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Bifurcations for a deterministic SIR epidemic model in discrete time

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Abstract

In this paper, we are concerned with the theoretical analysis of the bifurcations for a deterministic SIR epidemic model in discrete time. By deriving equations describing flows on the center manifolds, we discuss the transcritical bifurcation at the disease-free equilibrium point and the direction and stability of the flip bifurcation at the positive endemic equilibrium point. We give explicit conditions to check the stability of equilibrium points and the critical parameter for the emergence of a flip bifurcation. For illustrating the theoretical analysis, we also give some numerical simulation examples.

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1 Introduction

Since Kermack and McKendrick [1] proposed the Susceptible-Infective-Recovered model (or SIR for short) in 1927, a lot of glorious studies on the dynamics of epidemic models have been presented (see [2–5]). The basic and important research subjects for these systems are local and global stability of the disease-free equilibrium point and the endemic equilibrium point, existence of periodic solutions, persistence and extinction of the disease, *etc.* According to the dependence on the variable (*i.e.*, time), these systems were classified into two types: continuous-time systems and discrete-time systems.

For the epidemic models, there has been a lot of research focusing on the case of continuous time (see [2–11] and that cited therein). However, discrete-time models (also called difference equations) are also useful for modeling situations of epidemic. They cannot only have the basic features of the corresponding continuous-time models but also provide a substantial reduction of computer time (see [12]). What is more, a lot of discrete-time models are not trivial analogs of their continuous ones and simple models can even exhibit complex behavior. The following two logistic difference equations are such examples that have received much attention (see [13, 14]):

$$x_{n+1} = (1 + r)x_n(1 - x_n)$$

and

$$x_{n+1} = x_n \exp\big(r(1 - x_n)\big).$$

As the value of r increases above 2, we have period doubling and eventually chaos.

In view of the above-mentioned reasons, Allen systematically compared the discrete-time SI, SIS, and SIR models with the corresponding continuous-time ones in [15] and later compared deterministic discrete-time SIS and SIR models with stochastic ones in [16]. She showed in [15] that the simple discrete-time SI and SIR models without positive feedback (*e.g.* recovery or births) to the susceptible class do not have a periodic solution. This behavior is qualitatively similar to that of the continuous counterparts. On the contrary, if there are some types of positive feedback to the susceptible class, the behavior in the discrete-time SI, SIS, and SIR models differs from that of their continuous analogs. The author also showed that for a sufficiently large contact rate the period-doubling and chaotic behavior for the SIS model is possible. However, for the case of the SIR model with positive feedback (births and deaths) the author only obtained the simulation results of periodic behavior. More specific and more in-depth questions, such as the kinds of periodic behavior and the conditions the periodic behavior arises from, have not been studied. Therefore, the following questions will be naturally asked on the discrete-time SIR model with births and deaths.

1. What kinds of periodic behavior may occur?
2. What restriction is enough to guarantee this periodic behavior?
3. Is the period-doubling behavior possible?

As far as we know, there was no literature of theoretical analysis to answer the above questions up till now. In this paper, we pay attention to the theoretical analysis of structural stabilities of the disease-free equilibrium point and the endemic equilibrium point under certain restrictive conditions of α, β, and γ. By applying center manifold theory, we find and prove the existence of a transcritical bifurcation at a disease-free equilibrium point and flip bifurcation (or period-doubling bifurcation) at a positive endemic equilibrium point. The transcritical bifurcation behavior (see Theorem 3.1) shows that when we have the restrictive condition $s = 0$ the SIR system has only one equilibrium point (disease-free equilibrium point), when s is slightly away from zero, another equilibrium point (endemic equilibrium point) occurs, and, moreover, their stabilities exchange at $s = 0$. The flip bifurcation behavior (see Theorem 4.1) demonstrates that when the restrictive parameter γ crosses over the critical value slightly with a given direction, two endemic equilibrium points appear and form a period-two orbit (cycle). From these results we properly answer the above question. For illustrating our theoretical conclusions, we also give some numerical simulation examples.

For the literature and more information on bifurcation of the equilibrium point of map, one should refer to the references [17–19] and those cited therein.

We consider the following deterministic epidemic model studied by Allen [15]:

$$\begin{cases} S_{n+1} = S_n(1 - \frac{\alpha \Delta t}{N} I_n) + \beta \Delta t(N - S_n), \\ I_{n+1} = I_n(1 - \gamma \Delta t - \beta \Delta t + \frac{\alpha \Delta t}{N} S_n), \\ R_{n+1} = R_n(1 - \beta \Delta t) + \gamma \Delta t I_n, \end{cases} \tag{1}$$

where S_n, I_n, and R_n represent susceptible, infective, and removed (or isolated) subgroups, respectively, n represents $n\Delta t$, $n = 0, 1, 2, \ldots$, Δt is a fixed time interval (*e.g.*, 1 hour or 1 day). It is assumed that $S_0 > 0$, $I_0 > 0$, $R_0 \geq 0$, and $S_0 + I_0 + R_0 = N$ and the parameters are positive, $\alpha > 0$, $\beta > 0$, $\gamma > 0$. To guarantee the solutions of system (1) to be non-negative for

all initial conditions, we further assume $\beta + \gamma \leq 1$ (see [15]). It is easy to see that $S_n + I_n + R_n = N$ for all time, *i.e.* the total population size remains constant. $\alpha \Delta t$ is the value of the force of infection (number of contacts that result in infection per susceptible individual in the time interval Δt), $\beta \Delta t$ is the number of births or deaths per individual during the time interval Δt (number of births = number of deaths) and $\gamma \Delta t$ is the removal number (number of individuals that recover in the time interval Δt). In addition, it is assumed that there are no deaths due to the disease, no recruitment, and no vertical transmission of the disease (all new-born members are susceptible) and that the individual's recovery leads to immunity.

In order to discuss the model (1) easily, some preliminary transformations will be made hereafter.

Dividing both sides of every equation of (1) by N and performing a scaling

$$n \to n\Delta t, \qquad \alpha \to \alpha \Delta t, \qquad \beta \to \beta \Delta t, \qquad \gamma \to \gamma \Delta t,$$

$$S_n \to \frac{S_n}{N}, \qquad I_n \to \frac{I_n}{N}, \qquad R_n \to \frac{R_n}{N},$$

we write (1) in the form

$$\begin{cases} S_{n+1} = (1 - \alpha I_n)S_n + \beta(1 - S_n), \\ I_{n+1} = (1 - \beta - \gamma)I_n + \alpha I_n S_n, \\ R_{n+1} = (1 - \beta)R_n + \gamma I_n, \end{cases} \tag{2}$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$, and $\beta + \gamma \leq 1$. In view of the relation $S_n + I_n + R_n = 1$, system (2) becomes the following one:

$$\begin{cases} I_{n+1} = (1 - \beta - \gamma)I_n + \alpha I_n(1 - I_n - R_n), \\ R_{n+1} = (1 - \beta)R_n + \gamma I_n. \end{cases} \tag{3}$$

Rewrite (3) as a planar map F:

$$\begin{bmatrix} I \\ R \end{bmatrix} \mapsto \begin{bmatrix} (1 + \alpha - \beta - \gamma)I - \alpha I^2 - \alpha IR \\ \gamma I + (1 - \beta)R \end{bmatrix}. \tag{4}$$

Set

$$s = \frac{\alpha - (\beta + \gamma)}{\beta + \gamma}\beta.$$

It is obvious that $s > -\beta$. If $-\beta < s \leq 0$ then the map (4) has only one equilibrium point $P(0,0)$; if $s > 0$ then it has two equilibrium points $P(0,0)$ and $Q(I^*, R^*)$, where

$$I^* = \frac{s}{\alpha}, \qquad R^* = \frac{\gamma s}{\alpha \beta}.$$

The organization of this paper is as follows. In the next section, we identify all cases of non-hyperbolic and hyperbolic equilibrium points, which is the fundament for all succeeding studies. In Section 3, we discuss the transcritical bifurcation at the disease-free equilibrium point of (1). Section 4 is devoted to the investigation of the direction and

stability of the flip bifurcation at the positive endemic equilibrium point by computing a center manifold. In Section 5, some simulations are made to demonstrate our results.

