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# Existence of positive periodic solutions for third-order differential equation with strong singularity

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**Abstract**

Sufficient conditions are presented for the existence of positive periodic solutions for a third-order nonlinear differential equation with singularity. Besides, an example is given to illustrate the results.

**MSC:** 34K13; 34B16; 34B18

**Keywords:** third order; singularity; positive periodic solution; topological degree theorem

## 1 Introduction

In this paper, we consider the following third-order differential equation with singularity:

$$x''' + f(t, x')x'' + h(t, x)x' = g(t, x), \tag{1.1}$$

where $f$, $h$ are continuous function and $T$-periodic about $t$, $h(t,x) \leq 0$, $g : [0, T] \times (0, \infty) \to \mathbb{R}$ is an $L^2$-Carathéodory function, *i.e.*, it is measurable in the first variable and continuous in the second variable, and for every $0 < r < s$ there exists $h_{r,s} \in L^2[0, \omega]$ such that $|f(t, x(t))| \leq h_{r,s}$ for all $x \in [r, s]$ and a.e. $t \in [0, \omega]$, $f$ is $\omega$-periodic function about $t$. Equation (1.1) is singular at 0, which means that $g(t,x)$ becomes unbounded when $x \to 0^+$. We say that (1.1) is of repulsive type (resp. attractive type) if $g(t,x) \to +\infty$ (resp. $g(t,x) \to -\infty$) when $x \to 0$.

The study of singular differential equations began with the paper of Taliaferro. In 1979, Taliaferro [1] discussed the model equation with singularity

$$y'' + \frac{q(t)}{y^\alpha} = 0, \quad 0 < t < 1, \tag{1.2}$$

subject to

$$y(0) = 0 = y(1),$$

and obtained the existence of a solution for the problem. Here $\alpha > 0$, $q \in C(0, 1)$ with $q > 0$ on $(0, 1)$ and $\int_0^1 t(1 - t)q(t)\,dt < \infty$. We call the equation a strong force condition if $\alpha \geq 1$ and we call it a weak force condition if $0 < \alpha < 1$.

Taliaferro's work has attracted the attention of many specialists in differential equations and they have contributed to the research of singular differential equations (see, *e.g.*, [2–10]). Among these results, some are obtained for a second-order equation with strong force condition; see, *e.g.*, [5, 9]. With a strong singularity, the energy near the origin becomes infinite and this fact is helpful for obtaining either *a priori* bounds, which are needed for a classical application of the degree theory, or the fast rotation, which is needed in recent versions of the Poincaré-Birkhoff theorem. Afterwards, in 2007 Torres [10] considered the periodic problem for a singular second-order equation with the weak force condition and showed that weak singularities may help periodic solutions to exist, which has driven the study of weak singularities (see [7]).

At the beginning, most of work concentrated on second-order singular differential equation, as in the references we mentioned above. Recently there have been published some results on third-order singular differential equation (see [11–17]). For example, in [13], Sun and Liu considered the singular nonlinear third-order periodic boundary value problem

$$u''' + \rho^3 u = f(t,u), \quad 0 \le t \le 2\pi \tag{1.3}$$

with $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0, 1, 2$, where $\rho \in (0, 1/\sqrt{3})$ and $f$ is singular at $t = 0$, $t = 1$, and $u = 0$. Under suitable growth conditions, it is proved by constructing a special cone in $C[0, 2\pi]$ and employing fixed point index theory that the problem has at least one solution or at least two positive solutions. Afterwards, Li [15] investigated the third-order ordinary differential equation

$$u'''(t) = f\big(t, u(t), u'(t), u''(t)\big), \quad t \in \mathbb{R}, \tag{1.4}$$

where $f \in C(\mathbb{R} \times (0,\infty) \times \mathbb{R} \times \mathbb{R})$ is $\omega$-periodic in $t$, and $f(t,u,v,w)$ may be singular at $u = 0$. By applying of a fixed point theorem in cones, the author obtained that existence results of positive $\omega$-periodic solutions for (1.4). Recently, Ren *et al.* [17] studied the third-order nonlinear singular differential equation

$$x'''(t) + ax''(t) + bx'(t) + cx(t) = f\big(t, x(t)\big) + e(t). \tag{1.5}$$

Using Green's function for a third-order differential equation and some fixed point theorems, *i.e.*, the Leray-Schauder alternative principle and Schauder's fixed point theorem, they established three new existence results of periodic solutions for (1.5).

