t,x)\left[s\left(\frac{s}{2}+1\right)x + (s+1)y\right] dt,
\end{aligned}$$

which together with $q = sx + y$, $Q(y,y) \le 0$ and Lemma 2.2 implies that

$$\begin{aligned}
&\Phi(q+x) - \Phi(x) \\
&\quad = \frac{1}{2}\left[Q\left((s+1)x^+,(s+1)x^+\right) - Q\left(x^+,x^+\right)\right] \\
&\qquad + \frac{1}{2}\left[Q\left((s+1)x^- + y,(s+1)x^- + y\right) - Q\left(x^-,x^-\right)\right] + \int_0^T \left[G(t,x) - G(t,q+x)\right] dt \\
&\quad = s\left(\frac{s}{2}+1\right)Q\left(x^+,x^+\right) + s\left(\frac{s}{2}+1\right)Q\left(x^-,x^-\right) + \frac{1}{2}Q(y,y) + (s+1)Q\left(x^-,y\right) \\
&\qquad + \int_0^T \left[G(t,x) - G(t,q+x)\right] dt \\
&\quad = \frac{1}{2}Q(y,y) + \int_0^T \left\{g(t,x)\left[s\left(\frac{s}{2}+1\right)x + (s+1)y\right] + G(t,x) - G(t,q+x)\right\} dt < 0.
\end{aligned}$$

So the proof is finished. □

Lemma 2.4 *The following statements hold true*:

(a) *There is* $\alpha > 0$ *such that* $c := \inf_{x\in\mathcal{M}} \Phi(x) \ge \inf_{S_\alpha} \Phi(x) > 0$, *where* $S_\alpha := \{x \in E^+ : \|x\| = \alpha\}$.

(b) $\|x^+\| \ge \max\{\|x^-\|, \sqrt{2c}\}$ *for every* $x \in \mathcal{M}$.

Proof (a) First, for $x \in E^+$, we have $\Phi(x) = \frac{1}{2}\|x\|^2 - \int_0^T G(t,x)\,dt$ and

$$\int_0^T G(t,x)\,dt = o\left(\|x\|^2\right) \quad \text{as } x \to 0$$

by (2.1), hence the second inequality follows if $\alpha > 0$ is sufficiently small.

Second, since for every $x \in \mathcal{M}$, there is $s > 0$ such that $sx^+ \in \hat{E}(x) \cap S_\alpha$. Therefore, by virtue of Lemma 2.3, $\Phi(x) \ge \Phi(sx^+) \ge \inf_{S_\alpha} \Phi(x)$ and the first inequality follows.

(b) For $x \in \mathcal{M}$, by Lemma 2.1, we have

$$c \le \Phi(x) = \frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right) - \int_0^T G(t,x)\,dt \le \frac{1}{2}\left(\|x^+\|^2 - \|x^-\|^2\right),$$

from which the conclusion follows. □

Lemma 2.5 *If* $\mathcal{V} \subset E^+ \setminus \{0\}$ *is a compact subset, then there exists* $R > 0$ *such that* $\Phi \le 0$ *on* $E(x) \setminus B_R(0)$ *for every* $x \in \mathcal{V}$.

Proof Without loss of generality, we may assume that $\|x\| = 1$ for every $x \in \mathcal{V}$. Suppose by contradiction that there exist $x^j \in \mathcal{V}$ and $q^j \in E(x^j)$, $j \in \mathbb{N}$, such that $\Phi(q^j) > 0$ for all j and $\|q^j\| \to \infty$ as $j \to \infty$. Passing to a subsequence, we may assume that $x^j \to x \in E^+$, $\|x\| = 1$. Set $y^j = q^j/\|q^j\| = s_j x^j + (y^j)^-$, then

$$0 \le \frac{\Phi(q^j)}{\|q^j\|^2} = \frac{1}{2}\left(s_j^2 - \left\|(y^j)^-\right\|^2\right) - \int_0^T \frac{G(t,q^j)}{(q^j)^2}\left(y^j\right)^2 dt. \tag{2.6}$$

Hence $\|(y^j)^-\|^2 \le s_j^2 = 1 - \|(y^j)^-\|^2$ and therefore $\frac{1}{\sqrt{2}} \le s_j \le 1$, for a subsequence, $s_j \to s > 0$, $y^j \rightharpoonup y$ and $y^j \to y$ a.e. $t \in [0,T]$. Therefore, $y = sx + y^- \neq 0$, hence $|q^j| = |y^j| \cdot \|q^j\| \to +\infty$, it follows from (G_3) and the Fatou lemma that

$$\int_0^T \frac{G(t,q^j)}{(q^j)^2}\left(y^j\right)^2 dt \to +\infty, \tag{2.7}$$

which contradicts (2.6). □

Lemma 2.6 *For each $x \notin E^-$, the set $\mathcal{M} \cap \hat{E}(x)$ consists of precisely one point $\hat{m}(x)$ which is the unique global maximum of $\Phi|_{\hat{E}(x)}$.*

Proof By Lemma 2.3, it suffices to show that $\mathcal{M} \cap \hat{E}(x) \neq \emptyset$. Since $\hat{E}(x) = \hat{E}(x^+)$, we may assume that $x \in E^+$, $\|x\| = 1$. By Lemma 2.5, there exists $R > 0$ such that $\Phi \le 0$ on $\hat{E}(x) \setminus B_R(0)$. By Lemma 2.4(a), $\Phi(tx) > 0$ for small $t > 0$. Therefore, $0 < \sup_{\hat{E}(x)} \Phi < \infty$. It is easy to see that Φ is weakly upper semicontinuous on $\hat{E}(x)$, therefore, $\Phi(x^0) = \sup_{\hat{E}(x)} \Phi(x)$ for some $x^0 \in \hat{E}(x) \setminus \{0\}$. This x^0 is a critical point of $\Phi|_{E(x)}$, so $\langle \Phi'(x^0), x^0\rangle = \langle \Phi'(x^0), y\rangle = 0$ for all $y \in E(x)$. Consequently, $x^0 \in \mathcal{M} \cap \hat{E}(x)$, as required. □

