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Existence results for impulsive neutral stochastic functional integro-differential inclusions with infinite delays

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Abstract

In this paper, we prove the existence of mild solutions for a class of impulsive neutral stochastic functional integro-differential inclusions with infinite delays in Hilbert spaces. The results are obtained by using the fixed-point theorem for multi-valued operators due to Dhage. An example is provided to illustrate the theory.

MSC: 93B05; 93E03

Keywords: impulsive equation; stochastic functional inclusion; mild solution; infinite delay

1 Introduction

In this paper, we shall consider the existence of mild solutions for impulsive neutral stochastic functional integro-differential inclusions with infinite delay of the following form:

$$d\left[x(t) - g\left(t, x_t, \int_0^t a(t,s,x_s)\,ds\right)\right] dt$$

$$\in \left[Ax(t) + f(t,x_t)\right] dt + F(t,x_t)\,dw(t), \quad t \in J = [0,b], t \neq t_k, \tag{1.1}$$

$$\Delta x(t_k) = x\left(t_k^+\right) - x\left(t_k^-\right) = I_k\left(x\left(t_k^-\right)\right), \quad k = 1,2,\dots,m, \tag{1.2}$$

$$x(t) = \phi(t) \in L^2(\Omega, \mathcal{B}_h) \quad \text{for a.e. } t \in J_0 = (-\infty, 0], \tag{1.3}$$

where the state $x(\cdot)$ takes values in a separable real Hilbert space H with inner product $(\cdot,\cdot)$ and norm $|\cdot|$, A is the infinitesimal generator of a compact analytic resolvent operator $S(t)$, $t \geq 0$, in the Hilbert space H. Suppose that $\{w(t) : t \geq 0\}$ is a given K-valued Brownian motion or Wiener process with a finite trace nuclear covariance operator $Q \geq 0$ and $L(K,H)$ denotes the space of all bounded linear operators from K into H. Further $a : D \times \mathcal{B}_h \to H$, $g : J \times \mathcal{B}_h \times H \to H$, $f : J \times \mathcal{B}_h \to H$ and $F : J \times \mathcal{B}_h \to \mathcal{P}(L_Q(K,H))$ are given functions, where $D = \{(t,s) \in J \times J : s \leq t\}$, $\mathcal{P}(L_Q(K,H))$ is the family of all nonempty subsets of $L_Q(K,H)$ and $L_Q(K,H)$ denotes the space of all Q-Hilbert-Schmidt operators from K into H, which will be defined in Section 2. Here, $I_k \in C(H,H)$ $(k = 1,2,\dots,m)$ are bounded functions. Furthermore, the fixed times t_k satisfies $0 = t_0 < t_1 < t_2 < \cdots < t_m < b$, $x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of $x(t)$ at $t = t_k$. $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-))$ represents the jump in the state x at time t_k, where I_k determines the size of jump. The histories $x_t : \Omega \to \mathcal{B}_h$, $t \geq 0$, which are defined by setting $x_t = \{x(t+s) : s \in (-\infty, 0]\}$, belong to the abstract phase space $\mathcal{B}_h$, which will be defined in Section 2. The initial data $\phi = \{\phi(t) : -\infty < t \leq 0\}$ is an $\mathcal{F}_0$-measurable, $\mathcal{B}_h$-valued random variables independent of $\{w(t) : t \geq 0\}$ with finite second moment.

The theory of impulsive integro-differential inclusions has become an active area of investigation due to their applications in the fields such as mechanics, electrical engineering, medicine biology, ecology, and so on (see [1, 2] and references therein).

The existence of impulsive neutral stochastic functional integro-differential equations or inclusions with infinite delays have attracted great interest of researchers. For example, Lin and Hu [3] consider the existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions. Hu and Ren [4] studied the existence results for impulsive neutral stochastic functional integro-differential equations with infinite delays.

Motivated by the previous mentioned papers, we prove the existence of solutions for impulsive neutral stochastic functional integro-differential inclusions with infinite delays.

2 Preliminaries

Throughout this paper, $(H, |\cdot|)$ and $(K, |\cdot|_K)$ denote two real separable Hilbert spaces. Let $(\Omega, \mathcal{F}, P; F)$ ($F = \{\mathcal{F}_t\}_{t\geq 0}$) be a complete filtered probability space satisfying the requirement that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. An H-valued random variable is an $\mathcal{F}$-measurable function $x(t) : \Omega \to H$ and the collection of random variables $S = \{x(t,w) : \Omega \to H | t \in J\}$ is called a stochastic process. Suppose that $\{w(t) : t \geq 0\}$ is a cylindrical K-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $T_r Q = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies $Qe_i = \lambda_i e_i$. So, actually, $w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i$, where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener process. We assume that $\mathcal{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ is the σ-algebra generated by w and $\mathcal{F}_T = \mathcal{F}$. Let $\psi \in L(K,H)$ and define

$$|\psi|_Q^2 = T_r\bigl(\psi Q \psi^*\bigr) = \sum_{n=1}^{\infty} \bigl|\sqrt{\lambda_n}\psi e_n\bigr|^2.$$

If $|\psi|_Q < \infty$, then ψ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K,H)$ denote the space of all Q-Hilbert-Schmidt operator $\psi : K \to H$. The completion $L_Q(K,H)$ of $L(K,H)$ with respect to the topology induced by the norm $|\cdot|_Q$, where $|\psi|_Q^2 = \langle \psi, \psi \rangle$ is a Hilbert space with the above norm topology.

Let $A : D(A) \to H$ be the infinitesimal generator of a compact, analytic resolvent operator $S(t)$, $t \geq 0$. Let $0 \in \rho(A)$. Then it is possible to define the fractional power $(-A)^\alpha$ for $0 < \alpha \leq 1$ as a closed linear operator with its domain $D((-A)^\alpha)$ being dense in H. We denote by H_α the Banach space $D(-A^\alpha)$ endowed with the norm $\|x\|_\alpha = \|(-A)^\alpha x\|$, which is equivalent to the graph norm of $(-A)^\alpha$.

Lemma 2.1 ([5]) *The following properties hold*:

(i) *If* $0 < \beta < \alpha \leq 1$, *the* $H_\alpha \subset H_\beta$ *and the embedding is continuous and compact whenever the resolvent operator of* A *is compact*.

