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# Fractional integral problems for fractional differential equations via Caputo derivative

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**Abstract**
In this paper, we study the existence and uniqueness of solutions for fractional boundary value problems involving nonlocal fractional integral boundary conditions, by means of standard fixed point theorems. Some illustrative examples are also presented.
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**Keywords:** fractional differential equations; nonlocal boundary conditions; fixed point theorems

## 1 Introduction

Differential equations with fractional order have recently proved to be valuable tools for the description of hereditary properties of various materials and systems. Many phenomena in engineering, physics, continuum mechanics, signal processing, electro-magnetics, economics, and science describes efficiently by fractional order differential equations. For a reader interested in the systematic development of the topic, we refer the books [1–5]. Many researchers have studied the existence theory for nonlinear fractional differential equations with a variety of boundary conditions; for instance, see the papers [6–21], and the references therein.

In this paper, we study the existence and uniqueness of solutions for the following boundary value problem for the fractional differential equation with nonlocal fractional integral boundary conditions

$$\begin{cases} {}^cD^q u(t) = f(t, u(t)), \quad 1 < q \le 2, 0 < t < T, \\ \sum_{i=1}^m \theta_i I^{\alpha_i} u(T) = \omega, \\ \sum_{j=1}^n \lambda_j I^{\beta_j} u(\eta_j) = \sum_{k=1}^l \mu_k (I^{\gamma_k} u(T) - I^{\gamma_k} u(\xi_k)), \end{cases} \tag{1.1}$$

where ${}^cD^q$ denotes the Caputo fractional derivative of order $q$, $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $\eta_j, \xi_k \in (0, T)$, $\theta_i, \lambda_j, \mu_k \in \mathbb{R}$, for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, l$, $\omega \in \mathbb{R}$, and $I^\phi$ is the Riemann-Liouville fractional integral of order $\phi > 0$ ($\phi = \alpha_i, \beta_j, \gamma_k$, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, l$).

The significance of studying problem (1.1) is that the boundary conditions are very general and include many conditions as special cases. In particular, if $\alpha_i = \beta_j = \gamma_k = 1$, for all $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$, $k = 1, 2, \dots, l$ then the boundary conditions reduce to

$$\begin{cases} (\theta_1 + \theta_2 + \cdots + \theta_m) \int_0^T u(s)\,ds = \omega, \\ \lambda_1 \int_0^{\eta_1} u(s)\,ds + \cdots + \lambda_n \int_0^{\eta_n} u(s)\,ds = \mu_1 \int_{\xi_1}^T u(s)\,ds + \cdots + \mu_l \int_{\xi_l}^T u(s)\,ds. \end{cases} \tag{1.2}$$

Note that the condition (1.2) does not contain the values of the unknown function $u$ at the left side and right side of the boundary points $t = 0$ and $t = T$, respectively.

We develop some existence and uniqueness results for the boundary value problem (1.1) by using standard techniques from fixed point theory. The paper is organized as follows: in Section 2, we recall some preliminary facts that we need in the sequel and Section 3 contains our main results. Finally, Section 4 provides some examples for the illustration of the main results.

## 2 Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [2, 3] and present preliminary results needed in our proofs later.

**Definition 2.1** For an at least $n$-times differentiable function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $q$ is defined as

$$^{c}D^{q}g(t) = \frac{1}{\Gamma(n-q)}\int_0^t (t-s)^{n-q-1}g^{(n)}(s)\,ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number $q$.

**Definition 2.2** The Riemann-Liouville fractional integral of order $q$ is defined as

$$I^{q}g(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{g(s)}{(t-s)^{1-q}}\,ds, \quad q > 0,$$

provided the integral exists.

**Lemma 2.1** *For $q > 0$, the general solution of the fractional differential equation $^{c}D^{q}u(t) = 0$ is given by*

$$u(t) = c_0 + c_1 t + \cdots + c_{n-1}t^{n-1},$$

*where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q]+1$).*

In view of Lemma 2.1, it follows that

$$I^{q\,c}D^{q}u(t) = u(t) + c_0 + c_1 t + \cdots + c_{n-1}t^{n-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$ ($n = [q]+1$).

