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Multiple positive solutions for third-order p -Laplacian functional dynamic equations on time scales

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Abstract

In this paper we consider the following boundary value problems for p -Laplacian functional dynamic equations on time scales: $[\Phi_p(u^{\Delta\nabla}(t))]^\nabla + a(t)f(u(t), u(\mu(t))) = 0$, $t \in (0, T)_{\mathbf{T}}$, $u(t) = \varphi(t)$, $t \in [-r, 0]_{\mathbf{T}}$, $u^\Delta(0) = u^{\Delta\nabla}(T) = 0$, $u(T) + B_0(u^\Delta(\eta)) = 0$. By using the well-known Leggett-Williams fixed point theorem, some existence criteria of at least three positive solutions are established.

MSC: 39K10; 34B15

Keywords: time scale; p -Laplacian functional dynamic equations; boundary value problem; positive solution; fixed point

1 Introduction

The theory of dynamic equations on time scales was introduced by Stefan Hilger in 1988 [1]. Let \mathbf{T} be a time scale, *i.e.*, \mathbf{T} is a nonempty closed subset of \mathbb{R} . Let $0, T$ be points in \mathbf{T} , an interval $[0, T]_{\mathbf{T}}$ denoting a time scale interval, that is, $[0, T]_{\mathbf{T}} := [0, T] \cap \mathbf{T}$. Other types of intervals are defined similarly.

Recently, there has been much attention paid to the existence of positive solutions for boundary value problems on time scales [2–22] since the theory of dynamic equations on time scales has been a new important mathematical branch (see, for example, [23–28]). However, to the best of our knowledge, there is not much concern for p -Laplacian functional dynamic equations on time scales [16, 17].

For convenience, throughout this paper we denote $\Phi_p(s)$ as the p -Laplacian operator, *i.e.*, $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, $(\Phi_p)^{-1} = \Phi_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

In [4], Anderson, Avery and Henderson considered the following BVP on time scales:

$$\begin{aligned} [\Phi_p(u^\Delta(t))]^\nabla + c(t)f(u) &= 0, & t \in (a, b)_{\mathbf{T}}, \\ u(a) - B_0(u^\Delta(v)) &= 0, & u^\Delta(b) = 0, \end{aligned}$$

where $v \in (a, b)_{\mathbf{T}}$, $f \in C_{\text{Id}}([0, +\infty), [0, +\infty))$, $c \in C_{\text{Id}}([a, b], [0, +\infty))$, and $K_m x \leq B_0(x) \leq K_M x$ for some positive constants K_m, K_M . They established the existence result of at least one positive solution by a fixed point theorem of a cone expansion and a compression of functional type.

In [17], by using a double fixed point theorem due to Avery *et al.* [29], Song and Xiao considered the existence of at least twin positive solutions to the following p -Laplacian

functional dynamic equations on time scales:

$$\begin{cases} [\Phi_p(u^{\Delta\nabla}(t))]^\nabla + a(t)f(u(t), u(\mu(t))) = 0, & t \in (0, T)_{\mathbf{T}}, \\ u(t) = \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u^{\Delta}(0) = u^{\Delta\nabla}(T) = 0, & u(T) + B_0(u^{\Delta}(\eta)) = 0, \end{cases} \quad (1.1)$$

where $\eta \in (0, \rho(T))_{\mathbf{T}}$, $-r, 0, T \in \mathbf{T}$.

In [30], Zhao, Wang and Ge considered the existence of at least three positive solutions to the following p -Laplacian problem:

$$\begin{aligned} [\Phi_p(u'(t))]^\nabla + a(t)f(u, u') &= 0, \quad t \in [0, 1], \\ u'(0) = u(1) &= 0. \end{aligned}$$

The main tool used in [30] is the Leggett-Williams fixed point theorem.

Motivated by the results mentioned above, in this paper, we shall show that the BVP (1.1) has at least three positive solutions by using the Leggett-Williams fixed point theorem [31].