(ii) *For every* $0 < \alpha < 1$, *there exists a positive constant* c_α *such that*

$$\bigl\|(-A)^\alpha S(t)\bigr\| \leq \frac{C_\alpha}{t^\alpha}, \quad t > 0.$$

Now, we define the abstract phase space $\mathcal{B}_h$. *Assume that* $h:(-\infty,0]\to(0,\infty)$ *is a continuous function with* $l=\int_{-\infty}^0 h(t)\,dt<\infty$. *For any* $a>0$ *we define*

$$\mathcal{B}_h=\Big\{\psi:(-\infty,0]\to H:\big(E\big|\psi(\theta)\big|^2\big)^{\frac{1}{2}} \text{ is a bounded and measurable function on } [-a,0] \text{ and } \int_{-\infty}^0 h(s)\sup_{s\le\theta\le 0}\big(E\big|\psi(\theta)\big|^2\big)^{\frac{1}{2}}\,ds<\infty\Big\}.$$

If $\mathcal{B}_h$ *is endowed with the norm*

$$\|\psi\|_{\mathcal{B}_h}=\int_{-\infty}^0 h(s)\sup_{s\le\theta\le 0}\big(E\big|\psi(\theta)\big|^2\big)^{\frac{1}{2}}\,ds \quad \text{for all } \psi\in\mathcal{B}_h,$$

then $(\mathcal{B}_h,\|\cdot\|_{\mathcal{B}_h})$ *is a Banach space* [6]. *Now, we consider the space*

$$\mathcal{B}_b=\big\{x:(-\infty,b]\to H \text{ such that } x_k\in C(J_k,H) \text{ and there exist } x\big(t_k^+\big) \text{ and } x\big(t_k^-\big) \text{ with } x(t_k)=x\big(t_k^-\big), x_0=\phi\in L^2(\Omega,\mathcal{B}_h) \text{ on } (-\infty,0], k=1,2,\ldots,m\big\},$$

where x_k *is the restriction of* x *to* $J_k=(t_k,t_{k+1}]$, $k=0,1,\ldots,m$. *Let* $\|\cdot\|_b$ *be a seminorm in* $\mathcal{B}_b$ *defined by*

$$\|x\|_b=\|x_0\|_{\mathcal{B}_h}+\sup_{0\le s\le b}\big(E\big|x(s)\big|^2\big)^{\frac{1}{2}},\quad x\in\mathcal{B}_b.$$

Lemma 2.2 ([7]) *Assume that* $x\in\mathcal{B}_b$, *then for* $t\in J$, $x_t\in\mathcal{B}_h$. *Moreover*

$$l\big(E\big|x(t)\big|^2\big)^{\frac{1}{2}}\le\|x_t\|_{\mathcal{B}_h}\le\|x_0\|_{\mathcal{B}_h}+l\sup_{0\le s\le t}\big(E\big|x(t)\big|^2\big)^{\frac{1}{2}},$$

where $l=\int_{-\infty}^0 h(s)\,ds<\infty$.

We use the notation $\mathcal{P}(H)$ for the family of all subsets H and denote

$$\mathcal{P}_{cl}(H)=\big\{Y\in\mathcal{P}(H):Y \text{ is closed}\big\},$$
$$\mathcal{P}_{bd}(H)=\big\{Y\in\mathcal{P}(H):Y \text{ is bounded}\big\},$$
$$\mathcal{P}_{cv}(H)=\big\{Y\in\mathcal{P}(H):Y \text{ is convex}\big\},$$
$$\mathcal{P}_{cp}(H)=\big\{Y\in\mathcal{P}(H):Y \text{ is compact}\big\}.$$

A multi-valued mapping $\Gamma:H\to\mathcal{P}(H)$ is called upper semicontinuous (u.s.c) if for any $x\in H$, the set $\Gamma(x)$ is a nonempty closed subset of H and if for each open set G of H containing $\Gamma(x)$, there exists an open neighborhood N of x such that $\Gamma(N)\subseteq G$. Γ is said to be completely continuous if $\Gamma(B)$ is relatively compact for every bounded subset of $B\subseteq H$. If the multi-valued mapping Γ is completely continuous with nonempty compact values, then Γ is u.s.c. if and only if Γ has a closed graph, *i.e.*, $x_n\to x$, $y_n\to y$, $y_n\in\Gamma(x_n)$ imply $y\in\Gamma(x)$.

Definition 2.1 The multi-valued mapping $F : J \times \mathcal{B}_h \to \mathcal{P}(H)$ is said to be L^2-Carathéodory if

(i) $t \mapsto F(t, v)$ is measurable for each $v \in \mathcal{B}_h$,
(ii) $v \mapsto F(t, v)$ is u.s.c. for almost all $t \in J$ and $v \in \mathcal{B}_h$,
(iii) for each $q > 0$, there exists $h_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,v)\|^2 = \sup_{f \in F(t,v)} E(|f|^2) \le h_q(t),$$

for all $\|v\|^2_{\mathcal{B}_h} \le q$ and for a.e. $t \in J$.

The following lemma is crucial in the proof of our main result.

Lemma 2.3 ([8]) *Let I be a compact interval and H be a Hilbert space. Let F be an L^2-Carathéodory multi-valued mapping with $N_{F,x} \neq \phi$ and let Γ be a linear continuous mapping from $L^2(I, H)$ to $C(I, H)$. Then the operator*

$$\Gamma \circ N_F : C(I,H) \to \mathcal{P}_{cp,cv}(H), \qquad x \mapsto (\Gamma \circ N_F)(x) = \Gamma(N_{F,x})$$

is a closed graph operator in $C(I, H) \times C(I, H)$, where $N_{F,x}$ is known as the selectors set from F; it is given by

$$\sigma \in N_{F,x} = \{\sigma \in L^2(L(K,H)) : \sigma(t) \in F(t,x) \text{ for a.e. } t \in J\}.$$

Theorem 2.1 ([9]) *Let X be a Banach space, $\Phi_1 : X \to \mathcal{P}_{cl,cv,bd}(X)$ and $\Phi_2 : X \to \mathcal{P}_{cp,cv}(X)$ be two multi-valued operators satisfying*:

(a) *Φ_1 is a contraction,*
(b) *Φ_2 is u.s.c. and completely continuous.*

Then either

(i) *the operator inclusion $\lambda x \in \Phi_1 x + \Phi_2 x$ has a solution for $\lambda = 1$, or*
(ii) *the set $G = \{x \in X : \lambda x \in \Phi_1 x + \Phi_2 x, \lambda > 1\}$ is unbounded.*

Lemma 2.4 ([10]) *Let $v, w : [0, b] \to [0, \infty)$ be continuous functions. If w is nondecreasing and there are constants $\theta > 0$, $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + \theta \int_0^t \frac{v(s)}{(t-s)^{1-\alpha}}\, ds, \quad t \in J,$$

then

$$v(t) \le e^{\frac{\theta^n \Gamma(\alpha)^n t^{n\alpha}}{\Gamma(n\alpha)}} \sum_{j=0}^{n-1} \left(\frac{\theta b^\alpha}{\alpha}\right)^j w(t)$$

for every $t \in J$ and every $n \in N$ such that $n\alpha > 1$ and $\Gamma(\cdot)$ is the Gamma function.

3 Main result

Let $J_1 = (-\infty, b]$. First, we present the definition of the mild solution of problem (1.1)-(1.3).

Definition 3.1 A stochastic process $x : J_1 \times \Omega \to H$ is called a mild solution of problem (1.1)-(1.3) if

(i) $x(t)$ is measurable and $\mathcal{F}_t$-adapted for each $t \geq 0$,

(ii) $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, $k = 1, 2, \dots, m$,

(iii) $x(t) \in H$ has càdlàg paths on $t \in J$ a.e. and there exists a function $\sigma \in N_{F,x}$ such that

$$x(t) = S(t)\big[\phi(0) - g(0,\phi,0)\big] + g\left(t, x_t, \int_0^t a(t,s,x_s)\,ds\right) + \int_0^t AS(t-s)g\left(s, x_s, \int_0^s a(s,\tau,x_\tau)\,d\tau\right)ds + \int_0^t AS(t-s)f(s,x_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(x(t_k^-)\big), \quad t \in J,$$

(iv) $x_0(\cdot) = \phi \in L^2(\Omega, \mathcal{B}_h)$ on $J_0 = (-\infty, 0]$ satisfies $\|\phi\|_{\mathcal{B}_h}^2 < \infty$.