2 Non-hyperbolic and hyperbolic cases

In this section, s, γ will be taken as two parameters and the non-hyperbolic and hyperbolic cases will be discussed in the parameter space of s, γ. For the discussion of the property of equilibrium point $P(0,0)$ we define the notation first:

$$\beta_1 = 1 - \beta,$$
$$\ell_1 = \{(s,\gamma) \mid s = 0, 0 < \gamma \leq \beta_1\},$$
$$\mathcal{D}_1 = \{(s,\gamma) \mid 0 \leq \gamma \leq \beta_1, 0 \leq s\},$$
$$\mathcal{D}_2 = \{(s,\gamma) \mid 0 \leq \gamma \leq \beta_1, -\beta \leq s \leq 0\}.$$

It is obvious that the domain $\{(s,\gamma) \in \mathbb{R}^2 \mid -\beta < s, 0 < \gamma \leq \beta_1\}$ is divided by the line ℓ_1 into two districts $\mathcal{D}_1$ and $\mathcal{D}_2$ for equilibrium point P (see Figure 1).

Lemma 2.1 *Assume that $0 < \beta < 1$. The equilibrium point P of* (4) *has the following properties*:

(1) *It is non-hyperbolic if and only if (s,γ) lies on the line ℓ_1.*

(2) (a) *If $(s,\gamma) \in \mathcal{D}_1$, it is a saddle node*; (b) *if $(s,\gamma) \in \mathcal{D}_2$, it is a stable node.*

*Proof* The Jacobian matrix of (4) at P is

$$DF(P) = \begin{bmatrix} \alpha + \beta_1 - \gamma & 0 \\ \gamma & \beta_1 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = \alpha + \beta_1 - \gamma, \qquad \lambda_2 = \beta_1.$$

(1) From the assumption $0 < \beta < 1$, we see that $0 < \lambda_2 < 1$. Then non-hyperbolicity happens in the case $\lambda_1 = \pm 1$. In view of $0 < \alpha$ and $0 < \beta + \gamma \leq 1$, we know $\lambda_1 = -1$ is impossible. From $\lambda_1 = 1$, we get $\alpha - \beta - \gamma = 0$, *i.e.* $s = 0$, implying that (s,γ) lies on ℓ_1.

(2) (a) When $s > 0$ (referred to the case $\mathcal{D}_1$), the equilibrium point P is a saddle node since $\lambda_1 > 1$. (b) When $-\beta < s < 0$ (referred to the case $\mathcal{D}_2$), the eigenvalue $0 \leq \lambda_1 < 1$, then the equilibrium point P is a stable node. The proof is complete. □

Figure 1 Districts for equilibrium point *P*.

Figure 2 Districts for equilibrium point Q.
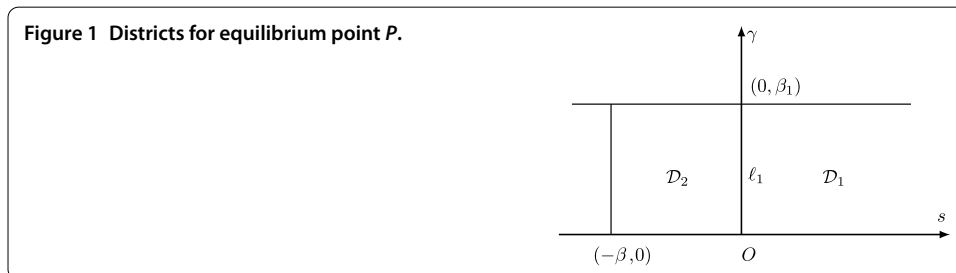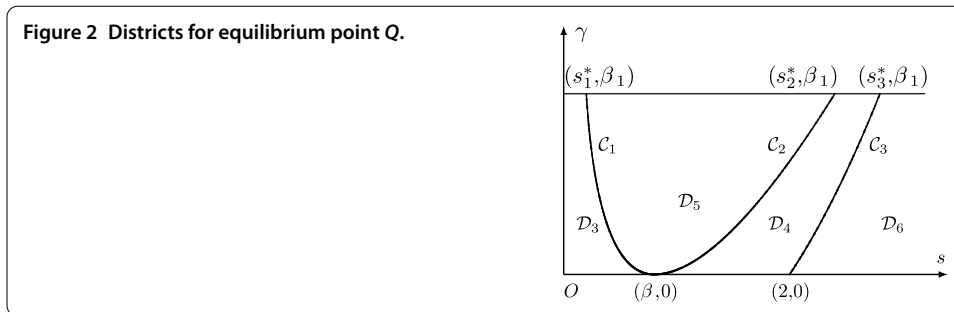
For the discussion of the property of the equilibrium point $Q(I^*, R^*)$ we define the notation

$$s_1^* = (1 - \sqrt{\beta_1})^2,$$

$$s_2^* = (1 + \sqrt{\beta_1})^2,$$

$$s_3^* = 2(1 + \beta_1),$$

$$\mathcal{C}_1 = \{(s, \gamma) \mid \gamma = (s - \beta)^2/(4s), s_1^* \le s \le \beta\},$$

$$\mathcal{C}_2 = \{(s, \gamma) \mid \gamma = (s - \beta)^2/(4s), \beta < s \le s_2^*\},$$

$$\mathcal{C}_3 = \{(s, \gamma) \mid \gamma = (2 - \beta)(s - 2)/s, 2 \le s \le s_3^*\},$$

$$\mathcal{D}_3 = \{(s, \gamma) \mid 0 \le \gamma \le \beta_1, 0 \le s \le s_1, \text{ where } s_1 \text{ satisfies: } (s_1, \gamma) \in \mathcal{C}_1\},$$

$$\mathcal{D}_4 = \{(s, \gamma) \mid 0 \le \gamma \le \beta_1, s_2 \le s \le s_3, \text{ where } s_i \text{ satisfy: } (s_i, \gamma) \in \mathcal{C}_i, i = 2, 3\},$$

$$\mathcal{D}_5 = \{(s, \gamma) \mid 0 \le \gamma \le \beta_1, s_1 \le s \le s_2, \text{ where } s_i \text{ satisfy: } (s_i, \gamma) \in \mathcal{C}_i, i = 1, 2\},$$

$$\mathcal{D}_6 = \{(s, \gamma) \mid 0 \le \gamma \le \beta_1, s_3 \le s, \text{ where } s_3 \text{ satisfies: } (s_3, \gamma) \in \mathcal{C}_3\}.$$

Obviously, the domain $\{(s, \gamma) \in \mathbb{R}^2 \mid 0 < s, 0 < \gamma \le \beta_1\}$ is divided by the curves $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ into four districts $\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$, and $\mathcal{D}_6$ for equilibrium point Q (see Figure 2).

