Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Complex linear differential equations with certain analytic coefficients of $[p,q]$-order in the unit disc

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Abstract

In this paper, the authors study some growth properties of analytic functions of $[p,q]$-order in the disc and apply them to investigating the growth and zeros of solutions of complex linear differential equations with analytic coefficients of $[p,q]$-order satisfying certain growth conditions in the unit disc, and they obtain some results which are generalizations and improvements of some previous results.

MSC: 30D35; 34M10

Keywords: linear differential equations; unit disc; $[p,q]$-order; $[p,q]$-exponent of convergence of zero-sequence

1 Notations and results

We assume that readers are familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions in the unit disc (see [1–4]). Firstly, we introduce some notations. Let us define inductively, for $r \in (0,+\infty)$, $\exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r)$, $i \in \mathbb{N}$. For all r sufficiently large in $(0,+\infty)$, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r)$, $i \in \mathbb{N}$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover, we denote the linear measure of a set $E \subset [0,1)$ by $mE = \int_E dt$, and the upper and lower density of $E \subset [0,1)$ are defined, respectively, by

$$\overline{\mathrm{dens}}_\Delta E = \overline{\lim_{r\to 1^-}} \frac{m(E\cap [r,1))}{1-r}, \qquad \underline{\mathrm{dens}}_\Delta E = \underline{\lim}_{r\to 1^-} \frac{m(E\cap [r,1))}{1-r}.$$

The complex oscillation theory of linear differential equations

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = 0 \tag{1.1}$$

and

$$f^{(k)} + A_{k-1}(z)f^{(k-1)} + \cdots + A_0(z)f = F(z) \tag{1.2}$$

in the unit disc has been developed since the 1980s (see [5]). After that, many important results have been obtained (see [6–11]). After the work of Liu *et al.* in [12], there has been an increasing interest in studying the interaction between the analytic coefficients of $[p,q]$-order and the solutions of (1.1) and (1.2) (see [13–16]). In this paper, the authors continue to focus on studying the growth and zeros of solutions of (1.1), (1.2) with analytic coefficients of $[p,q]$-order which satisfy certain growth conditions in the unit disc.

We use p, q to denote positive integers, and we use $\Delta = \{z : |z| < 1\}$ to denote the unit disc. In the following, we recall some notations of meromorphic functions and analytic functions in Δ.

Definition 1.1 (see [3, 10]) Let $f(z)$ be a meromorphic function in Δ, and set

$$D(f) = \overline{\lim_{r\to 1^-}} \frac{T(r,f)}{-\log(1-r)}.$$

If $f(z)$ is an analytic function in Δ,

$$D_M(f) = \overline{\lim_{r\to 1^-}} \frac{\log M(r,f)}{-\log(1-r)}.$$

If $D(f) = \infty$, we say that f is admissible, if $D(f) < \infty$, we say that f is non-admissible. If $D_M(f) = \infty$, we say that f is of infinite degree, if $D_M(f) < \infty$, we say that f is of finite degree.

Definition 1.2 (see [7, 11]) The iterated p-order of a meromorphic function $f(z)$ in Δ is defined by

$$\sigma_p(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p T(r,f)}{-\log(1-r)}.$$

For an analytic function $f(z)$ in Δ, we also define

$$\sigma_{M,p}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r,f)}{-\log(1-r)}.$$

Remark 1.1 If $p = 1$, we denote $\sigma_1(f) = \sigma(f)$ and $\sigma_{M,1}(f) = \sigma_M(f)$, and we have $\sigma(f) \le \sigma_M(f) \le \sigma(f) + 1$ (see [3]) and $\sigma_{M,p}(f) = \sigma_p(f)$ for $p > 1$ (see [7, 11]).

Definition 1.3 (see [13–15]) Let $1 \le q \le p$ or $2 \le q = p + 1$, and $f(z)$ be a meromorphic function in Δ, then the $[p,q]$-order of $f(z)$ is defined by

$$\sigma_{[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})}.$$

For an analytic function $f(z)$ in Δ, we also define

$$\sigma_{M,[p,q]}(f) = \overline{\lim_{r\to 1^-}} \frac{\log_{p+1} M(r,f)}{\log_q(\frac{1}{1-r})}.$$

Definition 1.4 (see [13]) Let $1 \le q \le p$ or $2 \le q = p + 1$, we use $N(r, \frac{1}{f})$ $(\overline{N}(r, \frac{1}{f}))$ to denote the integrated counting function for the (distinct) zero-sequence of a meromorphic function $f(z)$ in Δ. Then the $[p,q]$-exponents of convergence of (distinct) zero-sequence of $f(z)$

about $N(r,\frac{1}{f})$ ($\overline{N}(r,\frac{1}{f})$) are defined, respectively, by

$$\lambda^{N}_{[p,q]}(f)=\varlimsup_{r\to 1^-}\frac{\log_p N(r,\frac{1}{f})}{\log_q(\frac{1}{1-r})},\qquad \overline{\lambda}^{N}_{[p,q]}(f)=\varlimsup_{r\to 1^-}\frac{\log_p \overline{N}(r,\frac{1}{f})}{\log_q(\frac{1}{1-r})}.$$

By the above definitions, the following propositions about the analytic function of $[p,q]$-order in the unit disc can easily be deduced.