In this article, we always assume that:

- (C₁) $f : [0, +\infty)^2 \rightarrow (0, +\infty)$ is continuous;
- (C₂) $a : \mathbf{T} \rightarrow (0, +\infty)$ is left dense continuous (i.e., $a \in C_{\text{ld}}(\mathbf{T}, (0, +\infty))$) and does not vanish identically on any closed subinterval of $[0, T]_{\mathbf{T}}$, where $C_{\text{ld}}(\mathbf{T}, (0, +\infty))$ denotes the set of all left dense continuous functions from \mathbf{T} to $(0, +\infty)$, $\min_{t \in [0, T]_{\mathbf{T}}} a(t) = m$, $\max_{t \in [0, T]_{\mathbf{T}}} a(t) = M$, and $m < M$;
- (C₃) $\varphi : [-r, 0]_{\mathbf{T}} \rightarrow [0, +\infty)$ is continuous and $r > 0$;
- (C₄) $\mu : [0, T]_{\mathbf{T}} \rightarrow [-r, T]_{\mathbf{T}}$ is continuous, $\mu(t) \leq t$ for all t ;
- (C₅) $B_0(v)$ is for continuous functions defined on \mathbb{R} such that there exist $B \geq 1$ and $A \geq 1$ such that

$$Bx \leq B_0(x) \leq Ax, \quad \text{for all } \mathbb{R}.$$

Remark 1.1 Insofar as we have time scales, the technique using the Leggett-Williams fixed point theorem in our paper is not the same as in previous works, since the result of our paper is based on the space $E = C_{\text{ld}}^{\Delta}([0, T]_{\mathbf{T}}, \mathbb{R})$ with $\|u\| = \max\{\max_{t \in [0, T]_{\mathbf{T}}} |u(t)|, \max_{t \in [0, T]_{\mathbf{T}^k}} |u^{\Delta}(t)|\}$. Furthermore, the novelty in our paper of the hypotheses utilized in the existence theorem is the condition (H₄) (see Theorem 3.1) where we impose the effect of $a(t)$, so we use the Leggett-Williams fixed point theorem in the space of $E = C_{\text{ld}}^{\Delta}([0, T]_{\mathbf{T}}, \mathbb{R})$. On the other hand, the existence of three solutions is important since we think that the more we have the existence of solutions in similar conditions the more interesting it is.

In the remainder of this section we list the following well-known definitions, which can be found in [24–27].

Definition 1.1 For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and the backward jump operator ρ , respectively,

$$\sigma(t) = \inf\{\tau \in \mathbf{T} | \tau > t\} \in \mathbf{T}, \quad \rho(r) = \sup\{\tau \in \mathbf{T} | \tau < r\} \in \mathbf{T}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered. If $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If \mathbf{T} has a right scattered minimum m , define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If \mathbf{T} has a left scattered maximum M , define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 1.2 For $x: \mathbf{T} \rightarrow \mathbb{R}$ and $t \in \mathbf{T}^k$, we define the delta derivative of $x(t)$, $x^\Delta(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|[x(\sigma(t)) - x(s)] - x^\Delta(t)[\sigma(t) - s]| < \varepsilon|\sigma(t) - s|,$$

for all $s \in U$. For $x: \mathbf{T} \rightarrow \mathbb{R}$ and $t \in \mathbf{T}_k$, we define the nabla derivative of $x(t)$, $x^\nabla(t)$, to be the number (when it exists), with the property that, for any $\varepsilon > 0$, there is a neighborhood V of t such that

$$|[x(\rho(t)) - x(s)] - x^\nabla(t)[\rho(t) - s]| < \varepsilon|\rho(t) - s|,$$

for all $s \in V$.

If $\mathbf{T} = \mathbb{R}$, then $x^\Delta(t) = x^\nabla(t) = x'(t)$. If $\mathbf{T} = \mathbb{Z}$, then $x^\Delta(t) = x(t+1) - x(t)$ is the forward difference operator, while $x^\nabla(t) = x(t) - x(t-1)$ is the backward difference operator.