Lemma 2.7 *Φ is coercive on $\mathcal{M}$, that is, $\Phi(x) \to \infty$ as $\|x\| \to \infty$, $x \in \mathcal{M}$.*

Proof Arguing by contradiction, suppose there exists a sequence $\{x^j\} \subset \mathcal{M}$ such that $\|x^j\| \to \infty$ and $\Phi(x^j) \le d$ for some $d \in [c,\infty)$. Let $y^j := x^j/\|x^j\|$. Then $y^j \rightharpoonup y$ and $y^j \to y$ a.e. $t \in [0,T]$ as $j \to \infty$ after passing to a subsequence. Suppose

$$\int_0^T \left|(y^j)^+\right|^p dt \to 0 \quad \text{as } j \to \infty, \forall p > 2.$$

Then it follows from (2.1) that $\int_0^T G(t, s(y^j)^+)\,dt \to 0$ for each $s \ge 0$. By Lemma 2.4(b), $\|(y^j)^+\|^2 \ge \frac{1}{2}$. Hence, by Lemma 2.3, we obtain

$$\begin{aligned} d \ge \Phi\left(x^j\right) \ge \Phi\left(s\left(y^j\right)^+\right) &= \frac{1}{2}s^2\left\|\left(y^j\right)^+\right\|^2 - \int_0^T G\left(t, s\left(y^j\right)^+\right) dt \\ &\ge \frac{1}{4}s^2 - \int_0^T G\left(t, s\left(y^j\right)^+\right) dt \to \frac{1}{4}s^2 \quad \text{as } j \to \infty. \end{aligned} \tag{2.8}$$

This yields a contradiction if $s > 2\sqrt{d}$. Hence,

$$\lim_{j\to\infty} \int_0^T \left|(y^j)^+\right|^p dt > 0, \quad \forall p > 2.$$

Since $(y^j)^+ \to y^+$ in $L^p([0,T],\mathbb{R})$ $(p>2)$, we have $y \neq 0$. Then

$$\lim_{j\to\infty} |x^j| = |y^j| \cdot \|y^j\| = +\infty,$$

it follows again from (G_3) and the Fatou lemma that

$$\int_0^T \frac{G(t,x^j)}{(x^j)^2} (y^j)^2 \, dt \to +\infty.$$

Hence we have

$$0 \le \frac{\Phi(x^j)}{\|x^j\|^2} = \frac{1}{2}\big(\|(y^j)^+\|^2 - \|(y^j)^-\|^2\big) - \int_0^T \frac{G(t,x^j)}{(x^j)^2} (y^j)^2 \, dt \to -\infty,$$

which is a contradiction. This contradiction establishes the lemma. □

Lemma 2.8 *The map* $\hat{m}: E^+ \setminus \{0\} \to \mathcal{M}$, $x \mapsto \hat{m}(x)$ (*see Lemma* 2.6) *is continuous.*

Proof Let $x \in E^+ \setminus \{0\}$, it suffices to show that for any sequence $\{x^j\} \subset E^+ \setminus \{0\}$ with $x^j \to x$, we have $\hat{m}(x^j) \to \hat{m}(x)$ for some subsequence.

Without loss of generality, we may assume that $\|x^j\| = \|x\| = 1$ for all j, so that $\hat{m}(x^j) = \|\hat{m}(x^j)^+\| x^j + \hat{m}(x^j)^-$. By Lemma 2.5 and Lemma 2.6, there exists $R > 0$ such that

$$\Phi\big(\hat{m}(x^j)\big) = \sup_{\hat{E}(x^j)} \Phi \le \sup_{B_R(0)} \Phi \le \sup_{x \in B_R(0)} \frac{\|x^+\|^2}{2} = \frac{R^2}{2} \quad \text{for every } j. \tag{2.9}$$

Therefore, by Lemma 2.7, $\|\hat{m}(x^j)\| \le C < \infty$. Passing to a subsequence, we may assume that

$$s_j := \big\|\hat{m}(x^j)^+\big\| \to s \quad \text{and} \quad \hat{m}(x^j)^- \rightharpoonup y^- \quad \text{in } E, \text{as } j \to \infty,$$

where $s \ge \sqrt{2c} > 0$ by Lemma 2.4(b). Therefore, we have

$$\hat{m}(x^j) \rightharpoonup sx + y^- \quad \text{and} \quad \hat{m}(x^j) \to sx + y^- \quad \text{a.e. on } [0,T], \text{as } j \to \infty. \tag{2.10}$$

Note that $\hat{m}(x) = \tau x + \hat{m}(x)^-$, where $\tau := \|\hat{m}(x)^+\|$. It follows from Lemma 2.6 that

$$\Phi\big(\hat{m}(x^j)\big) \ge \Phi\big(\tau x^j + \hat{m}(x)^-\big) \to \Phi\big(\tau x + \hat{m}(x)^-\big) = \Phi\big(\hat{m}(x)\big),$$

which together with Fatou's lemma and the weak lower semicontinuity of the norm implies that

$$\begin{aligned}
\Phi\big(\hat{m}(x)\big) &\le \lim_{j\to\infty} \Phi\big(\hat{m}(x^j)\big) \\
&= \lim_{j\to\infty} \left[\frac{1}{2}s_j^2 - \frac{1}{2}\big\|\hat{m}(x^j)^-\big\|^2 - \int_0^T G\big(t,\hat{m}(x^j)\big)\, dt\right] \\
&\le \frac{1}{2}s^2 - \frac{1}{2}\big\|y^-\big\|^2 - \int_0^T G\big(t, sx + y^-\big)\, dt \\
&= \Phi\big(sx + y^-\big).
\end{aligned}$$

However, Lemma 2.6 implies that $\Phi(sx+y^-) \le \Phi(\hat{m}(x))$. Hence all inequalities above must be equalities and it follows that $\hat{m}(x^j)^- \to y^-$, besides, $\hat{m}(x) = sx + y^-$ due to Lemma 2.6, so we have $\hat{m}(x^j) \to sx + y^- = \hat{m}(x)$. □

We now consider the functional

$$\hat{\Psi} : E^+ \setminus \{0\} \to \mathbb{R}, \qquad \hat{\Psi}(x) := \Phi\bigl(\hat{m}(x)\bigr), \tag{2.11}$$

which is continuous by Lemma 2.8. Here, we should mention that Lemmas 2.9 and 2.10 are due to Szulkin and Weth [19].