Now, we assume the following hypotheses:

(H1) A is the infinitesimal generator of a compact analytic resolvent operator $S(t)$, $t \geq 0$, in the Hilbert space H and there exist positive constants M and M_1 such that

$$\big\|S(t)\big\|^2 \leq M, \qquad \big\|A^{-\beta}\big\| \leq M_1, \quad t \in J.$$

(H2) $a : D \times \mathcal{B}_h \to H$, $D = \{(t,s) \in J \times J : t \geq s\}$ is a continuous function and there exists a constant M_a such that

$$E\left|\int_0^t \big[a(t,s,x) - a(t,s,y)\big]\,ds\right|^2 \leq M_a\|x-y\|_{\mathcal{B}_h}^2 \quad \text{for all } t \in J, x, y \in \mathcal{B}_h.$$

(H3) There exist constants $0 < \beta < 1$ and M_g such that g is H_β-valued, $(-A)^\beta g$ is continuous and

$$E\big|(-A)^\beta g(t,x_1,y_1) - (-A)^\beta g(t,x_2,y_2)\big|^2 \leq M_g\big[\|x_1 - x_2\|_{\mathcal{B}_h}^2 + E|y_1 - y_2|^2\big].$$

(H4) The function $f : J \times \mathcal{B}_h \to H$ satisfies the following conditions:

(i) $t \mapsto f(t,s)$ is measurable for each $x \in \mathcal{B}_h$;

(ii) $x \mapsto f(t,x)$ is continuous for almost all $t \in J$;

(iii) There exists a constant M_f such that

$$E\big|(-A)^\beta f(t,x) - (-A)^\beta f(t,y)\big|^2 \leq M_f\|x-y\|_{\mathcal{B}_h}^2$$

for all $x, y \in \mathcal{B}_h$, $t \in J$ and

$$E\big|f(t,x)\big|^2 \leq p(t)\psi\big(\|x\|_{\mathcal{B}_h}^2\big)$$

for almost all $t \in J$, where $p \in L^1(J,\mathbb{R})$, $\psi : \mathbb{R}_+ \to (0,\infty)$ is continuous and increasing with

$$\int_0^b \overline{\mu(s)}\,ds \leq \int_{B_0k_1}^\infty \frac{1}{\psi(s)}\,ds,$$

$$\overline{\mu}(t) = B_0k_3p(t),$$

$$k_1 = \frac{4\|\phi\|_{\mathcal{B}_h}^2 + l^2 F}{1 - 96l^2\|(-A)^{-\beta}\|^2 M_g(1 + 2M_a)},$$

$$k_2 = \frac{96bl^2 M_g(1 + 2M_a)c_{1-\beta}^2}{1 - 96l^2\|(-A)^{-\beta}\|^2 M_g(1 + 2M_a)},$$

$$k_3 = \frac{48Mbl^2}{1 - 96l^2\|(-A)^{-\beta}\|^2 M_g(1 + 2M_a)},$$

$$L_0 = 3l^2\left[M_g(1 + M_a)\left(\left\|(-A)^{-\beta}\right\|^2 + \frac{(C_{1-\beta}b^\beta)^2}{2\beta - 1}\right) + M_f\frac{(C_{1-\beta}b^\beta)^2}{2\beta - 1}\right] < 1,$$

$$B_0 = e^{k_2^n\Gamma(\beta)^n b^{n\beta}/\Gamma(n\beta)}\sum_{j=1}^{n-1}\left(\frac{k_2 b^\beta}{\beta}\right)^j,$$

$$c_1 = b^2\sup_{(t,s)\in D} a^2(t,s,0), \qquad c_2 = \left\|(-A)^\beta\right\|^2\sup_{t\in J}\left\|g(t,0,0)\right\|^2$$

and

$$\begin{aligned}\mathcal{F} &= 4M\left|\phi(0)\right|^2 + 96\left(M + \left\|(-A)^{-\beta}\right\|^2\right)c_2 + 192\left\|(-A)^{-\beta}\right\|^2 M_g c_1 \\ &\quad + \frac{192b^{2\beta}C_{1-\beta}^2}{2\beta - 1}(c_2 + 2M_g c_1) + 48M\|\mu\|_{L^1_{\text{loc}}(J,\mathbb{R}^+)}b^2\operatorname{Tr}(Q) \\ &\quad + 48Mm^2\sum_{k=1}^m d_k + 96M\left\|(-A)^{-\beta}\right\|^2 M_g\|\phi\|_{\mathcal{B}_h}^2.\end{aligned}$$

(H5) The multi-valued mapping $F : J \times \mathcal{B}_h \to \mathcal{P}_{bd,cl,cv}(L(K,H))$ is an L^2-Carathéodory function that satisfies the following conditions:

(i) For each $t \in J$, the function $F(t,\cdot) : \mathcal{B}_h \to \mathcal{P}_{bd,cl,cv}(L(K,H))$ is u.s.c. and for each fixed $x \in \mathcal{B}_h$, the function $F(\cdot,x)$ is measurable. For each $x \in \mathcal{B}_h$, the set

$$N_{F,x} = \left\{\sigma \in L^2(K,H) : \sigma(t) \in F(t,x) \text{ for a.e. } t \in J\right\}$$

is nonempty.

(ii) There exists a positive function $\mu \in L^1_{\text{loc}}(J,\mathbb{R}^+)$ such that

$$\left\|F(t,x)\right\|^2 = \sup_{\sigma\in F(t,x)} E|\sigma|^2 \le \mu(t).$$

(H6) $I_k \in C(H_\alpha, H_\alpha)$ and there exist positive constants d_k such that for each $x \in H_\alpha$,

$$\left|I_k(x)\right|^2 \le d_k, \quad k = 1, 2, \ldots, m.$$

We consider the mapping $\Phi : \mathcal{B}_h \to \mathcal{P}(\mathcal{B}_h)$ defined by

$$\Phi x(t) = \begin{cases}\phi(t), \quad t \in (-\infty, 0], \\ S(t)[\phi(0) - g(0,\phi,0)] + g(t,x_t,\int_0^t a(t,s,x_s)\,ds) \\ \quad + \int_0^t AS(t-s)g(s,x_s,\int_0^s a(s,\tau,x_\tau)\,d\tau)\,ds \\ \quad + \int_0^t AS(t-s)f(s,x_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) \\ \quad + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \quad t \in J,\end{cases}$$

where $\sigma \in N_{F,x}$. For each $\phi \in \mathcal{B}_h$, we define

$$\tilde{\phi}(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ S(t)\phi(0), & t \in J, \end{cases}$$