Based on the above work, in this paper we will study (1.1) and obtain the existence of periodic solutions by using topological degree theorem. The rest of this paper is organized as follows. In Section 2, we give some lemmas. In Section 3, by using topological degree theorem by Mawhin [18], some sufficient conditions are obtained for the existence of positive periodic solutions of (1.1). We, respectively, consider repulsive type and attractive type. In Section 4, an example is given to show the feasibility of the main result of this paper.

## 2 Some lemmas

**Lemma 2.1** [18, Theorem 2.4] *Let $X$, $Y$ be real normed spaces and $L : D(L) \subset X \to Y$ a linear Fredholm map of index zero. Assume that $\Omega \subset X$ is an open bounded set and*

*$N : \bar{\Omega} \to Y$ is an L-compact mapping. Assume that the following conditions are satisfied*:

(i) $Lx + \lambda Nx \neq 0$, *for each* $(x, \lambda) \in [(D(L) \setminus \ker L) \cap \partial\Omega] \times (0, 1)$;

(ii) $Nx \notin \operatorname{Im} L$, *for each* $x \in \ker L \cap \partial\Omega$;

(iii) $D_0(QN|_{\ker L}, \Omega \cap \ker L) \neq 0$, *where* $Q : Y \to Y$ *is a continuous projector such that* $\ker Q = \operatorname{Im} L$ *and* $D_0$ *is the Brouwer degree*,

*then the equation* $Lx + Nx = 0$ *has at least one solution in* $D(L) \cap \bar{\Omega}$.

For the sake of convenience, throughout this paper we will adopt the following notation:

$$|u|_\infty = \max_{t\in[0,T]} |u(t)|, \qquad |u|_0 = \min_{t\in[0,T]} |u(t)|,$$

$$|u|_p = \left(\int_0^T |u|^p \, dt\right)^{\frac{1}{p}}, \qquad \bar{h} = \frac{1}{T}\int_0^T h(t)\, dt.$$

**Lemma 2.2** [19] *If* $\omega \in C^1(\mathbb{R}, \mathbb{R})$ *and* $\omega(0) = \omega(T) = 0$, *then*

$$\int_0^T |\omega(t)|^p \, dt \le \left(\frac{T}{\pi_p}\right)^p \int_0^T |\omega'(t)|^p \, dt,$$

*where* $1 \le p < \infty$, $\pi_p = 2\int_0^{(p-1)/p} \frac{ds}{(1-\frac{s^p}{p-1})^{1/p}} = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)}$.

**Remark 2.1** When $p = 2$, $\pi_2 = 2\int_0^{(2-1)/2} \frac{ds}{(1-\frac{s^2}{2-1})^{1/2}} = \frac{2\pi(2-1)^{1/2}}{2\sin(\pi/2)} = \pi$.

**Lemma 2.3** [20] *If* $x \in C^2(\mathbb{R}, \mathbb{R})$ *with* $x(t + T) = x(t)$, *then*

$$|x'(t)|_2^2 \le \left(\frac{T}{2\pi}\right)^2 |x''(t)|_2^2.$$

**Lemma 2.4** *If* $x \in C^1(\mathbb{R}, \mathbb{R})$ *with* $x(t + T) = x(t)$, *and* $t_0 \in [0, T]$ *such that* $|x(t_0)| < d$, *then*

$$\left(\int_0^T |x(t)|^p \, dt\right)^{\frac{1}{p}} \le \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x'(t)|^p \, dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}}.$$