Definition 1.3 If $F^\Delta(t) = f(t)$, then we define the delta integral by

$$\int_a^t f(s) \Delta s = F(t) - F(a).$$

If $\Phi^\nabla(t) = f(t)$, then we define the nabla integral by

$$\int_a^t f(s) \nabla s = \Phi(t) - \Phi(a).$$

Throughout this papers, we assume that \mathbf{T} is a closed subset of \mathbb{R} with $0 \in \mathbf{T}_k$ and $T \in \mathbf{T}^k$.

Lemma 1.1 [10] *The following formulas hold:*

- (i) $(\int_a^t f(s) \Delta s)^\Delta = f(t)$,
- (ii) $(\int_a^t f(s) \Delta s)^\nabla = f(\rho(t))$,
- (iii) $(\int_a^t f(s) \nabla s)^\Delta = f(\sigma(t))$,
- (iv) $(\int_a^t f(s) \nabla s)^\nabla = f(t)$.

2 Preliminaries

In this section, we provide some background material from the theory of cones in Banach spaces and we then state the Leggett-Williams fixed point theorem.

Definition 2.1 Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

- (i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$;
- (ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \quad \text{if and only if} \quad y - x \in P.$$

Definition 2.2 Let E be a real Banach space and $P \subset E$ be a cone. A function $\alpha : P \rightarrow [0, \infty)$ is called a nonnegative continuous concave functional if α is continuous and

$$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $a, b, c > 0$ be constants, $P_c = \{x \in P : \|x\| < c\}$, $P(\alpha, a, b) = \{x \in P : a \leq \alpha(x), \|x\| \leq b\}$.

To prove our main results, we need the following theorem [31].

Theorem 2.1 (Leggett-Williams) *Let $A : \bar{P}_c \rightarrow \bar{P}_c$ be a completely continuous map and α be a nonnegative continuous concave functional on P such that $\alpha(x) \leq \|x\|, \forall x \in \bar{P}_c$. Suppose there exist a, b, d with $0 < a < b < d \leq c$, such that:*

- (i) $\{x \in P(\alpha, b, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Ax) > b$ for all $x \in P(\alpha, b, d)$;
- (ii) $\|Ax\| < a$ for all $x \in \bar{P}_a$;
- (iii) $\alpha(Ax) > b$, for all $x \in P(\alpha, b, c)$ with $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2, x_3 satisfying

$$\|x_1\| < a, \quad b < \alpha(x_2), \quad \|x_3\| > a, \quad \text{and} \quad \alpha(x_3) < b.$$

3 Main result

In this section we consider the existence of three positive solutions for the problem (1.1).

We say u is concave on $[0, T]_{\mathbb{T}}$ if $u^{\Delta \nabla}(t) \leq 0$ for $t \in [0, T]_{\mathbb{T}^k \cap \mathbb{T}_k}$.

We note that $u(t)$ is a solution of the problem (1.1) if and only if

$$u(t) = \begin{cases} \int_0^T (T-s)\Phi_q\left(\int_0^s a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s \\ \quad - B_0\left(\int_0^\eta \Phi_q\left(\int_0^s -a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s\right) \\ \quad + \int_0^t (t-s)\Phi_q\left(\int_0^s -a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s, & t \in [0, T]_{\mathbb{T}}, \\ \varphi(t), & t \in [-r, 0]_{\mathbb{T}}. \end{cases}$$

Let $E = C_{\text{id}}^\Delta([0, T]_{\mathbb{T}}, \mathbb{R})$ with $\|u\| = \max\{\max_{t \in [0, T]_{\mathbb{T}}} |u(t)|, \max_{t \in [0, T]_{\mathbb{T}^k}} |u^\Delta(t)|\}$, $P = \{u \in E : u \text{ is nonnegative, decreasing and concave on } [0, T]_{\mathbb{T}}\}$. So E is a Banach space with the norm $\|u\|$ and P is a cone in E . For each $u \in E$, extend $u(t)$ to $[-r, T]_{\mathbb{T}}$ with $u(t) = \varphi(t)$ for $t \in [-r, 0]_{\mathbb{T}}$.