Lemma 2.2 *The equilibrium point Q of* (4) *has the following properties*:
(1) *It is non-hyperbolic if and only if (s, γ) lies on the curve $\mathcal{C}_3$.*
(2) (a) *If $(s, \gamma) \in \mathcal{C}_1, \mathcal{C}_2, \mathcal{D}_3$ or $\mathcal{D}_4$, it is a stable node*; (b) *if $(s, \gamma) \in \mathcal{D}_6$, it is a saddle node*;
(c) *if $(s, \gamma) \in \mathcal{D}_5$, it is a stable focus.*

Proof Performing a coordinate shift as follows:

$$\tilde{I} = I - I^*, \qquad \tilde{R} = R - R^*,$$

and letting $\tilde{F}$ denote the transformed F, we translate the equilibrium point $Q(I^*, R^*)$ into $\tilde{Q}(0, 0)$ and discuss the equilibrium point $\tilde{Q}$ of the map $\tilde{F}$. The matrix of linearization of $\tilde{F}$ at $\tilde{Q}$ is

$$D\tilde{F}\big((0, 0)\big) = \begin{bmatrix} 1 - \frac{\alpha - \beta - \gamma}{\beta + \gamma}\beta & -\frac{\alpha - \beta - \gamma}{\beta + \gamma}\beta \\ \gamma & \beta_1 \end{bmatrix} = \begin{bmatrix} 1 - s & -s \\ \gamma & \beta_1 \end{bmatrix}$$

and its eigenvalues are

$$\lambda_1 = \frac{(1 - s + \beta_1) + \sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}}{2},$$

$$\lambda_2 = \frac{(1 - s + \beta_1) - \sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}}{2}.$$

(5)

(1) It is well known that $\tilde{Q}$ is hyperbolic if and only if none of the eigenvalues λ_1, λ_2 lies on the unit circle S^1. Denote $\Delta := (1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)$. In the case of $\Delta \geq 0$, λ_1 and λ_2 are both real. Then the non-hyperbolicity happens when $|\lambda_1|$ or $|\lambda_2|$ is 1. For whether $\lambda_1 = 1$ or $\lambda_2 = 1$, we get

$$(\beta + \gamma)s = 0.$$

However, for positive equilibrium point $\tilde{Q}$, we have $\beta + \gamma > 0$ and $s > 0$. Therefore, neither $\lambda_1 = 1$ nor $\lambda_2 = 1$ is possible. Next, we examine $\lambda_1 = -1$ and $\lambda_2 = -1$. From whether $\lambda_1 = -1$ or $\lambda_2 = -1$, we get

$$\gamma = (1 + \beta_1)(s - 2)/s.$$

By condition $0 < \gamma \leq \beta_1 = 1 - \beta$, we see that $2 < s \leq 2(1 + \beta_1)$. It is easy to check that $\lambda_2 = -1$ and $\lambda_1 = 1 - s + \beta_1 - \lambda_2 = 2 - s + \beta_1 \in (-1, 1)$ if and only if $(s, \gamma) \in \mathcal{C}_3$.

(2) When $\Delta \geq 0$ and $(s, \gamma) \notin \mathcal{C}_3$, the equilibrium point $\tilde{Q}$ is hyperbolic.

(a) If $\Delta = 0$, the matrix has a double real eigenvalue $\lambda_1 = \lambda_2 = (1 - s + \beta_1)/2$. It is obvious that $\lambda_1 = \lambda_2 < 1$. Considering the line $\gamma = \beta_1$ and the curve $4\gamma s = (1 - s - \beta_1)^2$, we can get two intersection points (s_1^*, β_1) and (s_2^*, β_1) where $s_1^* = (1 - \sqrt{\beta_1})^2$ and $s_2^* = (1 + \sqrt{\beta_1})^2$. Then $s_1^* \leq s \leq s_2^*$ as $\gamma \leq \beta_1$. This implies $\lambda_1 = \lambda_2 \geq (1 + \beta_1 - s_2^*)/2 = -\sqrt{\beta_1} > -1$. Therefore, the equilibrium point $\tilde{Q}$ is a stable node in the cases of $\mathcal{C}_1$ and $\mathcal{C}_2$.

If $\Delta > 0$, the eigenvalues λ_1 and λ_2 are different real numbers. We first discuss the case that $0 < s \leq s_1$, *i.e.* $(s, \gamma) \in \mathcal{D}_3$. In this case we have

$$\lambda_2 < \frac{1 - s + \beta_1}{2} < 1, \qquad \frac{d\lambda_2}{d\gamma} = \frac{s}{\sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}} > 0.$$

Since

$$\lim_{\gamma \to 0+} \lambda_2 = \lim_{\gamma \to 0+} \frac{(1 - s + \beta_1) - \sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}}{2} = \beta_1 > 0,$$

we have $0 < \lambda_2 < 1$ for $(s, \gamma) \in \mathcal{D}_3$. On the other hand, there also exists $0 < \lambda_1 < 1$ for $(s, \gamma) \in \mathcal{D}_3$. In fact, since

$$\lim_{\gamma \to 0+} \lambda_1 = \lim_{\gamma \to 0+} \frac{(1 - s + \beta_1) + \sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}}{2} = 1 - s < 1$$

and

$$\frac{d\lambda_1}{d\gamma} = -\frac{s}{\sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}} < 0,$$

we have $0 < \lambda_2 < \lambda_1 < 1$. Therefore, the equilibrium point $\tilde{Q}$ is a stable node as $(s, \gamma) \in \mathcal{D}_3$.

In the case $s_2 < s \le s_3$, *i.e.* $(s, \gamma) \in \mathcal{D}_4$, by a similar method to the above we easily get $-1 < \lambda_{1,2} < 1$ and the equilibrium point $\tilde{Q}$ is also a stable node.

(b) We discuss the case that $s_3 < s$, *i.e.* $(s, \gamma) \in \mathcal{D}_6$. In this case we have

$$
\begin{aligned}
\frac{d\lambda_2}{ds} &= \frac{1}{2}\left(\frac{1 - s - \beta_1 + 2\gamma}{\sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}} - 1 \right) \\
&\le \frac{1}{2}\left(\frac{-\beta}{\sqrt{(1 - s + \beta_1)^2 - 4(\beta_1 + (\gamma - \beta_1)s)}} - 1 \right) < 0.
\end{aligned}
$$

Then, in view of $\lambda_2|_{s=s_3} = -1$, we have $\lambda_2 < -1$ for $(s, \gamma) \in \mathcal{D}_6$. By a simple computation one derives $\lambda_1 > -1$ for $s > s_3^* = 2(1 + \beta_1)$ and $\gamma = \beta_1$. Moreover, we have

$$
\lim_{\gamma \to 0+} \lambda_1 = \beta_1 < 1, \qquad \frac{d\lambda_1}{d\gamma} < 0.
$$

Then $-1 < \lambda_1 < 1$ for $(s, \gamma) \in \mathcal{D}_6$. This means that the equilibrium point $\tilde{Q}$ is a saddle for $(s, \gamma) \in \mathcal{D}_6$. The proof is complete.