and then $\tilde{\phi} \in \mathcal{B}_h$. Let $x(t) = y(t) + \tilde{\phi}(t)$, $t \in (-\infty, b]$. Then it is easy to see that x satisfies (1.1)-(1.3) if and only if y satisfies $y_0 = 0$ and

$$\begin{aligned} y(t) = {} & -S(t)g(0,\phi,0) + g\left(t, y_t + \tilde{\phi}_t, \int_0^t a(t,s,y_s + \tilde{\phi}_s)\,ds\right) \\ & + \int_0^t AS(t-s)g\left(s, y_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)\,d\tau\right) ds \\ & + \int_0^t AS(t-s)f(s, y_s + \phi_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) \\ & + \sum_{0<t_k<t} S(t-t_k)I_k\big(y(t_k^-) + \tilde{\phi}(t_k^-)\big), \quad t \in J, \end{aligned}$$

where $\sigma \in N_{F,y}$. Let $\mathcal{B}_h' = \{y \in \mathcal{B}_h : y_0 = 0 \in \mathcal{B}_h\}$. For any $y \in \mathcal{B}_h'$,

$$\begin{aligned} \|y\|_b &= \|y_0\|_{\mathcal{B}_h} + \sup_{0 \le s \le b} \big(E|y(s)|^2\big)^{\frac{1}{2}} \\ &= \sup_{0 \le s \le b} \big(E|y(s)|^2\big)^{\frac{1}{2}} \end{aligned}$$

and thus $(\mathcal{B}_h', \|\cdot\|_b)$ is a Banach space. Set $\mathcal{B}_q = \{y \in \mathcal{B}_h' : \|y\|_b^2 \le q\}$ for some $q \ge 0$. Then $\mathcal{B}_q \subseteq \mathcal{B}_h'$ is uniformly bounded and for any $y \in \mathcal{B}_q$, from Lemma 2.2, we see that

$$\begin{aligned} \|y_t + \tilde{\phi}_t\|_{\mathcal{B}_h}^2 &\le 2\|y_t\|_{\mathcal{B}_h}^2 + 2\|\tilde{\phi}_t\|_{\mathcal{B}_h}^2 \\ &\le 4l^2 \sup_{0 \le s \le t} E|y(s)|^2 + 4\|y_0\|_{\mathcal{B}_h}^2 \\ &\quad + 4l^2 \sup_{0 \le s \le t} \|\tilde{\phi}(s)\|^2 + 4\|\tilde{\phi}_0\|_{\mathcal{B}_h}^2 \\ &\le 4l^2\big(q + M|\phi(0)|^2\big) + 4\|\tilde{\phi}\|_{\mathcal{B}_h}^2 \\ &:= q'. \end{aligned}$$

Define the operator $\tilde{\Phi} : \mathcal{B}_h' \to \mathcal{P}(\mathcal{B}_h')$ by

$$\tilde{\Phi}y(t) = \begin{cases} 0, \quad t \in (-\infty, 0], \\ -S(t)g(0,\phi,0) + g(t, y_t + \tilde{\phi}_t, \int_0^t a(t,s,y_s + \tilde{\phi}_s)\,ds) \\ \quad + \int_0^t AS(t-s)g(s, y_s + \tilde{\phi}, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)\,d\tau)\,ds \\ \quad + \int_0^t AS(t-s)f(s, y_s + \tilde{\phi}_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) \\ \quad + \sum_{0<t_k<t} S(t-t_k)I_k(y(t_k^-) + \tilde{\phi}(t_k^-)), \quad t \in J, \end{cases}$$

where $\sigma \in N_{F,y}$. Obviously, the operator Φ has a fixed point is equivalent to proving that $\tilde{\Phi}$ has a fixed point. Now, we decompose $\tilde{\Phi}$ as $\tilde{\Phi}_1 + \tilde{\Phi}_2$, where

$$\begin{aligned}\tilde{\Phi}_1 y(t) &= -S(t)g(0,\phi,0) + g\left(t, y_t + \tilde{\phi}_t, \int_0^t a(t,s,y_s + \tilde{\phi}_s)\,ds\right) \\ &\quad + \int_0^t AS(t-s)g\left(s, y_s + \tilde{\phi}_s, \int_0^s a(s,\tau,y_\tau + \tilde{\phi}_\tau)\,d\tau\right) ds \\ &\quad + \int_0^t AS(t-s)f(s,y_s + \tilde{\phi}_s)\,ds\end{aligned}$$

and

$$\tilde{\Phi}_2 y(t) = \int_0^s S(t-s)\sigma(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(y(t_k^-) + \tilde{\phi}(t_k^-)\big), \quad t \in J,$$

where $\sigma \in N_{F,y}$. In what follows, we show that the operators $\tilde{\Phi}_1$ and $\tilde{\Phi}_2$ satisfy all the conditions of Theorem 2.1.

Lemma 3.1 *Assume that the assumptions* (H1)-(H6) *hold. Then* $\tilde{\Phi}_1$ *is a contraction and* $\tilde{\Phi}_2$ *is u.s.c. and completely continuous.*

Proof We give the proof in several steps:

Step 1. $\tilde{\Phi}_1$ is a contraction.

Let $u, v \in \mathcal{B}_h'$. Then we have

$$\begin{aligned}&E\big|\tilde{\phi}_1 u(t) - \tilde{\phi}_1 v(t)\big|^2 \\ &\quad\le 3E\left|g\left(t, u_t + \tilde{\phi}_t, \int_0^t a(t,s,u_s + \tilde{\phi}_s)\,ds\right) - g\left(t, v_t + \tilde{\phi}_t, \int_0^t a(t,s,v_s + \tilde{\phi}_s)\,ds\right)\right|^2 \\ &\qquad + 3bE\left(\int_0^t \left|AS(t-s)\left[g\left(s, u_s + \tilde{\phi}_s, \int_0^s a(s,\tau,u_\tau + \tilde{\phi}_\tau)\,d\tau\right)\right.\right.\right. \\ &\qquad - \left.\left.\left. g\left(s, v_s + \tilde{\phi}_s, \int_0^s a(s,\tau,u_\tau + \tilde{\phi}_\tau)\,d\tau\right)\right]\right|^2 ds\right) \\ &\qquad + 3bE\left(\int_0^t \big|AS(t-s)\big[f(s,u_s + \tilde{\phi}_s) - f(s,v_s + \tilde{\phi}_s)\big]\big|^2 ds\right) \\ &\quad\le 3\big\|(-A)^{-\beta}\big\|^2 M_g\big(\|u_t - v_t\|_{\mathcal{B}_h}^2 + M_a\|u_t - v_t\|_{\mathcal{B}_h}^2\big) \\ &\qquad + 3b\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}} M_g\big(\|u_s - v_s\|_{\mathcal{B}_h}^2 + M_a\|u_s - v_s\|_{\mathcal{B}_h}^2\big)\,ds \\ &\qquad + 3b\int_0^t \frac{C_{1-\beta}^2}{(t-s)^{2(1-\beta)}} M_f\|u_s - v_s\|_{\mathcal{B}_h}^2\,ds \\ &\quad\le 3\big\|(-A)^{-\beta}\big\|^2 M_g(1 + M_a)\|u_t - v_t\|_{\mathcal{B}_h}^2 \\ &\qquad + 3M_g(1 + M_a)\frac{(C_{1-\beta}b^\beta)^2}{2\beta - 1}\|u_t - v_t\|_{\mathcal{B}_h}^2 \\ &\qquad + 3M_f\frac{(C_{1-\beta}b^\beta)^2}{2\beta - 1}\|u_t - v_t\|_{\mathcal{B}_h}^2\end{aligned}$$