Define $F : P \rightarrow E$ by

$$\begin{aligned} (Fu)(t) = & \int_0^T (T-s)\Phi_q\left(\int_0^s a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s \\ & - B_0\left(\int_0^\eta \Phi_q\left(\int_0^s -a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s\right) \\ & + \int_0^t (t-s)\Phi_q\left(\int_0^s -a(r)f(u(r), u(\mu(r)))\nabla r\right)\nabla s, \quad t \in [0, T]_{\mathbb{T}}. \end{aligned}$$

It is well known that this operator F is completely continuous.

We seek a fixed point, u_1 , of F in the cone P . Define

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_1(t), & t \in [0, T]_{\mathbf{T}}. \end{cases}$$

Then $u(t)$ denotes a positive solution of the problem (1.1).

Lemma 3.1 $F : P \rightarrow P$.

Proof The proof of the lemma is similar to that of [7, 17], so we omit it here. □

Let $l \in \mathbf{T}$ be fixed such that $0 < l < \eta < T$, and set

$$Y_1 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) \leq 0\}; \quad Y_2 = \{t \in [0, T]_{\mathbf{T}} : \mu(t) > 0\}; \quad Y_3 = Y_1 \cap [l, \eta]_{\mathbf{T}}.$$

Throughout this section, we assume $Y_3 \neq \emptyset$ and $\int_{Y_3} a(r) \nabla r > 0$.

Now we define the nonnegative continuous concave functional $\alpha : P \rightarrow [0, \infty)$ by

$$\alpha(u) = \min_{t \in [l, \eta]_{\mathbf{T}}} u(t), \quad \forall u \in P.$$

It is easy to see that $\alpha(u) = u(\eta) \leq \max_{t \in [0, T]_{\mathbf{T}}} |u(t)| \leq \|u\|$ if $u \in P$ and $\alpha(Fu) = (Fu)(\eta)$.

For convenience, we denote

$$\rho = T(B + 2T)\Phi_q \left(\int_0^T a(r) \nabla r \right), \quad \delta = A \int_{Y_3} \Phi_q \left(\int_0^s a(r) \nabla s \right) \nabla s.$$

We now state the growth conditions on f so that the problem (1.1) has at least three positive solutions.

Theorem 3.1 Let $0 < a < b \leq \frac{A\Phi_q(\frac{m}{M})}{T(B+2T)}d < d \leq c$, and suppose that f satisfies the following conditions:

- (H₁) $f(x, \varphi(s)) < \Phi_p(\frac{a}{\rho})$, for all $0 \leq x \leq a$, uniformly in $s \in [-r, 0]_{\mathbf{T}}$; $f(x_1, x_2) < \Phi_p(\frac{a}{\rho})$, for all $0 \leq x_i \leq a, i = 1, 2$,
- (H₂) $f(x, \varphi(s)) \leq \Phi_p(\frac{c}{\rho})$, for all $0 \leq x \leq c$, uniformly in $s \in [-r, 0]_{\mathbf{T}}$; $f(x_1, x_2) \leq \Phi_p(\frac{c}{\rho})$, for all $0 \leq x_i \leq c, i = 1, 2$,
- (H₃) $f(x, \varphi(s)) > \Phi_p(\frac{b}{\rho})$, for all $b \leq x \leq d$, uniformly in $s \in [-r, 0]_{\mathbf{T}}$,
- (H₄) $\Phi_q(\min_{x \in [0, c]} f(x, \varphi(s))) \cdot \Phi_q(\frac{M}{m}) \int_{Y_3} \Phi_q(\int_0^s a(r) \nabla r) \nabla s \geq \Phi_q(\max_{x_1, x_2 \in [0, c]} f(x_1, x_2)) \cdot \Phi_q(\int_0^T a(r) \nabla r)$, uniformly in $s \in [-r, 0]_{\mathbf{T}}$.