(c) In the case of $\Delta < 0$, λ_1 and λ_2 are a pair of conjugate complex. Since

$$
\begin{aligned}
|\lambda_1|^2 = |\lambda_2|^2 &= \frac{1}{4}\left[(1 - s + \beta_1)^2 + 4(\beta_1 + (\gamma - \beta_1)s) - (1 - s + \beta_1)^2 \right] \\
&= \beta_1 + (\gamma - \beta_1)s \le \beta_1 < 1,
\end{aligned}
$$

λ_1 and λ_2 lie inside of S^1 and the equilibrium point Q is a stable focus for the case $(\mathcal{D}_5)$. □

3 Transcritical bifurcation

In this section we consider the case that $(s, \gamma) \in \ell_1$, where the transcritical bifurcation at equilibrium point $P(0, 0)$ will happen.

Theorem 3.1 *A transcritical bifurcation occurs at the equilibrium point P when s = 0. More concretely, for a parameter s being slightly less than zero there are two equilibrium points: a stable point P and an unstable negative equilibrium point which coalesce at s = 0; for parameter s being slightly greater than zero there are also two equilibrium points: an unstable equilibrium point P and a stable positive equilibrium point Q. Thus an exchange of stability has occurred at s = 0.*

Proof For $(s, \gamma) \in \ell_1$, we have $\lambda_1 = 1$ and $0 < \lambda_2 = 1 - \beta < 1$. Consider s as the bifurcation parameter and write F as F_s to emphasize the dependence on s. One can easily see that the matrix $DF_s(0, 0)$ is

$$
\begin{bmatrix} 1 & 0 \\ \gamma & 1 - \beta \end{bmatrix}
$$

and it has eigenvectors

$$
\left(1, \frac{\gamma}{\beta} \right)^T, \qquad (0, 1)^T \tag{6}
$$

corresponding to λ_1 and λ_2, respectively, where T means the transpose of the matrices. Our goal is to determine the nature of the stability of $(0,0)$ for s near zero. First, we must put the matrix $DF_s(0,0)$ into a diagonal form.

Using the eigenvectors (6), we obtain the transformation

$$
\begin{bmatrix} I \\ R \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\gamma}{\beta} & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}
\tag{7}
$$

with inverse

$$
\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{\gamma}{\beta} & 1 \end{bmatrix} \begin{bmatrix} I \\ R \end{bmatrix}
\tag{8}
$$

which transforms system (4) into

$$
\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1-\beta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \frac{\beta+\gamma}{\beta} su - \alpha(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uv) \\ \frac{\beta\gamma+\gamma^2}{\beta^2} su + \alpha\frac{\gamma}{\beta}(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uv) \end{bmatrix}.
\tag{9}
$$

Rewrite system (9) in the suspended form

$$
\begin{bmatrix} u \\ v \\ s \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ s \end{bmatrix} + \begin{bmatrix} \frac{\beta+\gamma}{\beta} su - \alpha(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uv) \\ \frac{\beta\gamma+\gamma^2}{\beta^2} su + \alpha\frac{\gamma}{\beta}(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uv) \\ 0 \end{bmatrix}.
\tag{10}
$$

Thus, from the center manifold theory (see Theorem 2.1.4 in [18]), the stability of the equilibrium point $(u,v) = (0,0)$ near $s = 0$ can be determined by studying a one-parameter family of maps on a center manifold which can be represented as follows:

$$
W^c(0,0) = \left\{ (u,v,s) \in \mathbb{R}^3 \mid v = h(u,s), h(0,0) = 0, Dh(0,0) = 0 \right\}
$$

for sufficiently small u and s.

We now want to compute the center manifold and derive the mapping on the center manifold. We assume

$$
h(u,s) = au^2 + bus + cs^2 + \mathcal{O}(3)
\tag{11}
$$

near the origin, where $\mathcal{O}(3)$ means terms of order ≥ 3. By Theorem 2.1.4 in [18], the coefficients a, b, and c can be determined by the equation

$$
\begin{aligned}
\mathcal{N}(h(u,s)) := {} & h\left(u + \frac{\beta+\gamma}{\beta} su - \alpha\left(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uh(u,s) \right), s \right) \\
& - (1-\beta)h(u,s) + \frac{\beta\gamma+\gamma^2}{\beta^2} su + \alpha\frac{\gamma}{\beta}\left(\frac{\beta+\gamma}{\beta}u^2 + \frac{\gamma}{\beta}uh(u,s) \right) \\
= {} & 0.
\end{aligned}
\tag{12}
$$

Substituting (11) into (12) and comparing coefficients of u^2, us, and s^2 in (12), we get

$$\begin{cases} a - (1 - \beta)a - \alpha\frac{\gamma}{\beta}(1 + \frac{\gamma}{\beta}) = 0, \\ b - (1 - \beta)b + \frac{\gamma}{\beta} = 0, \\ c - (1 - \beta)c = 0, \end{cases}$$

from which we resolve

$$a = \frac{\alpha\gamma^3}{\beta}(\beta + \gamma), \qquad b = -\frac{\gamma}{\beta^2}, \qquad c = 0.$$

Therefore the expression of (11) is approximately determined. Substituting (11) into (10), we obtain a one-dimensional map reduced to the center manifold

$$u \mapsto \phi_s(u) = u + \frac{\beta + \gamma}{\beta}us - \alpha\frac{\beta + \gamma}{\beta}u^2 + \frac{\alpha\gamma^2}{\beta^4}(\beta + \gamma)u^3 - \frac{(\beta + \gamma)\gamma^2}{\beta^4}u^2s + \mathcal{O}(4). \quad (13)$$

It is easy to check that

$$\frac{\partial\phi_s}{\partial s}(0,0) = 0, \qquad \frac{\partial^2\phi_s}{\partial u\,\partial s}(0,0) \neq 0, \qquad \frac{\partial^2\phi_s}{\partial u^2}(0,0) \neq 0. \quad (14)$$

The condition (14) implies that in the study of the orbit structure near the bifurcation point terms of $\mathcal{O}(3)$ do not qualitatively affect the nature of the bifurcation, namely they do not affect the geometry of the curves of equilibrium points passing through the bifurcation point. Thus, (14) shows that the orbit structure of (13) near $(u,s) = (0,0)$ is qualitatively the same as the orbit structure near $(u,s) = (0,0)$ of the map

$$u \mapsto u + \frac{\beta + \gamma}{\beta}us - \alpha\frac{\beta + \gamma}{\beta}u^2. \quad (15)$$

The map (15) can be viewed as a truncated normal form for the transcritical bifurcation (see [18, p.365]). The stability of the two branches of equilibrium points lying on both sides of $s = 0$ are easily verified. □

Remark 3.1 (The biological explanation of Theorem 3.1) Because the epidemic model (1) cannot have a negative equilibrium point in real life, when $s < 0$ (*i.e.* $\alpha < \beta + \gamma$), (1) has only a disease-free equilibrium point $(N, 0, 0)$ which is stable. In this case, for any given initial value (S_0, I_0, R_0) with $I_0 > 0$, the state (S_n, I_n, R_n) will finally tend to $(N, 0, 0)$, namely, the final situation of epidemic is free from disease. However, when $s > 0$, a positive equilibrium point will occur. It is an endemic equilibrium point (S^*, I^*, R^*) and stable, meanwhile, the disease-free equilibrium point $(N, 0, 0)$ changes to unstable. For any given initial value (S_0, I_0, R_0) with $I_0 > 0$, the state (S_n, I_n, R_n) will finally tend to (S^*, I^*, R^*).