Lemma 2.9 $\hat{\Psi} \in C^1(E^+ \setminus \{0\}, \mathbb{R})$, *and*

$$\bigl\langle \hat{\Psi}'(w), z \bigr\rangle = \frac{\|\hat{m}(w)^+\|}{\|w\|} \bigl\langle \Phi'\bigl(\hat{m}(w)\bigr), z \bigr\rangle, \quad w, z \in E^+, w \neq 0.$$

Proof For $w \in E^+ \setminus \{0\}$, we put $x := \hat{m}(w) \in \mathcal{M}$, so we have $x = \frac{\|x^+\|}{\|w\|} w + x^-$. Let $z \in E^+$. Choose $\delta > 0$ such that $w_l := w + lz \in E^+ \setminus \{0\}$ for $|l| < \delta$, and put $x_l := \hat{m}(w_l) \in \mathcal{M}$. We may write $x_l = s_l w_l + x_l^-$ with $s_l > 0$. Then $s_0 = \frac{\|x^+\|}{\|w\|}$, and the function $(-\delta, \delta) \to \mathbb{R}$, $l \mapsto s_l$, is continuous by Lemma 2.8.

Note that $x^+ = s_0 w$, which together with Lemma 2.6 and the mean value theorem implies that

$$\begin{aligned} \hat{\Psi}(w_l) - \hat{\Psi}(w) &= \Phi(x_l) - \Phi(x) \\ &= \Phi\bigl(s_l w_l + x_l^-\bigr) - \Phi\bigl(s_0 w + x^-\bigr) \\ &= \Phi\bigl(s_l w_l + x_l^-\bigr) - \Phi\bigl(x^+ + x^-\bigr) \\ &\le \Phi\bigl(s_l w_l + x_l^-\bigr) - \Phi\bigl(s_l w + x_l^-\bigr) \\ &= s_l \bigl\langle \Phi'\bigl(s_l\bigl[w + \tau_l(w_l - w)\bigr] + x_l^-\bigr), (w_l - w) \bigr\rangle \\ &= s_0 l \bigl\langle \Phi'(x), z \bigr\rangle + o(l) \quad \text{as } l \to 0, \end{aligned} \tag{2.12}$$

where $\tau_l \in (0,1)$. Note that $x_l^+ = s_l w_l$. Similarly, we have

$$\begin{aligned} \hat{\Psi}(w_l) - \hat{\Psi}(w) &= \Phi\bigl(s_l w_l + x_l^-\bigr) - \Phi\bigl(s_0 w + x^-\bigr) = \Phi\bigl(x_l^+ + x_l^-\bigr) - \Phi\bigl(s_0 w + x^-\bigr) \\ &\ge \Phi\bigl(s_0 w_l + x^-\bigr) - \Phi\bigl(s_0 w + x^-\bigr) \\ &= s_0 \bigl\langle \Phi'\bigl(s_0\bigl[w + \eta_l(w_l - w)\bigr] + x^-\bigr), (w_l - w) \bigr\rangle \\ &= s_0 l \bigl\langle \Phi'(x), z \bigr\rangle + o(l) \quad \text{as } l \to 0, \end{aligned} \tag{2.13}$$

where $\eta_l \in (0,1)$. Therefore, combining (2.12) and (2.13), we conclude that

$$\partial_z \hat{\Psi}(w) = \lim_{l \to 0} \frac{\hat{\Psi}(w_l) - \hat{\Psi}(w)}{l} = s_0 \bigl\langle \Phi'(x), z \bigr\rangle = \frac{\|\hat{m}(w)^+\|}{\|w\|} \bigl\langle \Phi'\bigl(\hat{m}(w)\bigr), z \bigr\rangle.$$

Hence, $\partial_z \hat{\Psi}(w)$ is linear (and continuous) in z and depends continuously on w. So the assertion follows from Proposition 1.3 in [23]. □

Next we consider the unit sphere

$$S^+ := \{w \in E^+ : \|w\| = 1\} \quad \text{in } E^+.$$

We note that the restriction of the map $\hat{m}$ to S^+ has an inverse given by

$$\check{m} : \mathcal{M} \to S^+, \qquad \check{m}(x) = \frac{x^+}{\|x^+\|}. \tag{2.14}$$

We also consider the restriction $\Psi : S^+ \to \mathbb{R}$ of $\hat{\Psi}$ to S^+.

Lemma 2.10 *The following statements hold true*:

(a) $\Psi \in C^1(S^+)$ *and* $\langle \Psi'(w), z\rangle = \|\hat{m}(w)^+\| \langle \Phi'(\hat{m}(w)), z\rangle$ *for* $z \in T_w S^+ = \{y \in E^+ : (w, y) = 0\}$.

(b) $\{w^j\}$ *is a Palais-Smale sequence for* Ψ *if and only if* $\{\hat{m}(w^j)\}$ *is a Palais-Smale sequence for* Φ.

(c) *We have* $\inf_{S^+} \Psi = \inf_{\mathcal{M}} \Phi = c$. *Moreover*, $x \in S^+$ *is a critical point of* Ψ *if and only if* $\hat{m}(x) \in \mathcal{M}$ *is a critical point of* Φ, *and the corresponding critical values coincide.*

Proof (a) is a direct consequence of Lemma 2.9.