Then the problem (1.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_i(t), & t \in [0, T]_{\mathbf{T}}, i = 1, 2, 3, \end{cases}$$

where $\|u_1\| < a, b < \alpha(u_2), \|u_3\| > a$ and $\alpha(u_3) < b$.

Proof We first assert that $F : \bar{P}_c \rightarrow \bar{P}_c$.

Indeed, if $u \in \bar{P}_c$, then, in view of Lemma 3.1, we have $F\bar{P}_c \subset P$. Furthermore, $\forall u \in \bar{P}_c$, we have $0 \leq u \leq c$, and then from (H₂), we have

$$\begin{aligned} |Fu(t)| &= \int_0^T (T-s)\Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &\quad - B_0 \left(\int_0^\eta \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\ &\quad + \int_0^t (t-s)\Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &< \int_0^T T\Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &\quad + B \int_0^T \Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &\quad + \int_0^T T\Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &= T(B+2T)\Phi_q \left(\int_{Y_1} a(r)f(u(r), \varphi(\mu(r))) \nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r))) \nabla r \right) \\ &\leq T(B+2T)\Phi_q \left(\int_0^T a(r) \nabla r \right) \frac{c}{\rho} \\ &= c, \\ |(Fu)^\Delta(t)| &= \left| \int_0^t \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right| \\ &\leq \int_0^T \Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\ &= T\Phi_q \left(\int_{Y_1} a(r)f(u(r), \varphi(\mu(r))) \nabla r + \int_{Y_2} a(r)f(u(r), u(\mu(r))) \nabla r \right) \\ &\leq T\Phi_q \left(\int_0^T a(r) \nabla r \right) \frac{c}{\rho} \\ &= \frac{c}{B+2T} \\ &\leq c. \end{aligned}$$

Therefore, $\|Fu\| \leq c$, i.e., $F : \bar{P}_c \rightarrow \bar{P}_c$.

By (H₁) and in a way similar to above, we arrive at $F : \bar{P}_a \rightarrow P_a$.

Next, we assert that $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \emptyset$ and $\alpha(Au) > b$ for all $u \in P(\alpha, b, d)$.

Let $u = \frac{b+d}{2}$, then $u \in P$, $\|u\| = \frac{b+d}{2} \leq d$ and $\alpha(u) = \frac{b+d}{2} > b$. That is, $\{u \in P(\alpha, b, d) : \alpha(u) > b\} \neq \emptyset$.

Moreover, $\forall u \in P(\alpha, b, d)$, we have $b \leq u(t) \leq d$, $t \in [0, \eta]_{\mathbb{T}}$, then from (H₃), we see that

$$\begin{aligned} \alpha(Fu) &= (Fu)(\eta) \\ &= \int_0^T (T-s)\Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \end{aligned}$$

$$\begin{aligned}
 & -B_0 \left(\int_0^\eta \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 & + \int_0^\eta (t-s) \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\
 \geq & -B_0 \left(\int_0^\eta \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 \geq & A \left(\int_0^\eta \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 \geq & A \left(\int_l^\eta \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 \geq & A \left(\int_{Y_3} \Phi_q \left(\int_0^s a(r)f(u(r), \varphi(\mu(r))) \nabla r \right) \nabla s \right) \\
 > & A \int_{Y_3} \Phi_q \left(\int_0^s a(r) \nabla s \right) \nabla s \frac{b}{\delta} \\
 = & b,
 \end{aligned}$$

as required.

Finally, we assert that $\alpha(Fu) > b$, for all $u \in P(\alpha, b, c)$ and $\|Fu\| > d$.