*Proof* Let $\omega(t) = x(t + t_0) - x(t_0)$, and then $\omega(0) = \omega(T) = 0$. By Lemma 2.2 and Minkowski's inequality, we have

$$\begin{aligned}
\left(\int_0^T |x(t)|^p \, dt\right)^{\frac{1}{p}} &= \left(\int_0^T |\omega(t) + x(t_0)|^p \, dt\right)^{\frac{1}{p}} \\
&\le \left(\int_0^T |\omega(t)|^p \, dt\right)^{\frac{1}{p}} + \left(\int_0^T |x(t_0)|^p \, dt\right)^{\frac{1}{p}} \\
&\le \left(\frac{T}{\pi_p}\right)\left(\int_0^T |\omega'(t)|^p \, dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}} \\
&= \left(\frac{T}{\pi_p}\right)\left(\int_0^T |x'(t)|^p \, dt\right)^{\frac{1}{p}} + dT^{\frac{1}{p}}.
\end{aligned}$$

This completes the proof of Lemma 2.4. □

**Lemma 2.5** *If* $x \in C^2(\mathbb{R}, \mathbb{R})$ *with* $x(t+T) = x(t)$, *and* $t_0 \in [0, T]$ *such that* $|x(t_0)| < d$, *then*

$$\left(\int_0^T |x(t)|^2\,dt\right)^{\frac{1}{2}} \le \left(\frac{T^2}{2\pi^2}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} + dT^{\frac{1}{2}}.$$

*Proof* From Lemma 2.3 and Lemma 2.4, we know that, when $p = 2$, we have

$$\begin{aligned}\left(\int_0^T |x(t)|^2\,dt\right)^{\frac{1}{2}} &\le \left(\frac{T}{\pi}\right)\left(\int_0^T |x'(t)|^2\,dt\right)^{\frac{1}{2}} + dT^{\frac{1}{2}} \\ &\le \left(\frac{T}{\pi}\right)\left(\frac{T}{2\pi}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} + dT^{\frac{1}{2}} \\ &= \left(\frac{T^2}{2\pi^2}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} + dT^{\frac{1}{2}}.\end{aligned}$$

This completes the proof of Lemma 2.5. □

## 3 Main results

First we consider (1.1) when $g(t,x)$ is of attractive type. Assume that

$$\varphi(t) = \lim_{x\to+\infty} \sup \frac{g(t,x)}{x} \tag{3.1}$$

exists uniformly a.e. $t \in [0,T]$, *i.e.*, for any $\varepsilon > 0$ there is $g_\varepsilon \in L^2(0,T)$ such that

$$g(t,x) \le \big(\varphi(t) + \varepsilon\big)x + g_\varepsilon(t) \tag{3.2}$$

for all $x > 0$ and a.e. $t \in [0,T]$. Moreover, $\varphi \in C(\mathbb{R},\mathbb{R})$ and $\varphi(t+T) = \varphi(t)$.

For the sake of convenience, we list the following assumptions which will be used repeatedly in the sequel:

($H_1$) There exist two positive constants $D_1 < D_2$ such that

$$g(t,x) < 0, \quad \text{for all } 0 < x < D_1; \quad \text{and} \quad g(t,x) > 0, \quad \text{for all } x > D_2.$$

($H_2$) (Decomposition condition) $g(t,x) = g_0(x) + g_1(t,x)$, where $g_0 \in C((0,\infty);\mathbb{R})$ and $g_1 : [0,T] \times [0,\infty) \to \mathbb{R}$ is an $L^2$-Carathéodory function, *i.e.*, it is measurable in the first variable and continuous in the second variable, and for any $b > 0$ there is $h_b \in L^2(0,T;\mathbb{R}_+)$ such that

$$\big|g_1(t,x)\big| \le h_b(t), \quad \text{a.e. } t \in [0,T], \forall 0 \le x \le b.$$

($H_3$) (Strong force condition at $x = 0$) $\int_0^1 g_0(x)\,dx = -\infty$.

($H_4$) There exists a positive constant $A$ such that $|f(t,u)| \le A$, for all $(t,u) \in [0,T] \times \mathbb{R}$.

**Theorem 3.1** *Assume that* (3.2), $h(t,x) \le 0$, *and* ($H_1$)-($H_4$) *hold. We have the following condition*:

(H$_5$) $A(\frac{T}{2\pi}) + |\varphi^+|_\infty(\frac{T^3}{\pi^2}) < 1$.

*Then* (1.1) *has at least one positive* $T$*-periodic solution*.