$$\begin{aligned}
&\leq 3\left[M_g(1+M_a)\left(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}\right) + M_f\frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}\right] \\
&\quad \times \left[l^2 \sup_{s\in[0,b]} E\big|u(s)-v(s)\big|^2 + \|u_0\|^2_{\mathcal{B}_h} + \|v_0\|^2_{\mathcal{B}_h}\right] \\
&= 3l^2\left[M_g(1+M_a)\left(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}\right) + M_f\frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}\right]\sup_{s\in[0,b]} E\big|u(s)-v(s)\big|^2 \\
&= L_0 \sup_{s\in[0,b]} E\big|u(s)-v(s)\big|^2,
\end{aligned}$$

where $L_0 = 3l^2[M_g(1+M_a)(\|(-A)^{-\beta}\|^2 + \frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}) + M_f\frac{(C_{1-\beta}b^\beta)^2}{2\beta-1}] < 1$ and we have used the fact that $\|u_0\|^2_{\mathcal{B}_h} = 0$ and $\|v_0\|^2_{\mathcal{B}_h} = 0$. Taking the supremum over t, we obtain

$$\|\tilde{\phi}_1 u - \tilde{\phi}_1 v\|_b^2 \leq L_0\|u-v\|_b^2$$

and so $\tilde{\phi}_1$ is a contraction.

Now, we show that the operator $\tilde{\Phi}_2$ is completely continuous.

Step 2. $\tilde{\Phi}_2 y$ is convex for each $y \in \mathcal{B}'_h$.

In fact, if $u_1, u_2 \in \tilde{\Phi}_2(y)$, then there exist $\sigma_1, \sigma_2 \in N_{F,y}$ such that

$$u_i(t) = \int_0^t S(t-s)\sigma_i(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)$$

for $i = 1, 2$ and $t \in J$. Let $\lambda \in [0,1]$. Then for each $t \in J$, we have

$$\begin{aligned}
\lambda u_1(t) + (1-\lambda)u_2(t) &= \int_0^t S(t-s)\big[\lambda\sigma_1(s) + (1-\lambda)\sigma_2(s)\big]\,dw(s) \\
&\quad + \sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big).
\end{aligned}$$

Since $N_{F,y}$ is convex (because F has convex values), we obtain

$$\lambda u_1(t) + (1-\lambda)u_2(t) \in \tilde{\Phi}_2(y).$$

Step 3. $\tilde{\Phi}_2$ maps bounded sets into bounded sets in $\mathcal{B}'_h$.

It is enough to show that there exists a positive constant Λ such that for each $u \in \tilde{\Phi}_2 y$, $y \in \mathcal{B}_q = \{y \in \mathcal{B}'_h : \|y\|_b \leq q\}$ one has $\|u\|_b \leq \Lambda$. If $u \in \tilde{\Phi}_2(y)$, there exists $\sigma \in N_{F,y}$ such that for each $t \in J$

$$u(t) = \int_0^t S(s-t)\sigma(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)$$

and so

$$\begin{aligned}
E\big|u(t)\big|^2 &= E\left|\int_0^t S(t-s)\sigma(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)\right|^2 \\
&\leq 2E\left|\int_0^t S(t-s)\sigma(s)\,dw(s)\right|^2 + 2E\left|\sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)\right|^2
\end{aligned}$$

$$\begin{aligned}
&\leq 2\operatorname{Tr}(Q)Mb\int_0^b \mu(s)\,ds + 2Mm^2\sum_{k=1}^m d_k \\
&\leq 2\operatorname{Tr}(Q)Mb^2\|\mu\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}^+)} + 2Mm^2\sum_{k=1}^m d_k \\
&:= \Lambda.
\end{aligned}$$

Thus, for each $y \in \mathcal{B}_h'$, we get $\|u\|_b^2 \leq \Lambda$.

Step 4. $\tilde{\Phi}_2$ maps bounded sets into equicontinuous sets of $\mathcal{B}_h'$.

Let $0 < \tau_1 < \tau_2 \leq b$. For each $y \in \mathcal{B}_q = \{y \in \mathcal{B}_h' : \|y\|_b \leq q\}$ and $u \in \tilde{\Phi}_2(y)$. Let $\tau_1, \tau_2 \in J\backslash\{t_1, t_2, \ldots, t_m\}$. Then there exists $\sigma \in N_{F,y}$ such that for each $t \in J$,

$$u(t) = \int_0^t S(t-s)\sigma(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big).$$

Thus we have

$$\begin{aligned}
&E\big|u(\tau_2) - u(\tau_1)\big|^2 \\
&\quad = E\bigg|\int_0^{\tau_2} S(\tau_2 - s)\sigma(s)\,dw(s) + \sum_{0<t_k<\tau_2} S(\tau_2 - s)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big) \\
&\qquad - \int_0^{\tau_1} S(\tau_1 - s)\sigma(s)\,dw(s) - \sum_{0<t_k<\tau_1} S(\tau_1 - t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)\bigg|^2 \\
&\quad \leq 2E\bigg|\int_0^{\tau_1-\varepsilon} \big(S(\tau_2 - s)\sigma(s) - S(\tau_1 - s)\sigma(s)\big)\,dw(s) \\
&\qquad + \int_{\tau_1-\varepsilon}^{\tau_1} \big(S(\tau_2 - s)\sigma(s) - S(\tau_1 - s)\sigma(s)\big)\,dw(s) + \int_{\tau_1}^{\tau_2} S(\tau_2 - s)\sigma(s)\,dw(s)\bigg|^2 \\
&\qquad + 2E\bigg|\sum_{0<t_k<\tau_1} \big[S(\tau_2 - t_k) - S(\tau_1 - t_k)\big]I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big) \\
&\qquad + \sum_{\tau_1<t_k<\tau_2} S(\tau_2 - t_k)I_k\big(y\big(t_k^-\big) + \tilde{\phi}\big(t_k^-\big)\big)\bigg|^2 \\
&\quad \leq 6\varepsilon\operatorname{Tr}(Q)\int_0^{\tau_1-\varepsilon} \mu(s)\big\|S(\tau_2 - s) - S(\tau_1 - s)\big\|^2\,ds \\
&\qquad + 6\varepsilon\operatorname{Tr}(Q)\int_{\tau_1-\varepsilon}^{\tau_1} \mu(s)\big\|S(\tau_2 - s) - S(\tau_1 - s)\big\|^2\,ds \\
&\qquad + 6(\tau_2 - \tau_1)\operatorname{Tr}(Q)\int_{\tau_1}^{\tau_2} \mu(s)\big\|S(\tau_2 - s)\big\|^2\,ds \\
&\qquad + 4m^2\sum_{0<t_k<\tau_1} \big\|S(\tau_2 - s) - S(\tau_1 - s)\big\|^2 d_k \\
&\qquad + 4m^2M\sum_{\tau_1<t_k<\tau_2} d_k.
\end{aligned}$$

The right-hand side of the above inequality tends to zero as $\tau_1 \to \tau_2$ with ε sufficiently small, since $S(t)$ is strongly continuous and the compactness of $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology. Thus, the set $\{\tilde{\Phi}_2 y : y \in \mathcal{B}_q\}$ is equicontinuous. Here we consider the case $0 < \tau_1 < \tau_2 \le b$, since the case $\tau_1 < \tau_2 \le 0$ or $\tau_1 \le 0 \le \tau_2 \le b$ is simple.