Proposition 1.1 *Let $f(z)$ be an analytic function of $[p,q]$-order in Δ. Then the following five statements hold*:

(i) *If $p=q=1$, then $\sigma(f)\le\sigma_M(f)\le 1+\sigma(f)$.*
(ii) *If $p=q\ge 2$ and $\sigma_{[p,q]}(f)<1$, then $\sigma_{[p,q]}(f)\le\sigma_{M,[p,q]}(f)\le 1$.*
(iii) *If $p=q\ge 2$ and $\sigma_{[p,q]}(f)\ge 1$, or $p>q\ge 1$, then $\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f)$.*
(iv) *If $p\ge 1$ and $\sigma_{[p,p+1]}(f)>1$, then $D(f)=\infty$; if $\sigma_{[p,p+1]}(f)<1$, then $D(f)=0$.*
(v) *If $p\ge 1$ and $\sigma_{M,[p,p+1]}(f)>1$, then $D_M(f)=\infty$; if $\sigma_{M,[p,p+1]}(f)<1$, then $D_M(f)=0$.*

Proof (i), (iv), (v) hold obviously, we prove (ii) and (iii).

(ii) By the standard inequality $T(r,f)\le\log^+ M(r,f)\le\frac{1+3r}{1-r}T(\frac{1+r}{2},f)$ $(0<r<1)$ (see [1, 2, 4]), we get

$$\log_p T(r,f)\le\log^+_{p+1} M(r,f)\le\max\left\{\log_p\left(\frac{4}{1-r}\right),\log_p T\left(\frac{1+r}{2},f\right)\right\}. \tag{1.3}$$

If $p=q\ge 2$ and $\sigma_{[p,q]}(f)<1$, from (1.3) we have $\sigma_{[p,q]}(f)\le\sigma_{M,[p,q]}(f)\le 1$.

(iii) If $p=q\ge 2$ and $\sigma_{[p,q]}(f)\ge 1$, or $p>q\ge 1$, from (1.3), we have $\sigma_{[p,q]}(f)=\sigma_{M,[p,q]}(f)$. □

Proposition 1.2 *Let $f(z)$ be a meromorphic function of $[p,q]$-order in Δ. Then the following statements hold*:

(i) *If $p>q\ge 1$, then $\overline{\lambda}^{N}_{[p,q]}(f)=\overline{\lambda}^{n}_{[p,q]}(f)$.*
(ii) *If $p=q=1$, then $\overline{\lambda}^{N}(f)\le\overline{\lambda}^{n}(f)\le\overline{\lambda}^{N}(f)+1$.*
(iii) *If $p=q\ge 2$, then $\overline{\lambda}^{N}_{[p,p]}(f)\le\overline{\lambda}^{n}_{[p,p]}(f)\le\max\{\overline{\lambda}^{N}_{[p,p]}(f),1\}$. Furthermore, we have $\overline{\lambda}^{N}_{[p,p]}(f)=\overline{\lambda}^{n}_{[p,p]}(f)$ if $\overline{\lambda}^{N}_{[p,p]}(f)\ge 1$, and if $\overline{\lambda}^{N}_{[p,p]}(f)<1$ then $\overline{\lambda}^{N}_{[p,p]}(f)\le\overline{\lambda}^{n}_{[p,p]}(f)\le 1$.*

Proof Without loss of generality, assuming that $f(0)\neq 0$, by $\overline{N}(r,\frac{1}{f})=\int_0^r\frac{\overline{n}(t,\frac{1}{f})}{t}\,dt$, we have

$$\overline{n}\left(r,\frac{1}{f}\right)\le\frac{1}{\log(1+\frac{1-r}{2r})}\int_r^{r+\frac{1-r}{2}}\frac{\overline{n}(t,\frac{1}{f})}{t}\,dt\le\frac{1}{\log(1+\frac{1-r}{2r})}\overline{N}\left(\frac{1+r}{2},\frac{1}{f}\right)\quad(0<r<1),$$

since $\log(1+\frac{1-r}{2r})\sim\frac{1-r}{2r}$ $(r\to 1^-)$, we obtain

$$\varlimsup_{r\to 1^-}\frac{\log_p\overline{n}(r,\frac{1}{f})}{\log_q(\frac{1}{1-r})}\le\max\left\{\varlimsup_{r\to 1^-}\frac{\log_p\overline{N}(\frac{1+r}{2},\frac{1}{f})}{\log_q(\frac{1}{1-r})},\varlimsup_{r\to 1^-}\frac{\log_p(\frac{2r}{1-r})}{\log_q(\frac{1}{1-r})}\right\}.$$

By the above inequality, we obtain:

(i) if $p>q\ge 1$, then $\overline{\lambda}^{n}_{[p,q]}(f)\le\overline{\lambda}^{N}_{[p,q]}(f)$;

(ii) if $p = q = 1$, then $\overline{\lambda}^{\overline{n}}(f) \le \overline{\lambda}^{\overline{N}}(f) + 1$;
(iii) if $p = q \ge 2$, then $\overline{\lambda}^{\overline{n}}_{[p,p]}(f) \le \max\{\overline{\lambda}^{\overline{N}}_{[p,p]}(f), 1\}$.