4 Flip bifurcation

This section is devoted to the analysis for the case $(s, \gamma) \in \mathcal{C}_3$, where bifurcation happens at the equilibrium point $Q(I^*, R^*)$. From Section 2, we have, for $(s, \gamma) \in \mathcal{C}_3$,

$$\lambda_2 = -1, \qquad \lambda_1 = 1 - s + \beta_1 - \lambda_2 = \beta_1 + 2 - s,$$

$$\lambda_1 - \lambda_2 = \beta_1 + 3 - s, \qquad s = \frac{2(\beta_1 + 1)}{\beta_1 + 1 - \gamma}.$$

For convenience, we let

$$E_1 := \frac{\alpha(1 - \lambda_1)(1 - \lambda_2)}{(\lambda_1 - \lambda_2)(\lambda_1^2 - \lambda_2)},$$

$$E_2 := \frac{1}{\lambda_1 - \lambda_2}\left[(1 - \lambda_1)(\lambda_1 - \lambda_2) + \alpha(1 - \lambda_2)s - (\lambda_1 - \lambda_2)s - \alpha s^2\right],$$

$$E_3 := \frac{1}{\lambda_1 - \lambda_2}\left[(1 - \lambda_2)(\lambda_1 - \lambda_2) + 2\alpha(1 - \lambda_2)s - (\lambda_1 - \lambda_2)s - 2\alpha s^2\right].$$

Then we have the following theorem.

Theorem 4.1 *If $E_2^2 - E_1 E_3 \neq 0$, then a flip bifurcation occurs at the equilibrium point $Q(I^*, R^*)$ when $(s, \gamma) \in \mathcal{C}_3$, i.e. $\gamma = (\beta_1 + 1)(1 - 2/s)$ and $s \in (2, s_3^*)$. More concretely, for $E_2^2 - E_1 E_3 > 0$, an attractive 2-periodic orbit of map F emerges near the equilibrium point when $\gamma < (\beta_1 + 1)(1 - 2/s)$, but the 2-periodic orbit does not exist when $\gamma \geq (\beta_1 + 1)(1 - 2/s)$, for $E_2^2 - E_1 E_3 < 0$, a repellent 2-periodic orbit of map F emerges near the equilibrium point when $\gamma > (\beta_1 + 1)(1 - 2/s)$, but the 2-periodic orbit does not exist when $\gamma \leq (\beta_1 + 1)(1 - 2/s)$.*

Proof For $(s, \gamma) \in \mathcal{C}_3$, we have $\lambda_2 = -1$ and $\lambda_1 \in (-1, 1)$. Consider s as the bifurcation parameter and write $\tilde{F}$ as $\tilde{F}_s$ to emphasize the dependence on s, where $\tilde{F}$ defined as in Lemma 2.2 is the transformed F from (I^*, R^*) into $(0, 0)$. Then we have

$$D\tilde{F}_s(0, 0) = \begin{bmatrix} 1 - s & -s \\ \gamma & \beta_1 \end{bmatrix}.$$

The matrix $D\tilde{F}_s(0, 0)$ has eigenvectors $(s, 1 - s - \lambda_1)^T$ and $(s, 1 - s - \lambda_2)^T$ corresponding to λ_1 and λ_2, respectively, where T means the transpose of matrices. Hence the matrix $D\tilde{F}_s(0, 0)$ can be diagonalized by the change of variables $(I, R)^T = H(u, v)^T$, where

$$H = \begin{bmatrix} s & s \\ 1 - s - \lambda_1 & 1 - s - \lambda_2 \end{bmatrix}.$$

Therefore $\tilde{F}_s$ can be changed into the map $\Phi_s : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\begin{bmatrix} u \\ v \end{bmatrix} \mapsto \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} - \frac{\alpha}{\lambda_1 - \lambda_2}\begin{bmatrix} (1 - s - \lambda_2)A \\ -(1 - s - \lambda_1)A \end{bmatrix}, \tag{16}$$

where $A = s(u + v)^2 - (u + v)((1 - s - \lambda_1)u + (1 - s - \lambda_2)v)$.

Rewrite (16) in the suspended form

$$\begin{bmatrix} u \\ v \\ s \end{bmatrix} \mapsto \begin{bmatrix} \lambda_1 u \\ \lambda_2 v \\ s \end{bmatrix} - \frac{\alpha}{\lambda_1 - \lambda_2}\begin{bmatrix} (1 - s - \lambda_2)A \\ -(1 - s - \lambda_1)A \\ 0 \end{bmatrix} \tag{17}$$

so as to involve the parameter s explicitly in the discussion. Equivalently, the suspended system (17) has a two-dimensional center manifold of the form

$$v = h(u,s) = au^2 + bus + cs^2 + \mathcal{O}(3) \tag{18}$$

near the origin, where $\mathcal{O}(3)$ means terms of order ≥ 3. By Theorem 6 in [20, pp.34-35], these coefficients a, b, and c can be determined by the equation

$$
\begin{aligned}
\mathcal{N}\big(h(u,s)\big) := \, & h\left(\lambda_1 u - \frac{\alpha}{\lambda_1 - \lambda_2}(1 - s - \lambda_2)A, s\right) \\
& - \lambda_2 h(u,s) + \frac{\alpha}{\lambda_1 - \lambda_2}(1 - s - \lambda_1)A \\
= \, & 0,
\end{aligned} \tag{19}
$$

where

$$A = s\big(u + h(u,s)\big)^2 - \big(u + h(u,s)\big)\big((1 - s - \lambda_1)u + (1 - s - \lambda_2)h(u,s)\big).$$

Comparing coefficients of u^2, us, and s^2 in (19), we get

$$
\begin{cases}
a\lambda_1 - a\lambda_2 - (1 - \lambda_1)\frac{\alpha(1-\lambda_2)}{\lambda_1-\lambda_2} = 0, \\
b\lambda_1 - b\lambda_2 = 0, \\
c - \lambda_2 c = 0,
\end{cases}
$$

from which we solve

$$a = \frac{\alpha(1 - \lambda_1)(1 - \lambda_2)}{(\lambda_1^2 - \lambda_2)(\lambda_1 - \lambda_2)}, \qquad b = 0, \qquad c = 0.$$