*Proof* Let $X = \{x : \mathbb{R} \to \mathbb{R}$ is $C^2$ and satisfies $x(t+T) \equiv x(t)\}$, endowed with the $C^2$-norm. Let $Y = L^2(0,T;\mathbb{R})$ with the $L^2$-norm.

Let $D(L) = \{x \in X : x'''$ is absolutely continuous on $\mathbb{R}\}$ and let $L : D(L) \to Y$ be the operator defined by

$$(Lx)(t) = x'''(t), \quad t \in \mathbb{R}.$$

Define a nonlinear mapping $N : Y \to Y$ by

$$(Nx)(t) = f\big(t, x'\big)x''(t) + h(t,x)x'(t) - g\big(t, x(t)\big). \tag{3.3}$$

Then (1.1) can be converted to the abstract equation $Lx + Nx = 0$. Define the projectors $P : X \to X$ and $Q : Y \to Y$ by

$$Px = \frac{1}{T}\int_0^T x(s)\,ds; \qquad Qy = \frac{1}{T}\int_0^T y(s)\,ds. \tag{3.4}$$

The real number $Px$ and $Qy$ are seen as elements of $X$ and $Y$ inasmuch constant function. It is easy to see that $\ker L = \mathbb{R}$, $\operatorname{Im} L = \{y \in Y : \int_0^T y(t)\,dt = 0\}$, $\ker Q = \operatorname{Im} L$, $\operatorname{Im} P = \ker L$, and then $L$ is a Fredholm linear mapping with zero index.

Let $K$ denote the inverse of $L|_{\ker P \cap D(L)}$. Then we have

$$\begin{aligned}[Ky](t) = {}& \frac{t}{2}\int_0^T (T-s)y(s)\,ds - \frac{t}{2T}\int_0^T (T-s)^2 y(s)\,ds - \frac{t^2}{2T}\int_0^T (T-s)y(s)\,ds \\ & + \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds. \end{aligned} \tag{3.5}$$

From (3.3), (3.4), and (3.5), it follows that $QN$ and $K(I-Q)N$ are continuous, and $QN(\bar{\Omega})$ is bounded and then $K(I-Q)N(\bar{\Omega})$ is compact for any open bounded $\Omega \subset X$, which means $N$ is $L$-compact on $\bar{\Omega}$.

Now we consider the following (homotopy) family of (1.1):

$$x''' + \lambda f\big(t, x'\big)x'' + \lambda h(t,x)x' = \lambda g(t,x), \quad \lambda \in [0,1], \tag{3.6}$$

*i.e.*, the abstract equation $Lx + \lambda Nx = 0$. We need to show that the set of all possible solutions of the family of (3.6) is, *a priori*, bounded in $C^2(\mathbb{R},\mathbb{R})$ by a constant independent of $\lambda \in [0,1]$.

Suppose that $x$ is a solution to (3.6) for some $\lambda \in [0,1]$. Let $t^*$, $t_*$ be, respectively, the global maximum point and global minimum point of $x'(t)$ on $[0,T]$; Firstly, we consider $x'(t^*) = \max_{t\in\mathbb{R}} x'(t) = \max_{t\in[0,T]} x'(t)$. Since $\int_0^T x'(t)\,dt = 0$, we know that there exist two points $t_1$, $t_2$ such that $x'(t_1) < 0$, $x'(t_2) > 0$. So, we get $x'(t^*) > x'(t_2) > 0$. Because $x'(t^*)$ is the maximum value of $x'(t)$, $x''(t^*) = 0$. Furthermore, we conclude

$$x'''\big(t^*\big) \le 0. \tag{3.7}$$

So, we have

$$x'''(t^*) + \lambda h(t^*, x(t^*))x'(t^*) = \lambda g(t^*, x(t^*)),$$

since $x'''(t^*) \leq 0$ and $h(t^*, x(t^*))x'(t^*) \leq 0$, and we get

$$g(t^*, x(t^*)) \leq 0.$$

From $(H_1)$ we obtain

$$x(t^*) \leq D_2. \tag{3.8}$$

Similarly, we get

$$g(t_*, x(t_*)) \geq 0.$$

From $(H_1)$ we obtain

$$x(t_*) \geq D_1. \tag{3.9}$$

From (3.8) and (3.9), we know that there exists a point $\xi \in [0, T]$ such that

$$D_1 \leq x(\xi) \leq D_2. \tag{3.10}$$

Therefore, we have

$$|x(t)| = \left| x(\xi) + \int_\xi^t x'(s)\,ds \right| \leq D_2 + \int_0^T |x'(s)|\,ds. \tag{3.11}$$