On the other hand, by

$$\begin{aligned}\overline{N}\left(r, \frac{1}{f}\right) &= \int_{r_0}^{r} \frac{\overline{n}(t, \frac{1}{f})}{t}\, dt + \overline{N}\left(r_0, \frac{1}{f}\right)\\ &\le \overline{n}\left(r, \frac{1}{f}\right) \log\left(\frac{r}{r_0}\right) + \overline{N}\left(r_0, \frac{1}{f}\right) \quad (0 < r_0 < r < 1),\end{aligned}$$

we can easily get $\overline{\lambda}^{\overline{N}}_{[p,q]}(f) \le \overline{\lambda}^{\overline{n}}_{[p,q]}(f)$ $(p \ge q \ge 1)$. Therefore, the conclusions of Proposition 1.2 hold. □

In recent years, Belaïdi has investigated the growth of solutions of (1.1), (1.2) with analytic coefficients of $[p, q]$-order in the unit disc and obtained the following results.

Theorem A (see [13]) *Let $p \ge q \ge 1$ be integers and H_1 be a set of complex numbers satisfying $\overline{\text{dens}}_\Delta\{|z| : z \in H_1 \subseteq \Delta\} > 0$, and let $A_0, A_1, \ldots, A_{k-1}$ be analytic functions in Δ satisfying $\max\{\sigma_{M,[p,q]}(A_j) : j = 1, \ldots, k-1\} \le \sigma_{M,[p,q]}(A_0) = \sigma_1$. Suppose that there exists a real number α_1 satisfying $0 \le \alpha_1 < \sigma_1$ such that, for any given ε $(0 < \varepsilon < \sigma_1 - \alpha_1)$, we have*

$$\left|A_0(z)\right| \ge \exp_{p+1}\left\{(\sigma_1 - \varepsilon) \log_q\left(\frac{1}{1 - |z|}\right)\right\}$$

and

$$\left|A_j(z)\right| \le \exp_{p+1}\left\{\alpha_1 \log_q\left(\frac{1}{1 - |z|}\right)\right\} \quad (j = 1, 2, \ldots, k-1),$$

as $|z| \to 1^-$ for $z \in H_1$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0) = \sigma_1$.

Theorem B (see [15]) *Let $p \ge q \ge 1$ be integers and H_2 be a set of complex numbers satisfying $\overline{\text{dens}}_\Delta\{|z| : z \in H_2 \subseteq \Delta\} > 0$, and let $A_0, A_1, \ldots, A_{k-1}$ be analytic functions in Δ satisfying $\max\{\sigma_{[p,q]}(A_j) : j = 1, \ldots, k-1\} \le \sigma_{[p,q]}(A_0) = \sigma_2$. Suppose that there exists a real number β_1 satisfying $0 \le \beta_1 < \sigma_2$ such that, for any given ε $(0 < \varepsilon < \sigma_2 - \beta_1)$, we have*

$$T(r, A_0) \ge \exp_p\left\{(\sigma_2 - \varepsilon) \log_q\left(\frac{1}{1 - |z|}\right)\right\}$$

and

$$T(r, A_j) \le \exp_p\left\{\beta_1 \log_q\left(\frac{1}{1 - |z|}\right)\right\} \quad (j = 1, 2, \ldots, k-1),$$

as $|z| \to 1^-$ for $z \in H_2$. Then every solution $f \not\equiv 0$ of (1.1) satisfies $\sigma_{[p,q]}(f) = \sigma_{M,[p,q]}(f) = \infty$ and $\sigma_{[p,q]}(A_0) \le \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \le \max\{\sigma_{M,[p,q]}(A_j) : j = 0, 1, \ldots, k-1\}$. Furthermore, if $p > q$, then $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_{[p,q]}(A_0)$.

Theorem C (see [13]) *Suppose that the assumptions of Theorem* A *are satisfied, and let $F \not\equiv 0$ be an analytic function in Δ of $[p, q]$-order. Then the following two statements hold:*

(i) *If* $\sigma_{[p+1,q]}(F) < \sigma_{M,[p,q]}(A_0)$, *then every solution* f *of* (1.2) *satisfies* $\overline{\lambda}_{[p+1,q]}(f) = \lambda_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_{M,[p,q]}(A_0)$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,q]}(f_0) < \sigma_{M,[p,q]}(A_0)$.

(ii) *If* $\sigma_{[p+1,q]}(F) > \sigma_{M,[p,q]}(A_0)$, *then every solution* f *of* (1.2) *satisfies* $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$.

From Theorems A-C, we obtain the following results.

Theorem 1.1 *Let* H_3 *be a complex set satisfying* $\overline{\text{dens}}_{\Delta}\{|z| : z \in H_3 \subseteq \Delta\} > 0$. *If* $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *are analytic functions in* Δ *satisfying* $\max\{\sigma_{M,[p,q]}(A_j)|j = 0,1,\ldots,k-1\} \leq \sigma_3$ $(0 < \sigma_3 < \infty)$, *and if there exist two positive constants* α_2, β_2 $(0 < \beta_2 < \alpha_2)$ *such that, for all* $z \in H_3$ *and* $|z| \to 1^-$, *we have*

$$\bigl|A_0(z)\bigr| \geq \exp_p\left\{\alpha_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\}$$

and

$$\bigl|A_j(z)\bigr| \leq \exp_p\left\{\beta_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\} \quad (j = 1,\ldots,k-1).$$

Then the following statements hold:

(i) *If* $p \geq q \geq 1$, *then every solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_3$.