Thus the expression of (18) is determined, *i.e.*,

$$v = h(u,s) = E_1 u^2 + \mathcal{O}(3). \tag{20}$$

Substituting (20) into the first equation in (17), we obtain a one-dimensional map $u \mapsto \phi_s(u)$, where

$$
\begin{aligned}
\phi_s(u) = \, & \lambda_1 u - \frac{\alpha}{\lambda_1 - \lambda_2}(1 - s - \lambda_2)A \\
= \, & \lambda_1 u - \frac{\alpha(1 - s - \lambda_2)}{\lambda_1 - \lambda_2}\Big[s\big(u + h(u,s)\big)^2 - \big(u + h(u,s)\big)\big((1 - s - \lambda_1)u \\
& + (1 - s - \lambda_2)h(u,s)\big)\Big] \\
= \, & \lambda_1 u - \frac{1}{\lambda_1 - \lambda_2}\Big[(1 - \lambda_1)(\lambda_1 - \lambda_2) + \big(\alpha(1 - \lambda_2) - (\lambda_1 - \lambda_2)\big)s - \alpha s^2\Big]u^2 \\
& - \frac{E_1}{\lambda_1 - \lambda_2}\Big[(1 - \lambda_2)(\lambda_1 - \lambda_2) + \big(2\alpha(1 - \lambda_2) - (\lambda_1 - \lambda_2)\big)s - 2\alpha s^2\Big]u^3 + \mathcal{O}(4) \\
= \, & \lambda_1 u - E_2 u^2 - E_1 E_3 u^3 + \mathcal{O}(4).
\end{aligned} \tag{21}
$$

Here we note the dependence of λ_1 on s. From (21), one can check that

$$\left[\frac{\partial \phi_s}{\partial s}\frac{\partial^2 \phi_s}{\partial u^2} + 2\frac{\partial^2 \phi_s}{\partial u\,\partial s}\right]\Bigg|_{(u,s)=(0,s)} = \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}\left(\frac{2}{s} - 1\right) < 0 \tag{22}$$

and

$$\left[\frac{1}{2}\left(\frac{\partial^2 \phi_s}{\partial u^2}\right)^2 + \frac{1}{3}\frac{\partial^3 \phi_s}{\partial u^3}\right]\Bigg|_{(u,s)=(0,s)} = 2\left(E_2^2 - E_1 E_3\right) \neq 0, \tag{23}$$

as assumed in our theorem. Thus, the conditions (F_1) and (F_2) of Theorem 3.5.1 in [19] are checked by (22) and (23), respectively. Therefore a flip bifurcation occurs at $(u,s) = (0, \frac{2(\beta_1+1)}{\beta_1+1-\gamma})$ and a 2-periodic orbit arises as stated in the theorem. □

Remark 4.1 (The biological explanation of Theorem 4.1) When $(s,\gamma) \in \mathcal{C}_3$, the epidemic model (1) has only one positive equilibrium point, *i.e.* the endemic equilibrium point $Q(I^*, R^*)$. If the parameters (s,γ) cross the curve $\mathcal{C}_3$ slightly with a given direction, two new positive equilibrium points (assumed to be Q_1, Q_2) of model (1) will emerge and form a 2-periodic orbit, *i.e.* $F(Q_1) = Q_2$ and $F(Q_2) = Q_1$. Their stabilities are determined by the negative and positive values of $E_2^2 - E_1 E_3$, concretely, when $E_2^2 - E_1 E_3 > 0$ they are attractive, when $E_2^2 - E_1 E_3 < 0$ they are repellent.
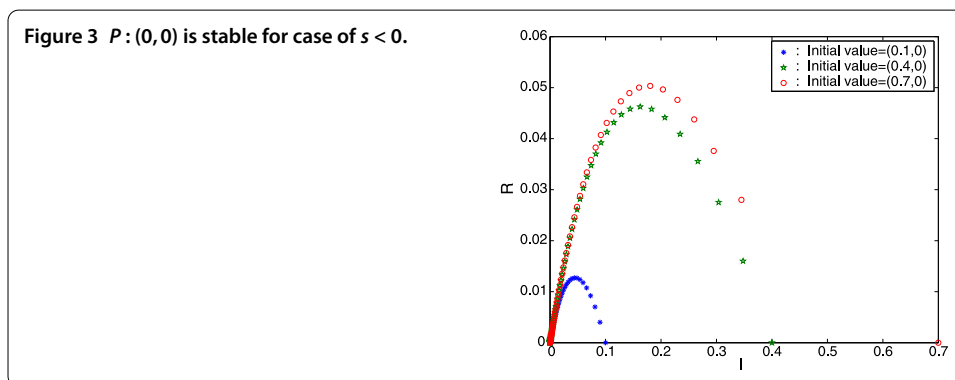
5 Simulations

In this section, we will give three simulation examples to illustrate the results obtained in the above sections.
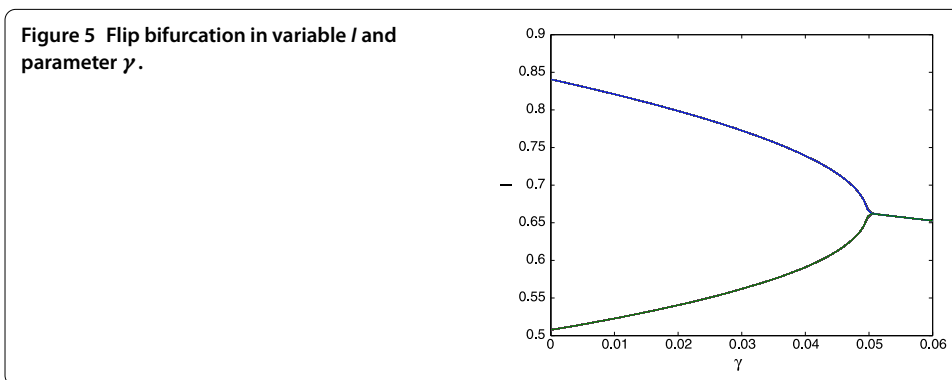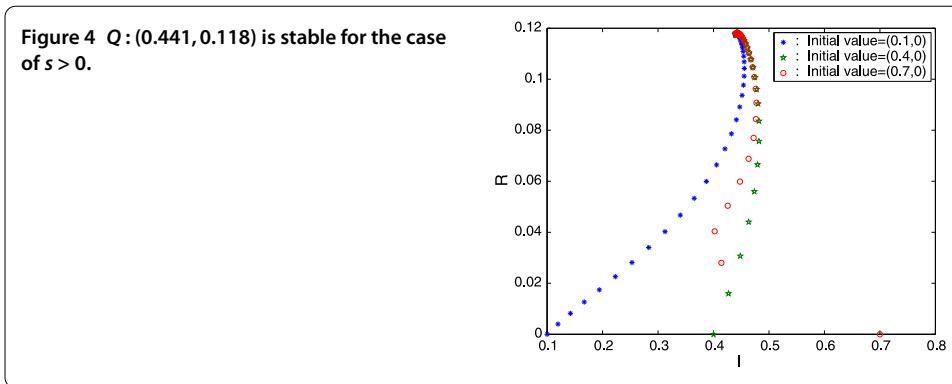
Example 5.1 Let $\beta = 0.15$, $\gamma = 0.04$, and choose three groups of initial values for (S_0, I_0, R_0) as follows:

$$(0.9, 0.1, 0), \qquad (0.6, 0.4, 0), \qquad (0.3, 0.7, 0).$$

If let $\alpha = 0.1$, we see that $s < 0$ and we have Figure 3. If let $\alpha = 0.43$, we see that $s > 0$ and we have Figure 4.