To prove (b), let $\{w^j\}$ be a sequence such that $C := \sup_j \Psi(w^j) = \sup_j \Phi(\hat{m}(w^j)) < \infty$, and let $x^j := \hat{m}(w^j) \in \mathcal{M}$. Since for every j we have an orthogonal splitting

$$E = E(w^j) \oplus T_{w^j} S^+ = E(x^j) \oplus T_{w^j} S^+, \quad \text{with respect to } (\cdot,\cdot)$$

and $\langle \Phi'(x^j), x^j\rangle = 0$, we have $\nabla\Phi(x^j) \in T_{w^j} S^+$ and using (a), we also have the following relation:

$$\|y\Psi'(w^j)\| = \sup_{z \in T_{w^j} S^+, \|z\|=1} \langle \Psi'(w^j), z\rangle = \sup_{z \in T_{w^j} S^+, \|z\|=1} \|(x^j)^+\| \langle \Phi'(x^j), z\rangle = \|(x^j)^+\| \cdot \|\Phi'(x^j)\|.$$

If $\Psi'(w^j) \to 0$ as $j \to \infty$, it follows from Lemma 2.4(b) that $\Phi'(x^j) \to 0$ as $j \to \infty$. On the other hand, if $\Phi'(x_j) \to 0$ as $j \to \infty$, it follows from Lemma 2.7 that $(x^j)^+$ is bounded, and hence $\Psi'(w^j) \to 0$ as $j \to \infty$. Hence, $\{w^j\}$ is a Palais-Smale sequence for Ψ if and only if $\{x^j\}$ is a Palais-Smale sequence for Φ.

The proof of (c) is similar to that of (b) and is omitted. □

Now, we complete the proof of Theorem 1.1.

Proof of Theorem 1.1 From Lemma 2.4(a), we know that $c > 0$. Moreover, if $x^0 \in \mathcal{M}$ satisfies $\Phi(x^0) = c$, then $\check{m}(x^0) \in S^+$ is a minimizer of Ψ and therefore a critical point of Ψ, so that x^0 is a critical point of Φ by Lemma 2.10. It remains to show that there exists a minimizer $x \in \mathcal{M}$ of $\Phi|_{\mathcal{M}}$. By Ekeland's variational principle [23], there exists a sequence $\{w^j\} \subset S^+$ such that $\Psi(w^j) \to c$ and $\Psi'(w^j) \to 0$ as $j \to \infty$. Put $x^j = \hat{m}(w^j) \in \mathcal{M}$, then $\Phi(x^j) \to c$ and $\Phi'(x^j) \to 0$ as $j \to \infty$ by Lemma 2.10(b). By Lemma 2.7, $\{x^j\}$ is bounded and hence $x^j \rightharpoonup x \in E$ and $x^j \to x$ a.e. $t \in [0, T]$ after passing to a subsequence. If

$$\int_0^T |x^j|^p \, dt \to 0 \quad \text{as } j \to \infty, \tag{2.15}$$

then by (2.1), the Hölder's inequality, and Sobolev's imbedding theorem, we have

$$\begin{aligned} \left|\int_0^T g(t,x^j)(x^j)^+\,dt\right| &\le \varepsilon \int_0^T |x^j|\cdot\left|(x^j)^+\right|dt + C_\varepsilon \int_0^T |x^j|^{p-1}\left|(x^j)^+\right|dt \\ &\le \varepsilon C\|x^j\|\cdot\left\|(x^j)^+\right\| + C'_\varepsilon\|x^j\|_{L^p}^{p-1}\cdot\left\|(x^j)^+\right\| \to 0 \quad \text{as } j\to\infty \end{aligned}$$

for some $C, C'_\varepsilon > 0$. It follows that $\int_0^T g(t,x^j)(x^j)^+\,dt = o(1)$ as $j\to\infty$. Therefore,

$$o(1) = \langle \Phi'(x^j), (x^j)^+\rangle = \left\|(x^j)^+\right\|^2 - \int_0^T g(t,x^j)(x^j)^+\,dt = \left\|(x^j)^+\right\|^2 - o(1).$$

Therefore, $\|(x^j)^+\| \to 0$, which contradicts Lemma 2.4(b). This contradiction shows that (2.15) cannot hold. Note that $x^j \to x$ in $L^p([0,T],\mathbb{R})$, so $x^j \rightharpoonup x \ne 0$ and $\Phi'(x) = 0$. Particularly, we see that $x \in \mathcal{M}$, which yields $\Phi(x) \ge c$.

On the other hand, by Lemma 2.1, the Fatou lemma and the boundedness of $\{x^j\}$, we get

$$\begin{aligned} c + o(1) &= \Phi(x^j) - \frac{1}{2}\langle \Phi'(x^j), x^j\rangle \\ &= \int_0^T \left[\frac{1}{2}g(t,x^j)x^j - G(t,x^j)\right]dt \\ &\ge \int_0^T \left[\frac{1}{2}g(t,x)x - G(t,x)\right]dt + o(1) \\ &= \Phi(x) - \frac{1}{2}\langle \Phi'(x), x\rangle + o(1) = \Phi(x) + o(1), \end{aligned}$$

which implies that $\Phi(x) \le c$. Therefore, we conclude that $\Phi(x) = c$. □

2.2 Proof of Theorem 1.2

In order to prove Theorem 1.2, we still need the following lemmas. In what follows, we always assume that (V_1), (G_1)-(G_4) and the nonlinearity $g(t,x)$ is odd in x with $x \in \mathbb{R}$ are satisfied.

Lemma 2.11 *The map* $\check{m}$ *defined in* (2.14) *is Lipschitz continuous.*

Proof For $x, y \in \mathcal{M}$, we have, by Lemma 2.4(b),

$$\begin{aligned} \left\|\check{m}(x) - \check{m}(y)\right\| &= \left\|\frac{x^+}{\|x^+\|} - \frac{y^+}{\|y^+\|}\right\| = \left\|\frac{x^+ - y^+}{\|x^+\|} + \frac{(\|y^+\| - \|x^+\|)y^+}{\|x^+\|\cdot\|y^+\|}\right\| \\ &\le \frac{2}{\|x^+\|}\left\|(x-y)^+\right\| \le \sqrt{\frac{2}{c}}\|x-y\|. \end{aligned}$$

The proof is completed. □

Remark 2.1 It is easy to see that both maps $\hat{m}$, $\check{m}$ are equivariant with respect to the $\mathbb{Z}$-action given by $x \mapsto x(\cdot - k)$ for $k \in \mathbb{Z}$. So, by Lemma 2.10(c), the orbits $\mathcal{O}(x) \subset \mathcal{M}$ consisting of critical points of Φ are in one-to-one correspondence with the orbits $\mathcal{O}(w) \subset S^+$ consisting of critical points of Ψ.