(ii) *If* $2 \leq q = p+1$ *and* $\sigma_3 > 1$, *then every solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_{[p+1,p+1]}(f) = \sigma_{M,[p+1,p+1]}(f) = \sigma_3$.

Theorem 1.2 *Let* H_4 *be a complex set satisfying* $\overline{\text{dens}}_{\Delta}\{|z| : z \in H_4 \subseteq \Delta\} > 0$. *If* $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *are analytic functions in* Δ *satisfying* $\max\{\sigma_{M,[p,q]}(A_j)|j = 0,1,\ldots,k-1\} \leq \sigma_4$ $(0 < \sigma_4 < \infty)$, *and there exist two positive constants* α_3, β_3 *such that, for all* $z \in H_4$ *and* $|z| \to 1^-$, *we have*

$$T\bigl(r, A_0(z)\bigr) \geq \exp_{p-1}\left\{\alpha_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\}$$

and

$$T\bigl(r, A_j(z)\bigr) \leq \exp_{p-1}\left\{\beta_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\} \quad (j = 1,\ldots,k-1).$$

Then the following statements hold:

(i) *If* $p \geq q \geq 2$ *and* $0 < \beta_3 < \alpha_3$, *then every solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_4$.

(ii) *If* $3 \leq q = p+1$, $0 < \beta_3 < \alpha_3$ *and* $\sigma_4 > 1$, *then every solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_{[p+1,p+1]}(f) = \sigma_{M,[p+1,p+1]}(f) = \sigma_4$.

(iii) *If* $p = 1$, $q = 2$, $0 < k\beta_3 < \alpha_3$ *and* $\sigma_4 > 1$, *then every solution* $f(z) \not\equiv 0$ *of* (1.1) *satisfies* $\sigma_{[2,2]}(f) = \sigma_{M,[2,2]}(f) = \sigma_4$.

Theorem 1.3 *Let* $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *be analytic functions in* Δ. *Suppose that* H_3, $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *satisfy the hypotheses in Theorem* 1.1, *then we have the following statements*:

(i) *Let* $1 \le q \le p$, *if* $\sigma_{[p+1,q]}(F) > \sigma_3$, *then all solutions of* (1.2) *satisfy* $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$; *if* $\sigma_{[p+1,q]}(F) \le \sigma_3$, *then all solutions of* (1.2) *satisfy* $\overline{\lambda}^N_{[p+1,q]}(f) = \lambda^N_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_3$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,q]}(f_0) < \sigma_3$.

(ii) *Let* $2 \le q = p + 1$, $\sigma_3 > 1$, *if* $\sigma_{[p+1,p+1]}(F) > \sigma_3$, *then all solutions of* (1.2) *satisfy* $\sigma_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(F)$; *if* $\sigma_{[p+1,p+1]}(F) \le \sigma_3$, *then all solutions of* (1.2) *satisfy* $\overline{\lambda}^N_{[p+1,p+1]}(f) = \lambda^N_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \sigma_3$, *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,p+1]}(f_0) < \sigma_3$.

Corollary 1.4 *Let* $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *be analytic functions in* Δ. *Suppose that* H_4, $A_j(z)$ $(j = 0,1,\ldots,k-1)$ *satisfy the hypotheses in Theorem* 1.2, *then we have the following statements*:

(i) *Let* $2 \le q \le p$, $0 < \beta_3 < \alpha_3$, *if* $\sigma_{[p+1,q]}(F) > \sigma_4$, *then all solutions of* (1.2) *satisfy* $\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F)$; *if* $\sigma_{[p+1,q]}(F) \le \sigma_4$, *then all solutions of* (1.2) *satisfy* $\overline{\lambda}^N_{[p+1,q]}(f) = \lambda^N_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f) = \sigma_4$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,q]}(f_0) < \sigma_4$.

(ii) *Let* $3 \le q = p + 1$, $0 < \beta_3 < \alpha_3$, $\sigma_4 > 1$, *if* $\sigma_{[p+1,p+1]}(F) > \sigma_4$, *then all solutions of* (1.2) *satisfy* $\sigma_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(F)$; *if* $\sigma_{[p+1,p+1]}(F) \le \sigma_4$, *then all solutions of* (1.2) *satisfy* $\overline{\lambda}^N_{[p+1,p+1]}(f) = \lambda^N_{[p+1,p+1]}(f) = \sigma_{[p+1,p+1]}(f) = \sigma_4$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[p+1,p+1]}(f_0) < \sigma_4$.

(iii) *Let* $p = 1$, $q = 2$, $0 < k\beta_3 < \alpha_3$, $\sigma_4 > 1$, *if* $\sigma_{[2,2]}(F) > \sigma_4$, *then all solutions of* (1.2) *satisfy* $\sigma_{[2,2]}(f) = \sigma_{[2,2]}(F)$; *if* $\sigma_{[2,2]}(F) \le \sigma_4$, *then all solutions of* (1.2) *satisfy* $\overline{\lambda}^N_{[2,2]}(f) = \lambda^N_{[2,2]}(f) = \sigma_{[2,2]}(f) = \sigma_4$ *with at most one exceptional solution* f_0 *satisfying* $\sigma_{[2,2]}(f_0) < \sigma_4$.