Multiplying by $x'(t)$ on both sides of (3.6) and integrating from 0 to $T$, we have

$$\begin{aligned}
&\int_0^T x'''(t)x'(t)\,dt + \lambda \int_0^T f(t, x')x''(t)x'(t)\,dt + \lambda \int_0^T h(t,x)\big(x'(t)\big)^2\,dt \\
&\quad = \lambda \int_0^T g(t, x(t))x'(t)\,dt.
\end{aligned}$$

Since $\int_0^T x'''(t)x'(t) = -\int_0^T |x''(t)|^2\,dt$, from $(H_4)$ and $h(t,x) \leq 0$, we have

$$\begin{aligned}
\int_0^T |x''(t)|^2\,dt &= \lambda \int_0^T f(t, x')x''(t)x'(t)\,dt + \lambda \int_0^T h(t,x)|x'(t)|^2\,dt \\
&\quad - \lambda \int_0^T g(t, x(t))x'(t)\,dt \\
&\leq \lambda \int_0^T f(t, x')x''(t)x'(t)\,dt - \lambda \int_0^T g(t, x(t))x'(t)\,dt \\
&\leq \int_0^T |f(t, x')||x''(t)||x'(t)|\,dt + \int_0^T |g(t, x(t))||x'(t)|\,dt
\end{aligned}$$

$$\leq A\int_0^T |x''(t)||x'(t)|\,dt + \int_0^T \big|g\big(t,x(t)\big)\big||x'(t)|\,dt$$
$$\leq A\int_0^T |x''(t)||x'(t)|\,dt + |x'|_\infty \int_0^T \big|g\big(t,x(t)\big)\big|\,dt.$$

For any $\varepsilon > 0$, let $g_\varepsilon \in L^2(0,T)$ be as in (3.2). Thus we have

$$g^+(t,x) \leq \big(\varphi^+(t) + \varepsilon\big)x(t) + g_\varepsilon^+(t).$$

Therefore,

$$\int_0^T |g^+(t,x)|\,dt \leq \big(|\varphi^+|_\infty + \varepsilon\big)\int_0^T |x(t)|\,dt + \int_0^T |g_\varepsilon^+(t)|\,dt.$$

Since $\int_0^T g(t,x)\,dt = 0$, we can get $\int_0^T |g(t,x)|\,dt = 2\int_0^T |g^+(t,x)|\,dt$. So, we have

$$\int_0^T |g(t,x)|\,dt \leq 2\big(|\varphi^+|_\infty + \varepsilon\big)\int_0^T |x(t)|\,dt + 2\int_0^T |g_\varepsilon^+(t)|\,dt. \tag{3.12}$$

From $x(0) = x(T)$, we know that there exists a point $\xi_1 \in [0,T]$ such that $x'(\xi_1) = 0$. So, we have

$$|x'|_\infty \leq \int_0^T |x''(t)|\,dt.$$

From (3.12) and the Hölder inequality, we have

$$\begin{aligned}\int_0^T |x''(t)|^2\,dt &\leq A\int_0^T |x''(t)||x'(t)|\,dt + 2\big(|\varphi^+|_\infty + \varepsilon\big)\int_0^T |x(t)|\,dt \int_0^T |x''(t)|\,dt \\ &\quad + 2\int_0^T |g_\varepsilon^+(t)|\,dt \int_0^T |x''(t)|\,dt \\ &\leq A\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |x'(t)|^2\,dt\right)^{\frac{1}{2}} \\ &\quad + 2\big(|\varphi^+|_\infty + \varepsilon\big)T\left(\int_0^T |x(t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} \\ &\quad + 2T\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |g_\varepsilon^+(t)|^2\,dt\right)^{\frac{1}{2}}.\end{aligned}$$