From Figures 3 and 4, we see that the conclusion of Theorem 3.1 is well verified by numerical simulation. Namely, for given various initial values for (S_0, I_0, R_0), if $s < 0$ slightly, there are a stable point $(0, 0)$ and an unstable negative point (I^*, R^*) which coalesce as

Figure 3 $P : (0, 0)$ is stable for case of $s < 0$.

Figure 4 $Q:(0.441, 0.118)$ is stable for the case of $s > 0$.

Figure 5 Flip bifurcation in variable *I* and parameter γ.

$s = 0$, if $s > 0$ slightly, point $(0,0)$ is unstable and positive point (I^*, R^*) is stable. Thus a transcritical bifurcation occurs at the equilibrium point $(0,0)$ when $s = 0$.
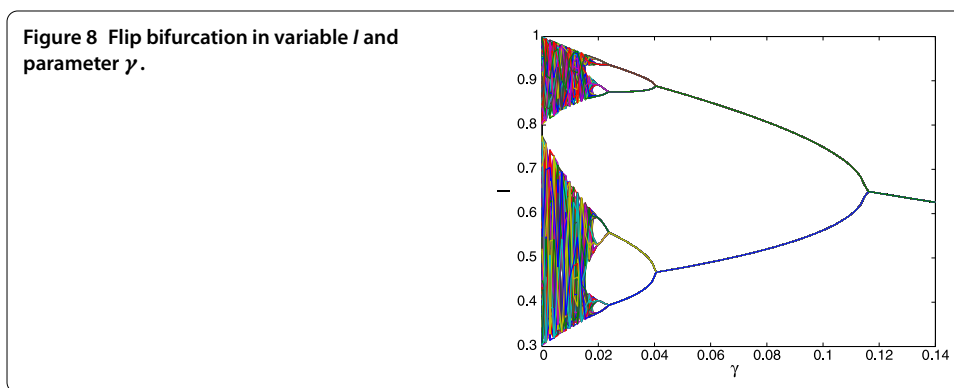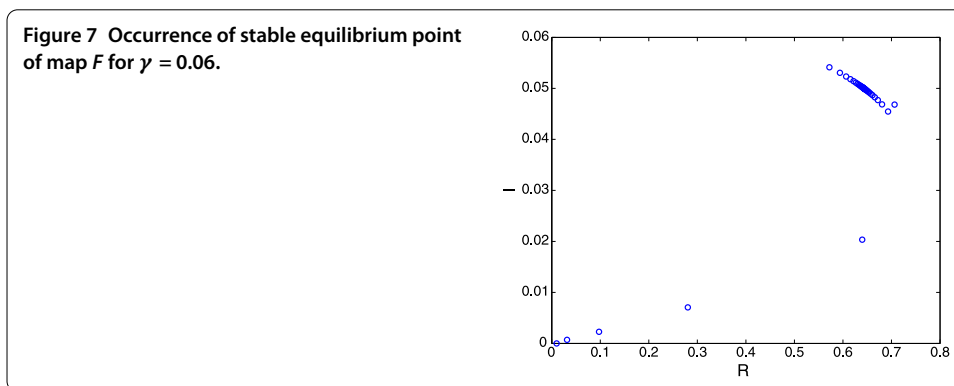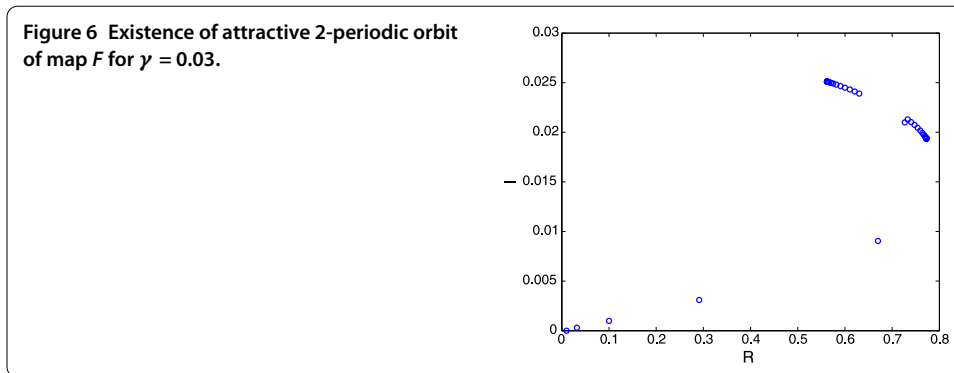
Example 5.2 Let $\alpha = 3.16$, $\beta = 0.90$, and initial value $(S_0, I_0, R_0) = (0.9, 0.1, 0)$. We easily solve the following equation:

$$\begin{cases} \gamma = (\beta_1 + 1)(1 - \frac{2}{s}), \\ s = \frac{\alpha - (\beta + \gamma)}{\beta + \gamma}\beta, \end{cases}$$

and we get a flip bifurcation parameters $\gamma_2 = 0.051\ldots$ and $s_2 = 2.094\ldots$. We also calculate $E_2^2 - E_1 E_3 = 11.257\ldots$. Then from Theorem 4.1 we know that if let $\gamma < 0.051\ldots$ an attractive 2-periodic orbit of map F emerges and if let $\gamma \geq 0.051\ldots$ the 2-periodic orbit does not exist, but a stable equilibrium point occurs. Figures 5, 6, and 7 illustrate this fact.