Remark 1.2 If a set $E \subset [0,1)$ satisfies $\overline{\operatorname{dens}}\, E > 0$, then $\int_E \frac{dt}{1-t} = +\infty$.

2 Preliminary lemmas

Lemma 2.1 (see [10]) *Let* $f(z)$ *be a meromorphic function in* Δ, *and let* $k \ge 1$ *be an integer. Then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = S(r,f),$$

where $S(r,f) = O\{\log^+ T(r,f) + \log(\frac{1}{1-r})\}$, *possibly outside a set* $E_1 \subset [0,1)$ *with* $\int_{E_1} \frac{dt}{1-t} < \infty$.

Remark 2.1 Throughout this paper, we use $E_1 \subset [0,1)$ to denote a set satisfying $\int_{E_1} \frac{dt}{1-t} < \infty$, not always the same at each occurrence.

Lemma 2.2 (see [9]) *Let* k *and* j *be integers satisfying* $k > j \ge 0$, *and let* $\varepsilon > 0$ *and* $d \in (0,1)$. *If* f *is a meromorphic function in* Δ *such that* $f^{(j)}$ *does not vanish identically, then*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\left(\frac{1}{1-|z|}\right)^{2+\varepsilon} \cdot \max\left\{\log\left(\frac{1}{1-|z|}\right), T\big(s(|z|),f\big)\right\}\right)^{k-j} \quad \big(|z| \notin E_1\big),$$

where $s(|z|) = 1 - d(1 - |z|)$.

Lemma 2.3 (see [14]) *Let* $1 \le q \le p$ *be integers. If* $A_0(z), \dots, A_{k-1}(z)$ *are analytic functions of* $[p,q]$*-order in the unit disc. Then every solution* f *of* (1.1) *satisfies* $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \le \max\{\sigma_{M,[p,q]}(A_j) | j = 0, 1, \dots, k-1\}$.

By a similar proof to Lemma 2.3, we have the following lemma.

Lemma 2.4 *If* $A_0(z), \dots, A_{k-1}(z)$ *are analytic functions of* $[p, p+1]$*-order in the unit disc with* $\max\{\sigma_{M,[p,p+1]}(A_j) | j = 0, 1, \dots, k-1\} < \infty$. *Then every solution* f *of* (1.1) *satisfies* $\sigma_{[p+1,p+1]}(f) \le \sigma_{M,[p+1,p+1]}(f) \le \max\{\sigma_{M,[p,p+1]}(A_j) | j = 0, 1, \dots, k-1\}$.

Lemma 2.5 *Let* $1 \le q \le p$ *or* $2 \le q = p+1$ *and* $f(z)$ *be an analytic function in* Δ *satisfying* $0 \le \sigma_{[p,q]}(f) = \sigma_5 \le \infty$ (*or* $0 \le \sigma_{M,[p,q]}(f) = \sigma_5 \le \infty$), *then there exists a set* $E_2 \subset [0,1)$ *satisfying* $\int_{E_2} \frac{dt}{1-t} = +\infty$ *such that, for all* $r \in E_2$, *we have*

$$\lim_{r \to 1^-} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})} = \sigma_5 \qquad \left(\lim_{r \to 1^-} \frac{\log_{p+1} M(r,f)}{\log_q(\frac{1}{1-r})} = \sigma_5\right).$$

Proof If $1 \le q \le p$, by Definition 1.3, there exists a sequence $\{r_n\}_{n=1}^{\infty} \to 1^-$ satisfying $1 - d(1-r_n) < r_{n+1}$ $(0 < d < 1)$ and

$$\lim_{n \to \infty} \frac{\log_p T(r_n,f)}{\log_q(\frac{1}{1-r_n})} = \sigma_{[p,q]}(f) = \sigma_5.$$

Therefore there exists an n_1 $(\in \mathbb{N})$ such that, for $n \ge n_1$ and for any $r \in E_2 = \bigcup_{n=n_1}^{\infty} [r_n, 1 - d(1-r_n)]$, we have

$$\frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})} \ge \frac{\log_p T(r_n,f)}{\log_q[\frac{1}{1-[1-d(1-r_n)]}]} = \frac{\log_p T(r_n,f)}{\log_q[\frac{1}{d(1-r_n)}]}.$$

Hence

$$\underline{\lim}_{r \to 1^-} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})} \ge \sigma_5 \quad (r \in E_2).$$

Since $\sigma_{[p,q]}(f) = \sigma_5$, for any $r \in E_2$, we have

$$\lim_{r \to 1^-} \frac{\log_p T(r,f)}{\log_q(\frac{1}{1-r})} = \sigma_5,$$

where

$$m_l E_2 = \sum_{n=n_1}^{\infty} \int_{r_n}^{1-d(1-r_n)} \frac{dt}{1-t} = \sum_{n=n_1}^{\infty} \log \frac{1}{d} = +\infty.$$

We also can prove $\lim_{r \to 1^-} \frac{\log_{p+1} M(r,f)}{\log_q(\frac{1}{1-r})} = \sigma_5$ $(r \in E_2)$ by the above proof.