Step 5. $\tilde{\Phi}_2$ maps $\mathcal{B}_q$ into a precompact set in H.

Let $0 < t \le b$ and $0 < \varepsilon < t$. For $y \in \mathcal{B}_q$ and $u \in \tilde{\Phi}_2(y)$, there exists $\sigma \in N_{F,y}$ such that

$$\begin{aligned} u(t) = {} & \int_0^{t-\varepsilon} S(t-s)\sigma(s)\,dw(s) + \int_{t-\varepsilon}^{t} S(t-s)\sigma(s)\,dw(s) \\ & + \sum_{0<t_k<t} S(t-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr). \end{aligned}$$

Define

$$u_\varepsilon(t) = S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)\sigma(s)\,dw(s) + \sum_{0<t_k<t-\varepsilon} S(t-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr).$$

Since $S(t)$ is a compact operator, the set $V_\varepsilon(t) = \{u_\varepsilon(t) : u_\varepsilon \in \tilde{\Phi}_2(\mathcal{B}_q)\}$ is relatively compact in H for each ε, $0 < \varepsilon < t$. Moreover,

$$\begin{aligned} & E\bigl|u(t) - u_\varepsilon(t)\bigr|^2 \\ & \quad = E\biggl|\int_0^{t-\varepsilon} S(t-s)\sigma(s)\,dw(s) + \int_{t-\varepsilon}^{t} S(t-s)\sigma(s)\,dw(s) \\ & \qquad + \sum_{0<t_k<t} S(t-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr) - S(\varepsilon)\int_0^{t-\varepsilon} S(t-\varepsilon-s)\sigma(s)\,dw(s) \\ & \qquad - \sum_{0<t_k<t-\varepsilon} S(t-t_k) I_k\bigl(y\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr)\biggr|^2 \\ & \quad \le 4Mb\,\mathrm{Tr}(Q)\varepsilon\|\mu\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}^+)} + 4m^2 M \sum_{t-\varepsilon<t_k<t} d_k. \end{aligned}$$

Therefore letting $\varepsilon \to 0$, we can see that there are relative compact sets arbitrarily close to the set $\{u(t) : u \in \tilde{\Phi}_2(B_q)\}$. Thus, the set $\{u(t) : u \in \tilde{\Phi}_2(B_q)\}$ is relatively compact in H. Hence, the Arzelá-Ascoli theorem shows that $\tilde{\Phi}_2$ is a compact multi-valued mapping.

Step 6. $\tilde{\Phi}_2$ has a closed graph.

Let $y_n \to y_*$, $u_n \in \tilde{\Phi}_2(y_n)$ and $u_n \to u_*$. We prove that $u_* \in \tilde{\Phi}_2(y_*)$.

Indeed, $u_n \in \tilde{\Phi}_2(y_n)$ means that there exists $\sigma_n \in N_{F,y_n}$ such that

$$u_n(t) = \int_0^t S(t-s)\sigma_n(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k) I_k\bigl(y_n\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr), \quad t \in J.$$

Thus we must prove that there exists $\sigma_* \in N_{F,y_*}$ such that

$$u_*(t) = \int_0^t S(t-s)\sigma_*(s)\,dw(s) + \sum_{0<t_k<t} S(t-t_k) I_k\bigl(y_*\bigl(t_k^-\bigr) + \tilde{\phi}\bigl(t_k^-\bigr)\bigr), \quad t \in J.$$

Since I_k, $k = 1, 2, \dots, m$, are continuous, we see that

$$\left\| \sum_{0<t_k<t} S(t-t_k) I_k\big(y_n(t_k^-) + \tilde{\phi}(t_k^-)\big) - \sum_{0<t_k<t} S(t-t_k) I_k\big(y_*(t_k^-) + \tilde{\phi}(t_k^-)\big) \right\|_b^2 \to 0$$

as $n \to \infty$. Consider the linear continuous operator $\Gamma : L^2(J, H) \to C(J, H)$ with $\Gamma(\sigma)(t) = \int_0^t S(t-s)\sigma(s)\,dw(s)$, where $\sigma \in N_{F,y}$. From Lemma 2.3, it follows that $\Gamma \circ N_F$ is a closed graph operator. Moreover, we have

$$u_n(t) - \sum_{0<t_k<t} S(t-t_k) I_k\big(y_n(t_k^-) + \tilde{\phi}(t_k^-)\big) \in \Gamma(N_{F,y_n}).$$

Since $y_n \to y_*$, from Lemma 2.3, we obtain

$$u_*(t) - \sum_{0<t_k<t} S(t-t_k) I_k\big(y_*(t_k^-) + \tilde{\phi}(t_k^-)\big) \in \Gamma(N_{F,y_*}).$$

That is, there exists a $\sigma_* \in N_{F,y_*}$ such that

$$\begin{aligned} u_*(t) - \sum_{0<t_k<t} S(t-t_k) I_k\big(y_*(t_k^-) + \tilde{\phi}(t_k^-)\big) &= \Gamma\big(\sigma_*(t)\big) \\ &= \int_0^t S(t-s)\sigma_*(s)\,dw(s). \end{aligned}$$

Therefore $\tilde{\Phi}_2$ has a closed graph and $\tilde{\Phi}_2$ is u.s.c. This completes the proof. □

Lemma 3.2 *Assume that the assumptions* (H1)-(H2) *hold. Then there exists a constant* $K > 0$ *such that* $\|y_t + \tilde{\phi}_t\|_{\mathcal{B}_h}^2 \le K$ *for all* $t \in J$, *where* K *is depends only on* b *and the functions* ψ *and* $\overline{\mu}$.