To continue the proof, we need the following notation. For $d \geq e \geq c$ we put

$$\Phi^d := \{x \in \mathcal{M} : \Phi(x) \leq d\}, \qquad \Phi_e := \{x \in \mathcal{M} : \Phi(x) \geq e\}, \qquad \Phi_e^d := \Phi^d \cap \Phi_e,$$
$$\Psi^d := \{w \in S^+ : \Psi(w) \leq d\}, \qquad \Psi_e := \{w \in S^+ : \Psi(w) \geq e\}, \qquad \Psi_e^d := \Psi^d \cap \Psi_e,$$
$$K := \{w \in S^+ : \Psi'(w) = 0\}, \qquad K_d := \{w \in K : \Psi(w) = d\},$$
$$\nu(d) := \sup\{\|x\| : x \in \Phi^d\}.$$

Note that $\nu(d) < \infty$ for every d due to Lemma 2.7. We may choose a subset $\mathcal{F}$ of K such that $\mathcal{F} = -\mathcal{F}$ and each orbit $\mathcal{O}(w) \subset K$ has a unique representative in $\mathcal{F}$. By Remark 2.1, it suffices to show that the set $\mathcal{F}$ is infinite. Suppose to the contrary that

$$\mathcal{F} \text{ is a finite set.} \tag{2.16}$$

Lemma 2.12 $\kappa := \inf\{\|v - w\| : v, w \in K, v \neq w\} > 0$.

Proof We can choose $v^j, w^j \in \mathcal{F}$ and $k^j, l^j \in \mathbb{Z}$ such that $v^j(\cdot - k^j) \neq w^j(\cdot - l^j)$ for all j and

$$\|v^j(\cdot - k^j) - w^j(\cdot - l^j)\| \to \kappa \quad \text{as } n \to \infty.$$

Let $m^j = k^j - l^j$. After passing to a subsequence, we have $v^j = v \in \mathcal{F}$, $w^j = w \in \mathcal{F}$ and either $m^j = m \in \mathbb{Z}$ for almost all j or $|m^j| \to \infty$. If the first case holds, we have

$$0 < \|v^j(\cdot - k^j) - w^j(\cdot - l^j)\| = \|v - w(\cdot - m)\| = \kappa \quad \text{for all } j.$$

If the second case holds, we have $w(\cdot - m^j) \rightharpoonup 0$, thus $\kappa = \lim_{j\to\infty} \|v - w(\cdot - m^j)\| \geq \|v\| = 1$, where $\|v\| = 1$ due to the definitions of K and S^+. Therefore, this lemma is proven. □

Lemma 2.13 *Let $d \geq c$. If $\{x^j\}, \{y^j\} \subset \Psi^d$ are two Palais-Smale sequences for Ψ, then either $\|x^j - y^j\| \to 0$ as $j \to \infty$ or $\limsup_{j\to\infty} \|x^j - y^j\| \geq \rho(d)$, where $\rho(d)$ depends on d but not on the particular choice of Palais-Smale sequences.*

Proof Let $q^j := \hat{m}(x^j)$ and $w^j := \hat{m}(y^j)$ for $j \in \mathbb{N}$. Then both sequences $\{q^j\}, \{w^j\} \subset \Phi^d$ are bounded Palais-Smale sequences for Φ by Lemma 2.7 and the definition of Ψ. Let p is the parameter in (G_1). We distinguish two cases.

Case 1. If

$$\|(q^j - w^j)^+\|_{L^p} \to 0, \quad p > 2. \tag{2.17}$$

Note that $\{q^j\}$ and $\{w^j\}$ are bounded Palais-Smale sequences for Φ, it follows from (2.1), Hölder's inequality, and Sobolev's inequality that

$$\begin{aligned}
&\|(q^j - w^j)^+\|^2 \\
&\quad = \langle \Phi'(q^j), (q^j - w^j)^+ \rangle - \langle \Phi'(w^j), (q^j - w^j)^+ \rangle \\
&\qquad + \int_0^T [g(t, q_n^j) - g(t, w_n^j)](q^j - w^j)^+ \, dt
\end{aligned}$$

$$\begin{aligned}
&\leq \varepsilon \| (q^j - w^j)^+ \| + \int_0^T [\varepsilon (|q_n^j| + |w_n^j|) + C_\varepsilon (|q_n^j|^{p-1} + |w_n^j|^{p-1})] |(q^j - w^j)^+| \, dt \\
&\leq (1 + C_0)\varepsilon \| (q^j - w^j)^+ \| + D_\varepsilon \| (q^j - w^j)^+ \|_{L^p}
\end{aligned} \tag{2.18}$$

for all $j \geq J_\varepsilon$, where $\varepsilon > 0$ is arbitrary, C_ε, D_ε, J_ε, and C_0 do not depend on the choice of ε. Therefore, by (2.17) and (2.18), we have $\|(q^j - w^j)^+\| \to 0$ as $j \to \infty$. Similarly, $\|(q^j - w^j)^-\| \to 0$ as $j \to \infty$. Therefore,

$$\| q^j - w^j \| \to 0 \quad \text{as } j \to \infty,$$

it follows from Lemma 2.11 that

$$\| x^j - y^j \| = \| \check{m}(q^j) - \check{m}(w^j) \| \to 0 \quad \text{as } j \to \infty.$$