By the above proof, this lemma also holds for the case $2 \le q = p+1$. □

Lemma 2.6 *Let* $A_j(z)$ $(j = 0, 1, \dots, k-1)$, $F(z) \not\equiv 0$ *be analytic functions in* Δ. *Then the following statements hold*:

(i) *If* $p \geq q \geq 1$ *and* $f(z)$ *is a solution of* (1.2) *satisfying* $\max\{\sigma_{[p,q]}(A_j), \sigma_{[p,q]}(F) | j = 0, 1, \ldots, k-1\} < \sigma_{[p,q]}(f)$, *then* $\overline{\lambda}^{N}_{[p,q]}(f) = \lambda^{N}_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

(ii) *If* $f(z)$ *is a solution of* (1.2) *satisfying* $\max\{\sigma_{[p,p+1]}(A_j), \sigma_{[p,p+1]}(F), 1 | j = 0, 1, \ldots, k-1\} < \sigma_{[p,p+1]}(f)$, *then* $\overline{\lambda}^{N}_{[p,p+1]}(f) = \lambda^{N}_{[p,p+1]}(f) = \sigma_{[p,p+1]}(f)$.

Proof (i) Suppose that $f(z) \not\equiv 0$ is a solution of (1.2). By (1.2), we get

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}\frac{f^{(k-1)}}{f} + \cdots + A_0\right), \tag{2.1}$$

and it is easy to see that if f has a zero at z_0 of order α ($\alpha > k$), and $A_0, \ldots, A_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \leq k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right)$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right). \tag{2.2}$$

By Lemma 2.1 and (2.1), we have

$$m\left(r, \frac{1}{f}\right) \leq m\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r, A_j) + O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\} \quad (r \notin E_1). \tag{2.3}$$

By (2.2)-(2.3), we get

$$\begin{aligned} T(r,f) &= T\left(r, \frac{1}{f}\right) + O(1) \\ &\leq k\overline{N}\left(r, \frac{1}{f}\right) + T(r,F) + \sum_{j=0}^{k-1} T(r, A_j) \\ &\quad + O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\} \quad (r \notin E_1). \end{aligned} \tag{2.4}$$

Since $\max\{\sigma_{[p,q]}(F), \sigma_{[p,q]}(A_j) | j = 0, 1, \ldots, k-1\} < \sigma_{[p,q]}(f)$, by Lemma 2.5 and Definition 1.3, there exists a set E_3 with $\int_{E_3} \frac{dt}{1-t} = +\infty$ such that

$$\max\left\{\frac{T(r,F)}{T(r,f)}, \frac{T(r,A_j)}{T(r,f)}\right\} \to 0 \quad \left(r \to 1^-, r \in E_3, j = 0, \ldots, k-1\right). \tag{2.5}$$

By (2.4)-(2.5), for all $|z| = r \in E_3 \setminus E_1$, we have

$$\left(1 - o(1)\right)T(r,f) \leq k\overline{N}\left(r, \frac{1}{f}\right) + O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\},$$

then we get $\sigma_{[p,q]}(f) \leq \overline{\lambda}^{N}_{[p,q]}(f)$. Therefore $\overline{\lambda}^{N}_{[p,q]}(f) = \lambda^{N}_{[p,q]}(f) = \sigma_{[p,q]}(f)$.

(ii) By a similar proof to case (i), we can easily obtain the conclusion of case (ii). □

Lemma 2.7 (see [17]) *Let $g:(0,1)\to R$ and $h:(0,1)\to\mathbb{R}$ be monotone increasing functions such that $g(r)\le h(r)$ holds outside of an exceptional set $E_1\subset[0,1)$ with $\int_{E_1}\frac{dt}{1-t}<\infty$. Then there exists a constant $d\in(0,1)$ such that if $s(r)=1-d(1-r)$, then $g(r)\le h(s(r))$ for all $r\in[0,1)$.*

Lemma 2.8 (see [18]) *Suppose that $f(z)$ is meromorphic in Δ with $f(0)=0$. Then*

$$m(r,f)\le\left[1+\varphi\left(\frac{r}{R}\right)\right]T\left(R,f'\right)+N\left(R,f'\right),\tag{2.6}$$

where $0<r<R<1$, $\varphi(t)=\frac{1}{\pi}\log\frac{1+t}{1-t}$.

Lemma 2.9 *Let $f(z)$ be an analytic function of $[p,q]$-order in Δ. Then the following statements hold*:

(i) *If $p\ge q\ge 1$, then $\sigma_{[p,q]}(f)=\sigma_{[p,q]}(f')$.*
(ii) *If $3\le q=p+1$, then $\sigma_{[p,p+1]}(f')\le\max\{\sigma_{[p,p+1]}(f),1\}$ and $\sigma_{[p,p+1]}(f)\le\max\{\sigma_{[p,p+1]}(f'),1\}$.*
(iii) *If $p=1$, $q=2$, then $\sigma_{[1,2]}(f')\le\max\{\sigma_{[1,2]}(f),1\}$ and $\sigma_{[1,2]}(f)\le 1+\sigma_{[1,2]}(f')$.*