Proof Let y be a possible solution of $y \in \lambda\tilde{\Phi}(y)$ for some $0 < \lambda < 1$. Then there exists $\sigma \in N_{F,y}$ such that for $t \in J$ we have

$$\begin{aligned} y(t) = {}& -\lambda S(t) g(0, \phi, 0) + \lambda g\left(t, y_t + \tilde{\phi}_t, \int_0^t a(t, s, y_s + \tilde{\phi}_s)\,ds\right) \\ &+ \lambda \int_0^t AS(t-s) g\left(s, y_s + \tilde{\phi}_s, \int_0^s a(s, \tau, y_\tau + \tilde{\phi}_\tau)\,d\tau\right) ds \\ &+ \lambda \int_0^t S(t-s) f(s, y_s + \tilde{\phi}_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) \\ &+ \lambda \sum_{0<t_k<t} S(t-t_k) I_k\big(y(t_k^-) + \tilde{\phi}(t_k^-)\big). \end{aligned}$$

Then, by the assumptions, we deduce that

$$\begin{aligned} E\big|y(t)\big|^2 \le E\bigg| & -S(t) g(0, \phi, 0) + g\left(t, y_t + \tilde{\phi}_t, \int_0^t a(t, s, y_s + \tilde{\phi}_s)\,ds\right) \\ &+ \int_0^t AS(t-s) g\left(s, y_s + \tilde{\phi}_s, \int_0^s a(s, \tau, y_\tau + \tilde{\phi}_\tau)\,d\tau\right) ds \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t S(t-s) f(s, y_s + \tilde{\phi}_s)\,ds + \int_0^t S(t-s)\sigma(s)\,dw(s) \\
&+ \sum_{0<t_k<t} S(t-t_k) I_k\big(y(t_k^-) + \tilde{\phi}(t_k^-)\big)\bigg|^2 \\
\le{}& 12\Bigg\{ 2M\big(\big\|(-A)^{-\beta}\big\|^2 M_g \|\phi\|^2_{\mathcal{B}_h} + c_2\big) \\
&+ 2\big\|(-A)^{-\beta}\big\|^2 \big[M_g\big(\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} + 2M_a\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} + 2c_1\big) + c_2\big] \\
&+ 2b\int_0^t \frac{c_{1-\beta}^2}{(t-s)^{2(1-\beta)}} \big[M_g\big(\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} + 2M_a\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} + 2c_1\big) + c_2\big]\,ds \\
&+ Mb\int_0^t p(s)\psi\big(\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h}\big)\,ds + M\|\mu\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}^+)} b^2\,\mathrm{Tr}(Q) + Mm^2\sum_{k=1}^m d_k \Bigg\} \\
={}& 24\big(M + \big\|(-A)^{-\beta}\big\|^2\big)c_2 + 48\big\|(-A)^{-\beta}\big\|^2 M_g c_1 + \frac{48b^{2\beta}c_{1-\beta}^2}{2\beta-1}(c_2 + 2M_g c_1) \\
&+ 12M\|\mu\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}^+)} b^2\,\mathrm{Tr}(Q) + 12Mm^2\sum_{k=1}^m d_k + 24M\big\|(-A)^{-\beta}\big\|^2 M_g\|\phi\|^2_{\mathcal{B}_h} \\
&+ 24\big\|(-A)^{-\beta}\big\|^2 M_g(1+2M_a)\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} \\
&+ 24bM_g(1+2M_a)c_{1-\beta}^2\int_0^t \frac{\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h}}{(t-s)^{2(1-\beta)}}\,ds \\
&+ 12Mb\int_0^t p(s)\psi\big(\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h}\big)\,ds.
\end{aligned}
$$

From Lemma 2.2 we see that

$$
\|y_t + \tilde{\phi}_t\|^2_{\mathcal{B}_h} \le 4l^2 \sup_{0\le s\le t} E\big|y(s)\big|^2 + 4l^2 M\big|\tilde{\phi}(0)\big|^2 + 4\|\tilde{\phi}\|^2_{\mathcal{B}_h}.
$$

Thus, for any $t \in J$, we have

$$
\begin{aligned}
&\|y_t + \tilde{\phi}_t\|^2_{\mathcal{B}_h} \\
&\quad\le 4l^2 M\big|\tilde{\phi}(0)\big|^2 + 4\|\tilde{\phi}\|^2_{\mathcal{B}_h} + 96l^2\big(M + \big\|(-A)^{-\beta}\big\|^2\big)c_2 \\
&\qquad+ 192l^2\big\|(-A)^{-\beta}\big\|^2 M_g c_1 + \frac{192l^2 b^{2\beta} C_{1-\beta}^2}{2\beta-1}(c_2 + 2M_g c_1) \\
&\qquad+ 48M\|\mu\|_{L^1_{\mathrm{loc}}(J,\mathbb{R}^+)} b^2 l^2\,\mathrm{Tr}(Q) + 48Ml^2 m^2\sum_{k=1}^m d_k \\
&\qquad+ 96Ml^2\big\|(-A)^{-\beta}\big\|^2 M_g\|\phi\|^2_{\mathcal{B}_h} \\
&\qquad+ 96l^2\big\|(-A)^{-\beta}\big\|^2 M_g(1+2M_a)\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h} \\
&\qquad+ 96bl^2 M_g(1+2M_a)C_{1-\beta}^2\int_0^t \frac{\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h}}{(t-s)^{2(1-\beta)}}\,ds \\
&\qquad+ 48Mbl^2\int_0^t p(s)\psi\big(\|y_s + \tilde{\phi}_s\|^2_{\mathcal{B}_h}\big)\,ds
\end{aligned}
$$

$$
\begin{aligned}
&= 4\|\phi\|_{\mathcal{B}_h}^2 + l^2\mathcal{F} + 96l^2\big\|(-A)^{-\beta}\big\|^2 M_g(1+2M_a)\sup_{0\le s\le t}\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}^2 \\
&\quad + 96bl^2M_g(1+2M_a)C_{1-\beta}^2\int_0^t \frac{\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}^2}{(t-s)^{2(1-\beta)}}\,ds \\
&\quad + 48Mbl^2\int_0^t p(s)\psi\big(\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}^2\big)\,ds.
\end{aligned}
$$

Let $\nu(t) = \sup_{0\le s\le t}\|y_s+\tilde{\phi}_s\|_{\mathcal{B}_h}^2$. Then the function $\nu(t)$ is nondecreasing in J. Thus, we obtain

$$
\begin{aligned}
\nu(t) &\le 4\|\phi\|_{\mathcal{B}_h}^2 + l^2\mathcal{F} + 96l^2\big\|(-A)^{-\beta}\big\|^2 M_g(1+2M_a)\nu(t) \\
&\quad + 96bl^2M_g(1+2M_a)C_{1-\beta}^2\int_0^t \frac{\nu(s)}{(t-s)^{2(1-\beta)}}\,ds \\
&\quad + 48Mbl^2\int_0^t p(s)\psi\big(\nu(s)\big)\,ds.
\end{aligned}
$$

From this we derive that

$$
\begin{aligned}
\nu(t) &\le \frac{4\|\phi\|_{\mathcal{B}_h}^2 + l^2\mathcal{F}}{1-96l^2\|(-A)^{-\beta}\|^2M_g(1+2M_a)} \\
&\quad + \frac{96bl^2M_g(1+2M_a)C_{1-\beta}^2}{1-96l^2\|(-A)^{-\beta}\|^2M_g(1+2M_a)}\int_0^t \frac{\nu(s)}{(t-s)^{1-\beta}}\,ds \\
&\quad + \frac{48Mbl^2}{1-96l^2\|(-A)^{-\beta}\|^2M_g(1+2M_a)}\int_0^t p(s)\psi\big(\nu(s)\big)\,ds \\
&\le k_1 + k_2\int_0^t \frac{\nu(s)}{(t-s)^{1-\beta}}\,ds + k_3\int_0^t p(s)\psi\big(\nu(s)\big)\,ds.
\end{aligned}
$$