Case 2. If

$$\lim_{j\to\infty} \| (q^j - w^j)^+ \|_{L^p} > 0, \quad p > 2. \tag{2.19}$$

Since $\{q^j\}$ and $\{w^j\}$ are bounded, we may pass to a subsequence such that

$$\begin{aligned}
&q^j = (q^j)^+ + (q^j)^- \rightharpoonup q = q^+ + q^- \in E = E^+ \oplus E^-, \\
&w^j = (w^j)^+ + (w^j)^- \rightharpoonup w = w^+ + w^- \in E = E^+ \oplus E^-, \\
&\Phi'(q) = \Phi'(w) = 0
\end{aligned}$$

and

$$\| (q^j)^+ \| \to \alpha_1, \qquad \| (w^j)^+ \| \to \alpha_2,$$

where $\sqrt{2c} \leq \alpha_i \leq \nu(d)$ for $i = 1, 2$ by Lemma 2.4(b). Note that $(q^j - w^j)^+ \to q^+ - w^-$ in $L^p([0,T], \mathbb{R})$, thus by (2.19), $q^+ \neq w^+$, thus $q \neq w$. We first consider the case where $q \neq 0$ and $w \neq 0$, so that $q, w \in \mathcal{M}$ and

$$x := \check{m}(q) \in K, \qquad y := \check{m}(w) \in K, \quad x \neq y.$$

Then by (2.14), the definition of Ψ^d and the weak lower semicontinuity of the norm, we have

$$\liminf_{j\to\infty} \| x^j - y^j \| = \liminf_{j\to\infty} \left\| \frac{(q^j)^+}{\|(q^j)^+\|} - \frac{(w^j)^+}{\|(w^j)^+\|} \right\| \geq \left\| \frac{q^+}{\alpha_1} - \frac{w^+}{\alpha_2} \right\| = \|\beta_1 x - \beta_2 y\|,$$

where $\beta_1 := \frac{\|q^+\|}{\alpha_1} \geq \frac{\sqrt{2c}}{\nu(d)}$ and $\beta_2 := \frac{\|w^+\|}{\alpha_2} \geq \frac{\sqrt{2c}}{\nu(d)}$. Since $\|x\| = \|y\| = 1$, an elementary geometric argument and the inequalities above imply that

$$\liminf_{j\to\infty} \| x^j - y^j \| \geq \|\beta_1 x - \beta_2 y\| \geq \min\{\beta_1, \beta_2\} \|x - y\| \geq \frac{\kappa\sqrt{2c}}{\nu(d)},$$

where κ is defined in Lemma 2.12. It remains to consider the case where either $q = 0$ or $w = 0$. If $w = 0$, then $q \neq 0$, and

$$\liminf_{j\to\infty}\|x^j - y^j\| = \liminf_{j\to\infty}\left\|\frac{(q^j)^+}{\|(q^j)^+\|} - \frac{(w^j)^+}{\|(w^j)^+\|}\right\| \geq \left\|\frac{q^+}{\alpha_1}\right\| \geq \frac{\sqrt{2c}}{\nu(d)}.$$

The case $q = 0$ is treated similarly. The proof is finished. □

It is well known (see [24], Lemma II.3.9) that Ψ admits a pseudo-gradient vector field, *i.e.*, there exists a Lipschitz continuous map $H : S^+ \setminus K \to TS^+$ (where TS^+ is the tangent bundle) with $H(w) \in T_w S^+$ for all $w \in S^+ \setminus K$ and

$$\|H(w)\| < 2\|\nabla\Psi(w)\|, \qquad \langle H(w), \nabla\Psi(w)\rangle > \frac{1}{2}\|\nabla\Psi(w)\|^2, \quad \forall w \in S^+ \setminus K. \tag{2.20}$$

Let $\eta : \mathcal{G} \to S^+ \setminus K$ be the corresponding (Ψ-decreasing) flow defined by

$$\begin{cases} \frac{d}{dt}\eta(t,w) = -H(\eta(t,w)), \\ \eta(0,w) = w, \end{cases} \tag{2.21}$$

where

$$\mathcal{G} := \{(t,w) : w \in S^+ \setminus K, T^-(w) < t < T^+(w)\} \subset \mathbb{R} \times (S^+ \setminus K)$$

and $T^-(w) < 0$, $T^+(w) > 0$ are the maximal existence times of the trajectory $t \mapsto \eta(t,w)$ in negative and positive direction. Note that Ψ is strictly decreasing along trajectories of η.

For deformation type arguments, the following lemma is crucial.

Lemma 2.14 *For every $w \in S^+$ the limit $\lim_{t\to T^+(w)} \eta(t,w)$ exists and is a critical point of Ψ.*

Proof Fix $w \in S^+$ and put $d := \Psi(w)$.

Case 1: $T^+(w) < \infty$. For $0 \leq s < t < T^+(w)$, by (2.20), (2.21), and Lemma 2.10(c), we have

$$\begin{aligned}
\|\eta(t,w) - \eta(s,w)\| &\leq \int_s^t \|H(\eta(\tau,w))\|\,d\tau \\
&\leq 2\sqrt{2}\int_s^t \sqrt{\langle H(\eta(\tau,w)), \nabla\Psi(\eta(\tau,w))\rangle}\,d\tau \\
&\leq 2\sqrt{2(t-s)}\left(\int_s^t \langle H(\eta(\tau,w)), \nabla\Psi(\eta(\tau,w))\rangle\,d\tau\right)^{\frac{1}{2}} \\
&= 2\sqrt{2(t-s)}\left[\Psi(\eta(s,w)) - \Psi(\eta(t,w))\right]^{\frac{1}{2}} \\
&\leq 2\sqrt{2(t-s)}\left[\Psi(w) - c\right]^{\frac{1}{2}}.
\end{aligned}$$

Since $T^+(w) < \infty$, this implies that $\lim_{t\to T^+(w)} \eta(t,w)$ exists and then it must be a critical point of Ψ (otherwise the trajectory $t \mapsto \eta(t,w)$ could be continued beyond $T^+(w)$).