From (3.10) and Lemma 2.5, we have

$$\left(\int_0^T |x(t)|^2\,dt\right)^{\frac{1}{2}} \leq \left(\frac{T^2}{2\pi^2}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} + D_2\sqrt{T}. \tag{3.13}$$

From (3.13) and Lemma 2.3, we have

$$\begin{aligned}&\int_0^T |x''(t)|^2\,dt \\ &\quad \leq A\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}\cdot\left(\frac{T}{2\pi}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}\end{aligned}$$

$$
\begin{aligned}
&+ 2T\left(|\varphi^+|_\infty + \varepsilon\right)\left[\left(\frac{T^2}{2\pi^2}\right)\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} + D_2\sqrt{T}\right]\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} \\
&+ 2T\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}\left(\int_0^T |g_\varepsilon^+(t)|^2\,dt\right)^{\frac{1}{2}} \\
={}& A\left(\frac{T}{2\pi}\right)\int_0^T |x''(t)|^2\,dt + \left(|\varphi^+|_\infty + \varepsilon\right)\left(\frac{T^3}{\pi^2}\right)\int_0^T |x''(t)|^2\,dt \\
&+ \left[2T\left(|\varphi^+|_\infty + \varepsilon\right)D_2\sqrt{T} + 2T|g_\varepsilon^+|_2\right]\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}} \\
={}& \left[A\left(\frac{T}{2\pi}\right) + \left(|\varphi^+|_\infty + \varepsilon\right)\left(\frac{T^3}{\pi^2}\right)\right]\int_0^T |x''(t)|^2\,dt \\
&+ 2T\left[\left(|\varphi^+|_\infty + \varepsilon\right)D_2\sqrt{T} + |g_\varepsilon^+|_2\right]\left(\int_0^T |x''(t)|^2\,dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Since $\varepsilon$ sufficiently small, from ($H_5$) we know that $A(\frac{T}{2\pi}) + |\varphi^+|_\infty(\frac{T^3}{\pi^2}) < 1$. Thus, it is easy to see that there exists a positive constant $M_1'$ such that

$$\int_0^T |x''(t)|^2\,dt \le M_1'.$$

So, by the Hölder inequality, we have

$$|x'|_\infty \le \int_0^T |x''(t)|\,dt \le \sqrt{T}\left(\int_0^T |x''(t)|^2\right)^{\frac{1}{2}} \le \sqrt{T}M_1'^{\frac{1}{2}} := M_2. \tag{3.14}$$

From (3.11), we have

$$|x|_\infty \le D_2 + \int_0^T |x'(t)|\,ds \le D_2 + M_2 T := M_1. \tag{3.15}$$

On the other hand, from $x'(0) = x'(T)$, we know that there exists a point $\xi_2 \in [0, T]$ such that $x''(\xi_2) = 0$. From (3.2), (3.14), and (3.15), and by (3.6), we have

$$
\begin{aligned}
|x''|_\infty &\le \max_{t\in\mathbb{R}}\left|\int_{\xi_2}^t x'''(s)\,ds\right| \\
&\le \lambda\int_0^T |f(t,x')||x''(t)|\,dt + \lambda\int_0^T |h(t,x)||x'(t)|\,dt + \lambda\int_0^T |g(t,x)|\,dt \\
&\le \lambda\left(|f|_{M_2}\sqrt{T}M_1'^{\frac{1}{2}} + |h|_{M_1}TM_2 + 2\left(|\varphi^+|_\infty + \varepsilon\right)TM_1 + 2\sqrt{T}|g_\varepsilon^+|_2\right) \\
&:= \lambda M_3,
\end{aligned} \tag{3.16}
$$

where $|h|_{M_1} = \max_{|x|\le M_1}|h(t,x)|$, $|f|_{M_2} = \max_{|u|\le M_2}|f(t,u)|$.