By Lemma 2.4, we get

$$
\nu(t) \le B_0\left(k_1 + k_3\int_0^t p(s)\psi\big(\nu(s)\big)\,ds\right),
$$

where

$$
B_0 = e^{k_2^n\Gamma(\beta)^n b^{n\beta}/\Gamma(n\beta)}\sum_{j=1}^{n-1}\left(\frac{k_2b^\beta}{\beta}\right)^j.
$$

Let us take the right-hand side of the above inequality as $\mu(t)$. Then $\mu(0) = B_0k_1$, $\nu(t) \le \mu(t)$, $t\in J$ and

$$
\mu'(t) \le B_0k_3p(t)\psi\big(\nu(t)\big).
$$

Since ψ is nondecreasing, we have

$$
\begin{aligned}
\mu'(t) &\le B_0k_3p(t)\psi\big(\mu(t)\big) \\
&= \overline{\mu}(t)\psi\big(\mu(t)\big).
\end{aligned}
$$

It follows that

$$\begin{aligned}\int_{\mu(0)}^{\mu(t)} \frac{1}{\psi(s)}\,ds &\le \int_0^b \overline{\mu(s)}\,ds \\ &\le \int_{B_0K_1}^{\infty} \frac{1}{\psi(s)}\,ds,\end{aligned}$$

which indicates that $\mu(t) < \infty$. Thus, there exists a constant K such that $\mu(t) \le K$, $t \in J$. Furthermore, we see that $\|y_t + \tilde{\phi}_t\|^2_{\mathcal{B}_h} \le \nu(t) \le \mu(t) \le K$, $t \in J$. □

Theorem 3.1 *Assume that the assumptions* (H1)-(H6) *hold. The problem* (1.1)-(1.3) *has at least one mild solution on* J.

Proof Let us take the set

$$G(\Phi) = \{x \in \mathcal{B}_h : x \in \lambda\Phi(x) \text{ for some } \lambda \in (0,1)\}.$$

Then for any $x \in G(\Phi)$, we have

$$\|x_t\|^2_{\mathcal{B}_h} = \|y_t + \tilde{\phi}_t\|^2_{\mathcal{B}_h} \le K, \quad t \in J,$$

where $K > 0$ is a constant in Lemma 3.2. This show that G is bounded on J. Hence from Theorem 2.1 there exists a fixed point $x(t)$ for Φ on $\mathcal{B}_h$, which is a mild solution of (1.1)-(1.3) on J. □

4 An example

As an application of Theorem 3.1, we consider the impulsive neutral stochastic functional integro-differential inclusion of the following form:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(z(t,x) + g\left(t, z(t-h,x), \int_0^t a\big(t,s,z(s-h,x)\big)\,ds\right)\right) \\ &\quad \in \frac{\partial^2}{\partial x^2} z(t,x) + \Big(f\big(t,z(t-h,x)\big) + \big[Q_1\big(t,z(t-h,x)\big), Q_2\big(t,z(t-h,x)\big)\big]\Big)\,dw(t), \qquad (4.1)\\ &\quad 0 \le x \le \pi, t \in J, t \ne t_k,\end{aligned}$$

$$\Delta z(t_k,x) = z\big(t_k^+,x\big) - z\big(t_k^-,x\big) = I_k\big(z\big(t_k^-,x\big)\big), \quad k = 1,2,\ldots,m, \qquad (4.2)$$

$$z(t,0) = z(t,\pi) = 0, \quad t \in J, \qquad (4.3)$$

$$z(t,x) = \rho(t,x), \quad -\infty < t \le 0, 0 \le x \le \pi, \qquad (4.4)$$

where $J = [0,b]$, $k = 1,2,\ldots,m$, $z(t_k^+,x) = \lim_{h\to 0^+} z(t_k + h,x)$, $z(t_k^-,x) = \lim_{h\to 0^-} z(t_k + h,x)$, $Q_1, Q_2 : J \times \mathbb{R} \to \mathbb{R}$ are two given functions and $w(t)$ is a one-dimensional standard Wiener process. We assume that for each $t \in J$, $Q_1(t,\cdot)$ is lower semicontinuous and $Q_2(t,\cdot)$ is upper semicontinuous. Let $J_1 = (-\infty,b]$ and $H = L^2([0,\pi])$ with norm $\|\cdot\|$. Define $A : H \to H$ by $Av = v''$ with domain $D(A) = \{v \in H : v, v'$ are absolutely continuous, $v'' \in H, v(0) = v(\pi) = 0\}$. Then

$$Av = \sum_{n=1}^{\infty} n^2(v, v_n), \quad v \in D(A),$$

where $v_n = \sqrt{\frac{2}{\pi}}\sin(ns)$, $n = 1, 2, \ldots$, is the orthogonal set of eigenvectors in A. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)$, $t \geq 0$ in H given by

$$S(t)v = \sum_{n=1}^{\infty} e^{-n^2 t}(v, v_n)v_n, \quad v \in H.$$

For every $v \in H$, $(-A)^{-\frac{1}{2}}v = \sum_{n=1}^{\infty} \frac{1}{n}(v, v_n)v_n$ and $\|(-A)^{-\frac{1}{2}}\| = 1$. The operator $(-A)^{\frac{1}{2}}$ is given by

$$(-A)^{\frac{1}{2}}v = \sum_{n=1}^{\infty} n(v, v_n)v_n$$

on the space $D((-A)^{-\frac{1}{2}}) = \{v \in H : \sum_{n=1}^{\infty} n(v, v_n)v_n \in H\}$. Since the analytic semigroup $S(t)$ is compact [10], there exists a constant $M > 0$ such that $\|S(t)\| \leq M$ and satisfies (H1). Now, we give a special $\mathcal{B}_h$-space. Let $h(s) = e^{2s}$, $s < 0$. Then $l = \int_{-\infty}^{0} h(s)\,ds = \frac{1}{2}$ and let

$$\|\varphi\|_{\mathcal{B}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} \left(E\left|\varphi(\theta)\right|^2\right)^{\frac{1}{2}} ds.$$

It follows from [5] that $(\mathcal{B}_h, \|\cdot\|_{\mathcal{B}_h})$ is a Banach space. Hence for $(t, \phi) \in [0, b] \times \mathcal{B}_h$, let

$$\phi(\theta)x = \phi(\theta, x), \quad (\theta, x) \in (-\infty, 0] \times [0, \pi],$$
$$z(t)(x) = z(t, x)$$

and

$$F(t, \phi)(x) = \left[Q_1\left(t, \phi(\theta, x)\right), Q_2\left(t, \phi(\theta, x)\right)\right], \quad -\infty < \theta \leq 0, x \in [0, \pi].$$

Then (4.1)-(4.4) can be rewritten as the abstract form as the system (1.1)-(1.3). If we assume that (H2)-(H6) are satisfied, then the system (4.1)-(4.4) has a mild solution on $[0, b]$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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