To see this, $\forall u \in P(\alpha, b, c)$ and $\|Fu\| > d$, then $0 \leq u(t) \leq c$, $t \in [0, T]_{\mathbb{T}}$, then from (H_4) , we have

$$\int_{Y_3} \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \geq \Phi_q \left(\frac{m}{M} \right) \Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right).$$

So,

$$\begin{aligned}
 \alpha(Fu) &= (Fu)(\eta) \\
 &= \int_0^T (T-s) \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\
 &\quad - B_0 \left(\int_0^\eta \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 &\quad + \int_0^\eta (t-s) \Phi_q \left(\int_0^s -a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \\
 &\geq A \left(\int_0^\eta \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 &\geq A \left(\int_l^\eta \Phi_q \left(\int_0^s a(r)f(u(r), u(\mu(r))) \nabla r \right) \nabla s \right) \\
 &\geq A \left(\int_{Y_3} \Phi_q \left(\int_0^s a(r)f(u(r), \varphi(\mu(r))) \nabla r \right) \nabla s \right) \\
 &\geq A \Phi_q \left(\frac{m}{M} \right) \Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \\
 &= \frac{A \Phi_q \left(\frac{m}{M} \right)}{T(B+2T)} T(B+2T) \Phi_q \left(\int_0^T a(r)f(u(r), u(\mu(r))) \nabla r \right) \\
 &\geq \frac{A \Phi_q \left(\frac{m}{M} \right)}{T(B+2T)} \|Fu\|
 \end{aligned}$$

$$\begin{aligned}
 &> \frac{A\Phi_q(\frac{m}{M})}{T(B+2T)}d \\
 &\geq b.
 \end{aligned}$$

To sum up, all the hypotheses of Theorem 2.1 are satisfied. Hence F has at least three fixed points, *i.e.*, the problem (1.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \\ u_i(t), & t \in [0, T]_{\mathbf{T}}, i = 1, 2, 3, \end{cases}$$

where $\|u_1\| < a$, $b < \alpha(u_2)$, $\|u_3\| > a$, and $\alpha(u_3) < b$. □

4 Example

Let $\mathbf{T} = [-\frac{3}{4}, -\frac{1}{4}] \cup \{0, \frac{3}{4}\} \cup \{\frac{1}{2}\}^{\mathbb{N}_0}$, where \mathbb{N}_0 denotes the set of all nonnegative integers.

Consider the following p -Laplacian functional dynamic equation on the time scale \mathbf{T} :

$$\begin{cases} [\Phi_p(u^{\Delta\nabla}(t))]^\nabla + a(t)[\frac{6u^3(t)}{u^3(t)+u^3(t-\frac{3}{4})+1} + \frac{1}{5}] = 0, & t \in (0, 1)_{\mathbf{T}}, \\ u(t) = \varphi(t) \equiv 0, & t \in [-\frac{3}{4}, 0]_{\mathbf{T}}, \\ u^{\Delta}(0) = u^{\Delta\nabla}(T) = 0, & u(T) + B_0(u^{\Delta}(\eta)) = 0, \end{cases} \tag{4.1}$$

where $T = 1$, $p = \frac{3}{2}$, $B = 2$, $A = 3$, $\mu : [0, 1]_{\mathbf{T}} \rightarrow [-\frac{3}{4}, 1]_{\mathbf{T}}$, and $\mu(t) = t - \frac{3}{4}$, $r = \frac{3}{4}$, $\eta = \frac{1}{2}$, $l = \frac{1}{4}$, $f(u, \varphi(s)) = \frac{6u^3}{u^3+1} + \frac{1}{5}$, $f(u_1, u_2) = \frac{6u_1^3}{u_1^3+u_2^3+1} + \frac{1}{5}$, and

$$a(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}]_{\mathbf{T}}, \\ -\frac{99}{50}t + \frac{199}{100}, & t \in [\frac{1}{2}, 1]_{\mathbf{T}}. \end{cases}$$

We deduce that $Y_1 = [0, \frac{3}{4}]_{\mathbf{T}}$, $Y_2 = (\frac{3}{4}, 1]_{\mathbf{T}}$, $Y_3 = [\frac{1}{4}, \frac{1}{2}]_{\mathbf{T}}$. Then $\int_{Y_3} \Phi_q(\int_0^s a(r)\nabla r)\nabla s = \frac{1}{16}$, $\int_0^T a(r)\nabla r = \int_0^1 a(r)\nabla r = \frac{503}{800}$.