Next, multiplying (3.6) by $x'(t)$ we get

$$
\begin{aligned}
&x'''(t)x'(t) + \lambda f\left(t,x'\right)x''(t)x'(t) + \lambda h(t,x)\left(x'(t)\right)^2 \\
&\quad = \lambda\left(g_0\left(x(t)\right) + g_1\left(t,x(t)\right)\right)x'(t).
\end{aligned} \tag{3.17}
$$

Let $\tau \in [0, T]$, for any $\tau \leq t \leq T$, we integrate (3.17) on $[\tau, t]$ and get

$$\begin{aligned} \lambda \int_{x(\tau)}^{x(t)} g_0(u)\,du &= \lambda \int_\tau^t g_0\big(x(s)\big)x'(s)\,ds \\ &= x''(t)x'(t) - x''(\tau)x'(\tau) - \int_\tau^t \big|x''(s)\big|^2\,dt + \lambda \int_\tau^t f\big(s,x'\big)x''(s)x'(s)\,ds \\ &\quad + \lambda \int_\tau^t h(s,x)\big(x'(t)\big)^2\,dt - \lambda \int_\tau^t g_1\big(s,x(s)\big)x'(s)\,ds. \end{aligned} \tag{3.18}$$

By (3.14), (3.15), and (3.16) we have

$$\begin{aligned} &x''(t)x'(t) \leq \lambda M_3 M_2, \\ &\left|\int_\tau^t \big|x''(s)\big|^2\,ds\right| \leq \lambda^2 M_3^2 T, \\ &\left|\int_\tau^t f\big(s,x'\big)x''(s)x'(s)\,ds\right| \leq \lambda |f|_{M_2} M_3 M_2 T, \\ &\left|\int_\tau^t h(s,x)x'(s)x'(s)\,ds\right| \leq |h|_{M_1} M_2^2 T, \\ &\left|\int_\tau^t g_1\big(s,x(s)\big)x'(s)\,ds\right| \leq \sqrt{T} M_2 |g_{M_1}|_2, \end{aligned}$$

where $g_{M_1} = \max_{0\leq x\leq M_1} |g_1(t,x)| \in L^2(0,T)$ is as in $(H_2)$.

From these inequalities we can derive form (3.18) that

$$\left|\int_{x(\tau)}^{x(t)} g_0(u)\,du\right| \leq M_4', \tag{3.19}$$

for some constant $M_4'$ which is independent on $\lambda$, $x$ and $t$. In view of the strong force condition $(H_3)$, we know that there exists a constant $M_4 > 0$ such that

$$x(t) \geq M_4, \quad \forall t \in [\tau, T]. \tag{3.20}$$

The case $t \in [0, \tau]$ can be treated similarly.

From (3.8), (3.14), (3.15), (3.16), and (3.20), we let

$$\Omega = \big\{x \in X : E_1 < x(t) < E_2, \big|x'(t)\big| < E_3 \text{ and } \big|x''(t)\big| < E_4, \forall t \in [0,T]\big\}, \tag{3.21}$$

where $0 < E_1 < \min(M_4, D_1)$, $E_2 > \max(M_1, D_2)$, $E_3 > M_2$ and $E_4 > M_3$. Then the conditions (i) and (ii) of Lemma 2.1 are satisfied. For a constant $x \in \ker L$, $x > 0$, we have

$$QNx = \frac{1}{T}\int_0^T g(t,x)\,dt.$$

The degree condition $(H_1)$ shows that

$$D_0(QN|_{\ker L}, \Omega \cap \ker L) = 1.$$

Thus (iii) of Lemma 2.1 is also verified. Therefore $Lx + Nx = 0$ has at least one solution in $\bar{\Omega}$, which means (1.1) has at least one positive $T$-periodic solution. □

Next we consider (1.1) when $g(t,x)$ is of repulsive type.

**Theorem 3.2** *Assume that* (3.2), $h(t,x) \leq 0$, ($H_2$), *and* ($H_4$), ($H_5$) *are satisfied. We have the following condition*:

($H_1'$) *there exist two positive constants* $D_1 < D_2$ *such that*

$$g(t,x(t)) > 0, \quad \textit{for all } 0 < x(t) < D_1; \quad \textit{and} \quad g(t,x(t)) < 0, \quad \textit{for all } x(t) > D_2.$$

($H_3'$) (*Strong force condition at* $x = 0$) $\int_0^1 g_0(x)\,dx = +\infty$.