Proof By Lemma 2.1, we have

$$T\left(r,f'\right)\le 2T(r,f)+m\left(r,\frac{f'}{f}\right)\le 3T(r,f)+O\left\{\log\frac{1}{1-r}\right\}\quad(0<r<1,r\notin E_1).\tag{2.7}$$

By (2.7) and Lemma 2.7, it is easy to see $\sigma_{[p,q]}(f')\le\sigma_{[p,q]}(f)$ $(p\ge q\ge 1)$ and $\sigma_{[p,p+1]}(f')\le\max\{\sigma_{[p,p+1]}(f),1\}$. On the other hand, set $R=\frac{1+r}{2}$, $0<r<1$, by Lemma 2.8, we have

$$T(r,f)<\left(3+\log\frac{4}{(1-r)}\right)T\left(\frac{1+r}{2},f'\right).\tag{2.8}$$

By (2.8), we have, if $p\ge q\ge 1$, then $\sigma_{[p,q]}(f)\le\sigma_{[p,q]}(f')$ and if $3\le q=p+1$, then $\sigma_{[p,p+1]}(f)\le\max\{\sigma_{[p,p+1]}(f'),1\}$; and we can easily obtain the conclusion (iii) by (2.7) and (2.8). Therefore Lemma 2.9 holds. □

3 Proofs of Theorems 1.1-1.3

Proof of Theorem 1.1 (i) Let $H_5=\{|z|:z\in H_3\subseteq\Delta\}$, since $\overline{\text{dens}}_\Delta\{|z|:z\in H_3\subseteq\Delta\}>0$, then by Remark 1.2, H_5 is a set of r with $\int_{H_5}\frac{dt}{1-t}=+\infty$. For any $|z|=r\in H_5$ and $r\to 1^-$, we have

$$\begin{aligned}&\left|A_0(z)\right|\ge\exp_p\left\{\alpha_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\},\\&\left|A_j(z)\right|\le\exp_p\left\{\beta_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\}\quad(j=1,\ldots,k-1).\end{aligned}\tag{3.1}$$

If $f\not\equiv 0$, from (1.1), we get

$$|A_0|\le\left|\frac{f^{(k)}}{f}\right|+|A_{k-1}|\left|\frac{f^{(k-1)}}{f}\right|+\cdots+|A_1|\left|\frac{f'}{f}\right|.\tag{3.2}$$

By Lemma 2.2, for $|z| = r \notin E_1$, we get

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le \left(\frac{1}{1-r}\right)^M \cdot T\big(s(r),f\big)^j \quad (j = 1,\dots,k), \tag{3.3}$$

where M denotes a positive constant, not always the same at each occurrence. By (3.1)-(3.3), for all z satisfying $|z| = r \in H_5 \setminus E_1$ and $r \to 1^-$, we have

$$\begin{aligned}&\exp_p\left\{\alpha_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\}\\ &\quad\le k\cdot\exp_p\left\{\beta_2\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_3}\right\}\cdot\left(\frac{1}{1-r}\right)^M\cdot T\big(s(r),f\big)^k.\end{aligned} \tag{3.4}$$

If $p \ge q \ge 1$, by (3.4), then $\sigma_3 \le \sigma_{[p+1,q]}(f)$. On the other hand, by Lemma 2.3, we have $\sigma_{[p+1,q]}(f) \le \max\{\sigma_{M,[p,q]}(A_j) | j = 0,1,\dots,k-1\} \le \sigma_3$. Therefore every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) = \sigma_3$.

(ii) If $2 \le q = p+1$ and $\sigma_3 > 1$, by a similar proof to case (i), we obtain the conclusion. □

Proof of Theorem 1.2 (i) Let $H_6 = \{|z| : z \in H_4 \subseteq \Delta\}$, since $\overline{\mathrm{dens}}_\Delta\{|z| : z \in H_4 \subseteq \Delta\} > 0$, then by Remark 1.2, H_6 is a set of r with $\int_{H_6} \frac{dt}{1-t} = +\infty$. For any $|z| = r \in H_6$ and $r \to 1^-$, we have

$$\begin{aligned}&T(r,A_0) \ge \exp_{p-1}\left\{\alpha_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\},\\ &T(r,A_j) \le \exp_{p-1}\left\{\beta_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\} \quad (j = 1,\dots,k-1).\end{aligned} \tag{3.5}$$

If $f \not\equiv 0$, from (1.1), we get

$$-A_0(z) = \frac{f^{(k)}(z)}{f(z)} + \cdots + A_j(z)\frac{f^{(j)}(z)}{f(z)} + \cdots + A_1(z)\frac{f'(z)}{f(z)},$$

then

$$T(r,A_0) \le \sum_{i=1}^{k-1} T(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(k)}}{f}\right) + O(1). \tag{3.6}$$

By Lemma 2.1 and (3.6), there exists a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dt}{1-t} < \infty$ such that, for all z satisfying $|z| = r \notin E_1$, we have

$$T(r,A_0) \le \sum_{i=1}^{k-1} T(r,A_j) + O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\}. \tag{3.7}$$