Thus it is easy to see by calculation that $\rho = 4(\frac{503}{800})^2$, $\delta = \frac{3}{16}$.

Choose $a = \frac{1}{10}$, $b = 1$, $d = 42,000$, $c = 45,000$; then by $M = 1$, $m = \frac{1}{100}$ we have $0 < a < b < \frac{A\Phi_q(\frac{m}{M})}{T(B+2T)}d < d < c$, and then

$$f(u, \varphi(s)) \leq \frac{6}{1,001} + \frac{1}{5} \approx 0.2004 < \Phi_p(\frac{a}{\rho}) = \sqrt{\frac{\frac{1}{10}}{4(\frac{503}{800})^2}} \approx 0.2515, 0 \leq u \leq \frac{1}{10}, \text{ uniformly in } s \in [-r, 0]_{\mathbf{T}};$$

$$f(u_1, u_2) \leq \frac{6}{1,002} + \frac{1}{5} \approx 0.2004 < \Phi_p(\frac{a}{\rho}) = \sqrt{\frac{\frac{1}{10}}{4(\frac{503}{800})^2}} \approx 0.2515, 0 \leq u_i \leq \frac{1}{10}, i = 1, 2;$$

$$f(u, \varphi(s)) < 6.2 < \Phi_p(\frac{c}{\rho}) = \sqrt{\frac{45,000}{4(\frac{503}{800})^2}} \approx 165, 0 \leq u \leq 45,000, \text{ uniformly in } s \in [-\frac{3}{4}, 0]_{\mathbf{T}};$$

$$f(u_1, u_2) < 6.2 < \Phi_p(\frac{c}{\rho}) = \sqrt{\frac{45,000}{4(\frac{503}{800})^2}} \approx 165, 0 \leq u_i \leq 45,000, i = 1, 2;$$

$$f(u, \varphi(s)) \geq 3.2 > \Phi_p(\frac{b}{\delta}) = \sqrt{\frac{16}{3}} \approx 2.31, 1 \leq u \leq 42,000, \text{ uniformly in } s \in [-\frac{3}{4}, 0]_{\mathbf{T}};$$

$$\Phi_q(\min_{x \in [0, c]} f(x, \varphi(s))) \cdot \Phi_q(\frac{M}{m}) \int_{Y_3} \Phi_q(\int_0^s a(r)\nabla r)\nabla s = 25 > 15.4 >$$

$$\Phi_q(\max_{x_1, x_2 \in [0, c]} f(x_1, x_2)) \cdot \Phi_q(\int_0^T a(r)\nabla r), \text{ uniformly in } s \in [-\frac{3}{4}, 0]_{\mathbf{T}}.$$

Thus by Theorem 3.1, the BVP (4.1) has at least three positive solutions of the form

$$u(t) = \begin{cases} u_i(t), & t \in [0, T]_{\mathbf{T}}, i = 1, 2, 3, \\ \varphi(t), & t \in [-r, 0]_{\mathbf{T}}, \end{cases}$$

where $\|u_1\| < \frac{1}{10}$, $1 < \alpha(u_2)$, $\|u_3\| > \frac{1}{10}$, and $\alpha(u_3) < 1$.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The main idea of this paper was proposed by D-BW and D-BW prepared the manuscript initially and WG performed a part of steps of the proofs in this research. All authors read and approved the final manuscript.

Acknowledgements

The authors thank the referees and the editors for their helpful comments and suggestions. Research was supported by the Postdoctoral fund in China (Grant No. 2013M531717), the Excellent Young Teacher Training Program of Lanzhou University of Technology (Grant No. Q200907) and Natural Science Foundation of Gansu Province of China (Grant No. 1310RJYA080).

Received: 13 January 2014 Accepted: 28 April 2014 Published: 13 May 2014

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10.1186/1687-1847-2014-145

Cite this article as: Wang and Guan: Multiple positive solutions for third-order p -Laplacian functional dynamic equations on time scales. *Advances in Difference Equations* 2014, 2014:145

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