*Then* (1.1) *has at least one positive* $T$-*periodic solution*.

*Proof* Let $\lambda \in [0,1]$ and consider the following:

$$x''' + \lambda f(t,x')x'' + \lambda h(t,x)x' = \lambda g(t,x). \tag{3.22}$$

Let $t^*$, $t_*$ be, respectively, the global maximum point and global minimum point of $x'(t)$ on $[0,T]$. First, we consider $x'(t_*) = \min_{t\in\mathbb{R}} x'(t) = \min_{t\in[0,T]} x'(t)$. Since $\int_0^T x'(t)\,dt = 0$, we know that there exist two points $t_1$, $t_2$ such that $x'(t_1) < 0$, $x'(t_2) > 0$. So, we get $x'(t_*) < x'(t_1) < 0$. Because $x'(t_*)$ is the minimum value of $x'(t)$, $x''(t_*) = 0$. Furthermore, we can conclude

$$x'''(t_*) \geq 0. \tag{3.23}$$

So, we have

$$x'''(t_*) + \lambda h(t_*,x(t_*))x'(t_*) = \lambda g(t_*,x(t_*)),$$

since $x'''(t_*) \geq 0$ and $h(t_*,x(t_*))x'(t_*) \geq 0$, we get

$$g(t_*,x(t_*)) \geq 0.$$

Hence, from ($H_1'$) we know that there exists a positive constant $D_2$ such that

$$x(t_*) \leq D_2. \tag{3.24}$$

Similarly, we get

$$g(t^*,x(t^*)) \leq 0.$$

Hence, from ($H_1'$) we know that there exists a positive constant $D_1$ such that

$$x(t^*) \geq D_1. \tag{3.25}$$

From (3.24) and (3.25), we know that there exists a point $\zeta \in [0, T]$ such that

$$D_1 \le x(\zeta) \le D_2.$$

Therefore, we have

$$\left|x(t)\right| = \left|x(\zeta) + \int_\zeta^t x'(s)\,ds\right| \le D_2 + \int_0^T \left|x'(s)\right|ds. \tag{3.26}$$

The rest of the proof is the same as that of Theorem 3.1. □

## 4 Examples

Finally, we present some examples to illustrate our result.

**Example 4.1** Consider the three-order differential equation with singularity:

$$\begin{aligned} &x'''(t) + \frac{1}{5}\bigl(2\sin 2t \cos x'(t) + 1\bigr)x''(t) + \frac{1}{10}\bigl(-x^2(t) - 3\bigr)x'(t) \\ &\quad = \frac{1}{25}(\sin 2t + 3)x(t) - \frac{1}{x(t)^\kappa}, \end{aligned} \tag{4.1}$$

where $\kappa \ge 1$.

It is clear that $T = \pi$, $f(t,x') = \frac{1}{5}(2\sin 2t\cos x' + 1)$, $h(t,x) = \frac{1}{10}(-x^2 - 3) < 0$, $g(t,x) = \frac{1}{25}(\sin 2t + 3)x - \frac{1}{x^r}$, $\varphi(t) = \frac{1}{25}(\sin 2t + 3)$. It is obvious that ($H_1$)-($H_4$) hold. Now we consider the assumption ($H_5$). Since $A \le \frac{3}{5}$, $|\varphi^+|_\infty \le \frac{4}{25}$, we have

$$\begin{aligned} &A\left(\frac{T}{2\pi}\right) + \left|\varphi^+\right|_\infty\left(\frac{T^3}{\pi^2}\right) \\ &\quad \le \frac{3}{5} \times \frac{\pi}{2\pi} + \frac{4}{25} \times \left(\frac{\pi^3}{\pi^2}\right) \\ &\quad = \frac{3}{5} \times \frac{1}{2} + \frac{4}{25} \times \pi \\ &\quad = \frac{15 + 8\pi}{50} < 1. \end{aligned}$$

So by Theorem 3.1, we know (4.1) has at least one positive $\pi$-periodic solution.

#### Competing interests
The author declares that they have no competing interests.

#### Author's contributions
ZBC worked together in the derivation of the mathematical results. The author read and approved the final manuscript.

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