By (3.5), (3.7), for $|z| = r \in H_6 \setminus E_1$ and $r \to 1^-$, we have

$$\begin{aligned}&\exp_{p-1}\left\{\alpha_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\}\\ &\quad\le (k-1)\cdot\exp_{p-1}\left\{\beta_3\left[\log_{q-1}\left(\frac{1}{1-r}\right)\right]^{\sigma_4}\right\} + O\left\{\log^+ T(r,f) + \log\left(\frac{1}{1-r}\right)\right\}.\end{aligned}$$

If $p \geq q \geq 2$ and $0 < \beta_3 < \alpha_3$, then every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_4 \leq \sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f)$. On the other hand, by Lemma 2.3, all solutions of (1.1) satisfy $\sigma_{[p+1,q]}(f) = \sigma_{M,[p+1,q]}(f) \leq \max\{\sigma_{M,[p,q]}(A_j) | j = 0, 1, \ldots, k-1\} \leq \sigma_4$. Therefore every solution $f(z) \not\equiv 0$ of (1.1) satisfies $\sigma_{[p+1,q]}(f) = \sigma_4$.

(ii)-(iii) By a similar proof to case (i), we obtain the conclusions of (ii)-(iii). □

Proof of Theorem 1.3 (i) For $1 \leq q \leq p$, assume that f is a solution of (1.2), by the elementary theory of differential equations, thus all the solutions of (1.2) have the form

$$f = f^* + C_1 f_1 + C_2 f_2 + \cdots + C_k f_k,$$

where $C_1, \ldots, C_k$ are complex constants, $f_1, \ldots, f_k$ is a solution base of (1.1), f^* is a solution of (1.2) and has the form

$$f^* = D_1 f_1 + D_2 f_2 + \cdots + D_k f_k, \tag{3.8}$$

where $D_1, \ldots, D_k$ are certain analytic functions in Δ satisfying

$$D_j' = F \cdot G_j(f_1, \ldots, f_k) \cdot W(f_1, \ldots, f_k)^{-1} \quad (j = 1, \ldots, k), \tag{3.9}$$

where $G_j(f_1, \ldots, f_k)$ are differential polynomials in $f_1, \ldots, f_k$ and their derivative with constant coefficients, and $W(f_1, \ldots, f_k)$ is the Wronskian of $f_1, \ldots, f_k$.

If $\sigma_{[p+1,q]}(F) > \sigma_3$, by Lemma 2.3, Lemma 2.9, and (3.8)-(3.9), we find that all solutions of (1.2) satisfy

$$\sigma_{[p+1,q]}(f) \leq \max\left\{\sigma_{[p+1,q]}(f_j), \sigma_{[p+1,q]}(F) | j = 1, \ldots, k\right\} = \max\left\{\sigma_3, \sigma_{[p+1,q]}(F)\right\} \leq \sigma_{[p+1,q]}(F).$$

On the other hand, by a simple order comparison from (1.2), we see that all solutions of (1.2) satisfy $\sigma_{[p+1,q]}(f) \geq \sigma_{[p+1,q]}(F)$. Therefore all solutions of (1.2) satisfy

$$\sigma_{[p+1,q]}(f) = \sigma_{[p+1,q]}(F).$$

If $\sigma_{[p+1,q]}(F) \leq \sigma_3$, by the above proof in (3.8)-(3.9), we can find that all solutions of (1.2) satisfy $\sigma_{[p+1,q]}(f) \leq \sigma_3$. We affirm that (1.2) can only possess at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_3$. In fact, if f_* is another solution satisfying $\sigma_{[p+1,q]}(f_*) < \sigma_3$, then $\sigma_{[p+1,q]}(f_0 - f_*) < \sigma_3$. But $f_0 - f_*$ is a solution of (1.1) and satisfies $\sigma_{[p+1,q]}(f_0 - f_*) = \sigma_3$ by Theorem 1.1(i), this is a contradiction. Then $\sigma_{[p+1,q]}(f) = \sigma_3$ holds for all solutions of (1.2) with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_3$. By Lemma 2.6(i), we get

$$\overline{\lambda}^{N}_{[p+1,q]}(f) = \lambda^{N}_{[p+1,q]}(f) = \sigma_{[p+1,q]}(f)$$

holds for all solutions satisfying $\sigma_{[p+1,q]}(f) = \sigma_3$ with at most one exceptional solution f_0 satisfying $\sigma_{[p+1,q]}(f_0) < \sigma_3$.

(ii) For $2 \leq q = p + 1$, $\sigma_3 > 1$, by a similar proof to case (i), we draw the conclusions of case (ii). □

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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Acknowledgements

This project is supported by the Natural Science Foundation of Jiangxi Province in China (20132BAB211002) and by the Science and Technology Plan Program of Education Bureau of Jiangxi Province (GJJ14271, GJJ14272). Zu-Xing Xuan is supported in part by Beijing Natural Science Foundation (No. 1132013) and The Project of Construction of Innovative Teams and Teacher Career Development for Universities and Colleges under Beijing Municipality (CIT and TCD20130513).

Received: 20 March 2014 Accepted: 27 May 2014 Published: 12 June 2014

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doi:10.1186/1687-1847-2014-167
Cite this article as: Tu and Xuan: **Complex linear differential equations with certain analytic coefficients of $[p,q]$-order in the unit disc.** *Advances in Difference Equations* 2014 **2014**:167.