Case 2: $T^+(w) = \infty$. To prove that $\lim_{t\to\infty} \eta(t,w)$ exists, it clearly suffices to establish the following property:

$$\text{for every } \varepsilon > 0, \text{there exists } t_\varepsilon > 0 \text{ with } \|\eta(t_\varepsilon,w) - \eta(t,w)\| < \varepsilon \text{ for } t \geq t_\varepsilon. \tag{2.22}$$

We suppose by contradiction that (2.22) is not satisfied. Then there exists $0 < \varepsilon < \frac{1}{2}\rho(d)$ (where $\rho(d)$ is given in Lemma 2.13) and a sequence $\{t_n\} \subset [0, \infty)$ with $t_n \to \infty$ and $\|\eta(t_n, w) - \eta(t_{n+1}, w)\| = \varepsilon$ for every n. Choose the smallest $t_n^1 \in (t_n, t_{n+1})$ such that

$$\left\|\eta(t_n, w) - \eta\left(t_n^1, w\right)\right\| = \frac{\varepsilon}{3}, \quad 0 < \varepsilon < \frac{1}{2}\rho(d) \tag{2.23}$$

and let

$$\kappa_n := \min_{s \in [t_n, t_n^1]} \left\|\nabla\Psi\left(\eta(s, w)\right)\right\|.$$

Then by (2.20) and (2.21), we have

$$\begin{aligned}
\frac{\varepsilon}{3} &= \left\|\eta\left(t_n^1, w\right) - \eta(t_n, w)\right\| \le \int_{t_n}^{t_n^1} \left\|H\left(\eta(\tau, w)\right)\right\| d\tau \\
&\le 2\int_{t_n}^{t_n^1} \left\|\nabla\Psi\left(\eta(\tau, w)\right)\right\| d\tau \\
&\le \frac{2}{\kappa_n} \int_{t_n}^{t_n^1} \left\|\nabla\Psi\left(\eta(\tau, w)\right)\right\|^2 d\tau \\
&\le \frac{4}{\kappa_n} \int_{t_n}^{t_n^1} \left\langle H\left(\eta(\tau, w)\right), \nabla\Psi\left(\eta(\tau, w)\right)\right\rangle d\tau \\
&= \frac{4}{\kappa_n}\left[\Psi\left(\eta(t_n, w)\right) - \Psi\left(\eta\left(t_n^1, w\right)\right)\right].
\end{aligned}$$

Note that Ψ is strictly decreasing along trajectories of η, and it follows that

$$\Psi\left(\eta(t_n, w)\right) - \Psi\left(\eta\left(t_n^1, w\right)\right) \to 0$$

as $n \to \infty$, thus $\kappa_n \to 0$ and there exist $s_n^1 \in [t_n, t_n^1]$ such that $\nabla\Psi(w_n^1) \to 0$, where $w_n^1 := \eta(s_n^1, w)$. Similarly we find a largest $t_n^2 \in (t_n^1, t_{n+1})$ for which $\|\eta(t_{n+1}, w) - \eta(t_n^2, w)\| = \frac{\varepsilon}{3}$ and then $w_n^2 := \eta(s_n^2, w)$ satisfying $\nabla\Psi(w_n^2) \to 0$. As $\|w_n^1 - \eta(t_n, w)\| \le \frac{\varepsilon}{3}$ and $\|w_n^2 - \eta(t_{n+1}, w)\| \le \frac{\varepsilon}{3}$, $\{w_n^1\}$ and $\{w_n^2\}$ are two Palais-Smale sequences such that

$$\frac{\varepsilon}{3} \le \left\|w_n^1 - w_n^2\right\| \le 2\varepsilon < \rho(d),$$

which contradicts Lemma 2.13, hence (2.22) is true. Therefore, $\lim_{t\to\infty} \eta(t, w)$ exists, and obviously it must be a critical point of Ψ. $\square$

In the following, for a subset $P \subset S^+$ and $\delta > 0$, we put

$$U_\delta(P) := \left\{x \in S^+ : \|x - P\| < \delta\right\}. \tag{2.24}$$

Lemma 2.15 *Let $d \ge c$. Then for every $\delta > 0$ there exists $\varepsilon = \varepsilon(\delta) > 0$ such that*

(a) $\Psi_{d-\varepsilon}^{d+\varepsilon} \cap K = K_d$;

(b) $\lim_{t\to T^+(w)} \Psi(\eta(t, w)) < d - \varepsilon$ *for* $w \in \Psi^{d+\varepsilon} \setminus U_\delta(K_d)$.

Proof By (2.16), (a) is obviously satisfied for $\varepsilon > 0$ small enough. Without loss of generality, we may assume $U_\delta(K_d) \subset \Psi^{d+1}$ and $\delta < \rho(d+1)$. In order to find ε such that (b) holds, we let

$$\tau := \inf\{\|\nabla\Psi(w)\| : w \in U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)\}. \tag{2.25}$$

We claim that $\tau > 0$. Indeed, suppose by contradiction that there exists a sequence $\{x^j\} \subset U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)$ such that $\nabla\Psi(x^j) \to 0$. Passing to a subsequence, using the finiteness condition (2.16) and the $\mathbb{Z}$-invariance of Ψ, we may assume $\{x^j\} \subset U_\delta(w_0) \setminus U_{\frac{\delta}{2}}(w_0)$ for some $w_0 \in K_d$. Let $y^j \to w_0$. Then $\nabla\Psi(y^j) \to 0$ and

$$\frac{\delta}{2} \le \limsup_{j\to\infty}\|x^j - y^j\| \le \delta < \rho(d+1),$$

which contradicts Lemma 2.13. Hence $\tau > 0$. Let

$$A := \sup\{\|\nabla\Psi(w)\| : w \in U_\delta(K_d) \setminus U_{\frac{\delta}{2}}(K_d)\} \tag{2.26}$$

and choose $\varepsilon < \frac{\delta\tau^2}{8A}$ such that (a) holds. By Lemma 2.14 and (a), we know that the only way (b) can fail is that

$$\eta(t,w) \to \widetilde{w} \in K_d \quad \text{as } t \to T^+(w) \text{ for some } w \in \Psi^{d+\varepsilon} \setminus U_\delta(K_d). \tag{2.27}$$

In this case we let

$$\begin{aligned} t_1 &:= \sup\{t \in [0, T^+(w)) : \eta(t,w) \notin U_\delta(\widetilde{w})\}, \\ t_2 &:= \inf\{t \in (t_1, T^+(w)) : \eta(t,w) \in U_{\frac{\delta}{2}}(\widetilde{w})\}. \end{aligned}$$