Example 5.3 Let $\alpha = 3.37$, $\beta = 0.7$, and initial value $(S_0, I_0, R_0) = (0.9, 0.1, 0)$. Using similar method to Example 5.2, we also solve the flip bifurcation parameters $\gamma_2 = 0.115\ldots$ and $s_2 = 2.194\ldots$. Moreover, we calculate $E_2^2 - E_1 E_3 = 21.763\ldots$. Then from Theorem 4.1 we know that if we let $\gamma < 0.115\ldots$ slightly an attractive 2-periodic orbit of map F emerges and if let $\gamma \geq 0.115\ldots$ the 2-periodic orbit does not exist, but a stable equilibrium point occurs. Figures 8, 9, and 10 give the numerical illustrations of this conclusion.
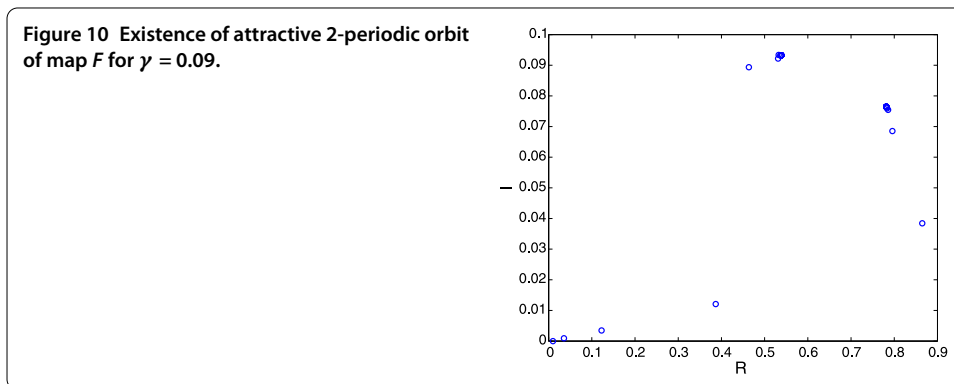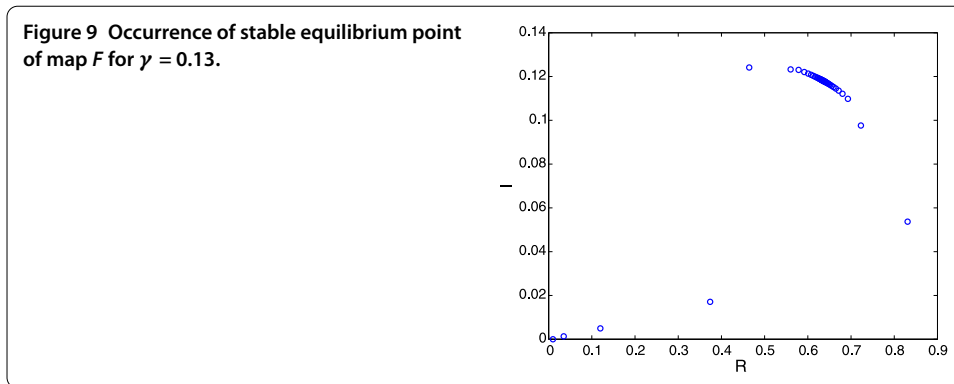
Additionally, we see from Figure 8 that the flip bifurcation giving a 4-periodic orbit occurs at parameter $\gamma_4 = 0.04\ldots$. The next period doubling takes place at $\gamma_8 = 0.024\ldots$, and so on. But from Figure 5 we do not see this phenomenon because of different parameters

Figure 6 Existence of attractive 2-periodic orbit of map *F* for *γ* = 0.03.

Figure 7 Occurrence of stable equilibrium point of map *F* for *γ* = 0.06.

Figure 8 Flip bifurcation in variable *I* and parameter *γ*.

being between in Examples 5.2 and 5.3. Indeed, a 4-periodic orbit, an 8-periodic orbit *etc.* may occur in the region $\gamma < 0$ in Example 5.2. However, $\gamma < 0$ exceeds the restriction of the parameter γ in our model. Therefore, in Example 5.2 we can only see the emergence of a stable equilibrium point and an attractive 2-periodic orbit.

6 Summary

Discrete-time epidemic models are useful for modeling situations of epidemic. They always exhibit richer and more complicated dynamical behaviors than continuous-time models, though some of them may be considered as approximations to the continuous-time models. Allen [15] gave a systematical comparison between the discrete-time models and the corresponding continuous-time models and showed the periodic behavior (which does not occur in the corresponding continuous cases) for the case of discrete-time model

Figure 9 Occurrence of stable equilibrium point of map *F* for $\gamma = 0.13$.

Figure 10 Existence of attractive 2-periodic orbit of map *F* for $\gamma = 0.09$.

SIR with births and deaths by numerical simulations. To reveal the reason for the resulting periodic behavior of the discrete-time models SIR with births and deaths, we give a sufficient theoretical investigation of this model. Our theoretical analysis focuses on the transcritical bifurcation at the disease-free equilibrium point and the period-doubling bifurcation at endemic equilibrium point. Our analytic conclusions well answer the questions presented in the Introduction section. Using our results, one can check the stability of the above-mentioned equilibrium points and calculate the critical parameter γ for the emergence of a flip bifurcation. Finally, we also present some numerical simulation examples for illustrating our theoretical analysis.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Each of the authors, XZ, XL, and WSW, contributed to each part of this study equally and read and approved the final version of the manuscript.

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References

1. Kermack, WO, McKendrick, AG: Contributions to the mathematical theory of epidemics. Proc. R. Soc. Lond. A **115**, 700-721 (1927)
2. Anderson, RM, May, RM: Infections Diseases of Humans: Dynamics and Control. Oxford University Press, Oxford (1991)
3. Diekmann, O, Heersterbeek, JAP: Mathematical Epidemiology of Infectious Diseases: Model Building Analysis, and Interpretation. Wiley, New York (2000)
4. Murray, JD: Mathematical Biology II, 3rd edn. Springer, Berlin (2003)
5. Ma, Z, Zhou, Y, Wang, W, Jin, Z: Mathematical Modelling and Research of Epidemic Dynamical Systems. Science Press, Beijing (2004)
6. Hethcote, HW, Stech, HW, van den Driessche, P: Periodicity and stability in epidemic models. In: Busenberg, SN, Cooke, KL (eds.) Differential Equations and Applications in Ecology, Epidemics and Population Problems, pp. 65-82. Academic Press, New York (1981)
7. Takeuchi, Y, Ma, W, Beretta, E: Global asymptotic properties of a delay SIR epidemic model with finite incubation times. Nonlinear Anal. **42**, 931-947 (2000)
8. Castillo-Chavez, C, Yakubu, A-A: Disperal, disease and life-history evolution. Math. Biosci. **173**, 35-53 (2001)
9. Medlock, J, Kot, M: Spreading disease: integro-differential equations old and new. Math. Biosci. **184**, 201-222 (2003)
10. Song, M, Ma, W: Asymptotic properties of a revised SIR epidemic model with density dependent birth rate and time delay. Dyn. Contin. Discrete Impuls. Syst. **13**, 199-208 (2006)
11. Yoshida, N, Hara, T: Global stability of a delayed SIR epidemic model with density dependent birth and death rates. J. Comput. Appl. Math. **201**, 339-347 (2007)
12. Robinson, RC: An Introduction to Dynamical Systems: Continuous and Discrete. Pearson Prentice Hall, Upper Saddle River (2004)
13. May, RM: Simple mathematical models with very complicated dynamics. Nature **261**, 459-467 (1976)
14. Rasband, SN: Chaotic Dynamics of Nonlinear Systems. Wiley-Interscience, New York (1990)
15. Allen, LJS: Some discrete-time SI, SIR and SIS epidemic models. Math. Biosci. **124**, 83-105 (1994)
16. Allen, LJS, Burgin, AM: Comparison of deterministic and stochastic SIS and SIR models in discrete time. Math. Biosci. **163**, 1-33 (2000)
17. Kuznetsov, YA: Elements of Applied Bifurcation Theory. Springer, New York (1995)
18. Wiggins, S: Introduction to Applied Nonlinear Dynamical Systems and Chaos. Springer, New York (1990)
19. Guckenheimer, J, Holmes, P: Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields. Springer, New York (1983)
20. Carr, J: Applications of Center Manifold Theory. Springer, New York (1981)