Then by (2.20), (2.21), and (2.26), we have

$$\frac{\delta}{2} = \|\eta(t_1,w) - \eta(t_2,w)\| \le \int_{t_1}^{t_2}\|H(\eta(s,w))\|\,ds \le 2\int_{t_1}^{t_2}\|\nabla\Psi(\eta(s,w))\|\,ds \le 2A(t_2 - t_1),$$

which together with (2.20), (2.21), and (2.25) imply that

$$\begin{aligned} \Psi(\eta(t_2,w)) - \Psi(\eta(t_1,w)) &= -\int_{t_1}^{t_2}\langle\nabla\Psi(\eta(s,w)), H(\eta(s,w))\rangle\,ds \\ &\le -\frac{1}{2}\int_{t_1}^{t_2}\|\nabla\Psi(\eta(s,w))\|^2\,ds \\ &\le -\frac{1}{2}\tau^2(t_2 - t_1) \le -\frac{\delta\tau^2}{8A}. \end{aligned}$$

Therefore, $\Psi(\eta(t_2,w)) \le d + \varepsilon - \frac{\delta\tau^2}{8A} < d$, thus $\eta(t,w) \nrightarrow \widetilde{w}$, which contradicts our assumption (2.27). □

Now, we complete the proof of Theorem 1.2.

Proof of Theorem 1.2 For $j \in \mathbb{N}$, we consider the family Σ_j of all closed and symmetric subsets $A \subset S^+$, that is, $A = -A = \overline{A}$ with $\gamma^*(A) \geq j$, where γ^* denotes the usual Krasnoselskii genus (see, *e.g.*, [24, 25]), that is,

$$\gamma^*(A) := \min\{i \in \mathbb{N} : \exists \text{ odd continuous } \varphi : A \to \mathbb{R}^i \setminus \{0\}\}.$$

In particular, if there does not exist a finite i, we set $\gamma^*(\emptyset) := \infty$. Finally, we set $\gamma^*(\emptyset) := 0$.

For the usual Krasnoselskii genus, let A and B are closed and symmetric subsets, then we have the following properties (see [25]):

1. *Mapping property*: If there exists an odd map $f \in C(A,B)$, then $\gamma^*(A) \leq \gamma^*(B)$.
2. *Monotonicity property*: If $A \subset B$, then $\gamma^*(A) \leq \gamma^*(B)$.
3. *Subadditivity*: $\gamma^*(A \cup B) \leq \gamma^*(A) + \gamma^*(B)$.
4. *Continuity property*: If A is compact, then $\gamma^*(A) < \infty$ and there is a $\delta > 0$ such that $\overline{U_\delta(A)}$ is a closed and symmetric subset and $\gamma^*(\overline{U_\delta(A)}) = \gamma^*(A)$, where $U_\delta(\cdot)$ is defined in (2.24).

We consider the nondecreasing sequence of Lusternik-Schnirelman values for Ψ defined by

$$c_k := \inf\{d \in \mathbb{R} : \gamma^*(\Psi^d) \geq k, k \in \mathbb{N}\}.$$

Obviously, $c_k \leq c_{k+1}$. Next, we claim that

$$K_{c_k} \neq \emptyset \quad \text{and} \quad c_k < c_{k+1} \quad \text{for all } k \in \mathbb{N}. \tag{2.28}$$

To prove this claim, we let $k \in \mathbb{N}$ and let $d = c_k$. By Lemma 2.12, we know $\gamma^*(K_d) = 0$ or 1 (depending on whether K_d is empty or not). By the continuity property 4 of the genus, there exists $\delta > 0$ such that

$$\gamma^*(\overline{U}) = \gamma^*(K_d), \tag{2.29}$$

where $U := U_\delta(K_d)$ and $\delta < \frac{\kappa}{2}$. Choose $\varepsilon = \varepsilon(\delta) > 0$ such that Lemma 2.15 holds, then for every $w \in \Psi^{d+\varepsilon} \setminus U$, there exists $t \in [0, T^+(w))$ such that $\Psi(\eta(t,w)) < d - \varepsilon$. Thus, we may define the following entrance time map:

$$e : \Psi^{d+\varepsilon} \setminus U \to [0,\infty), \qquad e(w) := \inf\{t \in [0, T^+(w)) : \Psi(\eta(t,w)) \leq d - \varepsilon\},$$

which satisfies $e(w) < T^+(w)$ for every $w \in \Psi^{d+\varepsilon} \setminus U$. Note that $d - \varepsilon$ is not a critical value of Ψ by Lemma 2.15. By $g(t,-x) = -g(t,x)$ for all $(t,x) \in [0,T] \times \mathbb{R}$, (2.21) and the definition of Ψ, we know e is a continuous (and even) map. Thus, by (2.21), we have

$$h : \Psi^{d+\varepsilon} \setminus U \to \Psi^{d-\varepsilon}, \qquad h(w) = \eta(e(w), w)$$

is odd and continuous. Therefore, by the properties 1-3 of the genus and the definition of $d = c_k$, we have

$$\gamma^*(\Psi^{d+\varepsilon}) - \gamma^*(\overline{U}) \leq \gamma^*(\Psi^{d+\varepsilon} \setminus U) \leq \gamma^*(\Psi^{d-\varepsilon}) \leq k - 1,$$

it follows from (2.29) that

$$\gamma^*\big(\Psi^{d+\varepsilon}\big) \leq \gamma^*(\overline{U}) + k - 1 = \gamma^*(K_d) + k - 1,$$

that is,

$$\gamma^*(K_d) \geq \gamma^*\big(\Psi^{d+\varepsilon}\big) - (k-1).$$

It follows from the definition of $d = c_k$ and of c_{k+1} that $\gamma^*(K_d) \geq 1$ if $c_{k+1} > c_k$ and $\gamma^*(K_d) > 1$ if $c_{k+1} = c_k$. Since $\gamma^*(\mathcal{F}) = \gamma^*(K_d) \leq 1$, we get (2.28) holds.

Therefore, (2.28) implies that there is an infinite sequence $(\pm w_k)$ of pairs of geometrically distinct critical points of Ψ with $\Psi(w_k) = c_k$, which contradicts (2.16). The proof is finished. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The main idea of this paper was proposed by G-WC and G-WC prepared the manuscript initially and JW performed a part of steps of the proofs in this research. All authors read and approved the final manuscript.

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