For convenience we set

$$\begin{aligned}
\Omega_1 &= \sum_{i=1}^{m}\theta_i\frac{T^{\alpha_i+1}}{\Gamma(\alpha_i+2)}, \qquad \Omega_2 = \sum_{i=1}^{m}\theta_i\frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)},\\
\Omega_3 &= \sum_{j=1}^{n}\lambda_j\frac{\eta_j^{\beta_j+1}}{\Gamma(\beta_j+2)}, \qquad \Omega_4 = \sum_{j=1}^{n}\lambda_j\frac{\eta_j^{\beta_j}}{\Gamma(\beta_j+1)},\\
\Omega_5 &= \sum_{k=1}^{l}\mu_k\frac{T^{\gamma_k+1}-\xi_k^{\gamma_k+1}}{\Gamma(\gamma_k+2)}, \qquad \Omega_6 = \sum_{k=1}^{l}\mu_k\frac{T^{\gamma_k}-\xi_k^{\gamma_k}}{\Gamma(\gamma_k+1)}
\end{aligned} \tag{2.2}$$

and

$$\Delta = \Omega_1(\Omega_6 - \Omega_4) - \Omega_2(\Omega_5 - \Omega_3). \tag{2.3}$$

**Lemma 2.2** *Let $\Delta \neq 0$, $1 < q \leq 2$, $\alpha_i, \beta_j, \gamma_k > 0$, $\eta_j, \xi_k \in (0,T)$ for $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $k = 1,2,\ldots,l$ and $h \in C([0,T],\mathbb{R})$. Then the problem*

$$^cD^q u(t) = h(t), \quad t \in (0,T), \tag{2.4}$$

$$\begin{aligned} &\sum_{i=1}^m \theta_i I^{\alpha_i} u(T) = \omega, \\ &\sum_{j=1}^n \lambda_j I^{\beta_j} u(\eta_j) = \sum_{k=1}^l \mu_k \bigl(I^{\gamma_k} u(T) - I^{\gamma_k} u(\xi_k)\bigr), \end{aligned} \tag{2.5}$$

*has a unique solution given by*

$$\begin{aligned} u(t) = {}& I^q h(t) + \frac{(\Omega_6 - \Omega_4)t - (\Omega_5 - \Omega_3)}{\Delta}\left(\omega - \sum_{i=1}^m \theta_i I^{\alpha_i+q} h(T)\right) \\ &+ \frac{\Omega_1 - \Omega_2 t}{\Delta}\left(\sum_{j=1}^n \lambda_j I^{\beta_j+q} h(\eta_j) - \sum_{k=1}^l \mu_k \bigl(I^{\gamma_k+q} h(T) - I^{\gamma_k+q} h(\xi_k)\bigr)\right). \end{aligned} \tag{2.6}$$

*Proof* Using Lemma 2.1, (2.4) can be expressed as an equivalent integral equation,

$$u(t) = I^q h(t) + c_1 t + c_2. \tag{2.7}$$

Taking the Riemann-Liouville fractional integral of order $p > 0$ for (2.7), we have

$$I^p u(t) = I^{p+q} h(t) + c_1 \frac{t^{p+1}}{\Gamma(p+2)} + c_2 \frac{t^p}{\Gamma(p+1)}. \tag{2.8}$$

From the first condition of (2.5) and (2.8) with $p = \alpha_i$, it follows that

$$c_1 \sum_{i=1}^m \theta_i \frac{T^{\alpha_i+1}}{\Gamma(\alpha_i+2)} + c_2 \sum_{i=1}^m \theta_i \frac{T^{\alpha_i}}{\Gamma(\alpha_i+1)} = \omega - \sum_{i=1}^m \theta_i I^{\alpha_i+q} h(T).$$

According to the above process, the second condition of (2.5) and (2.8) with $p = \beta_j$ and $p = \gamma_k$ imply that

$$\begin{aligned} &c_1\left(\sum_{k=1}^l \mu_k \frac{T^{\gamma_k+1} - \xi_k^{\gamma_k+1}}{\Gamma(\gamma_k+2)} - \sum_{j=1}^n \lambda_j \frac{\eta_j^{\beta_j+1}}{\Gamma(\beta_j+2)}\right) \\ &\quad + c_2\left(\sum_{k=1}^l \mu_k \frac{T^{\gamma_k} - \xi_k^{\gamma_k}}{\Gamma(\gamma_k+1)} - \sum_{j=1}^n \lambda_j \frac{\eta_j^{\beta_j}}{\Gamma(\beta_j+1)}\right) \\ &= \sum_{j=1}^n \lambda_j I^{\beta_j+q} h(\eta_j) - \sum_{k=1}^l \mu_k \bigl(I^{\gamma_k+q} h(T) - I^{\gamma_k+q} h(\xi_k)\bigr). \end{aligned}$$

Solving the system of linear equations for constants $c_1$, $c_2$, we have

$$\begin{aligned} c_1 &= \frac{\Omega_6-\Omega_4}{\Delta}\left(\omega-\sum_{i=1}^{m}\theta_i I^{\alpha_i+q}h(T)\right) \\ &\quad -\frac{\Omega_2}{\Delta}\left(\sum_{j=1}^{n}\lambda_j I^{\beta_j+q}h(\eta_j)-\sum_{k=1}^{l}\mu_k\bigl(I^{\gamma_k+q}h(T)-I^{\gamma_k+q}h(\xi_k)\bigr)\right), \\ c_2 &= -\frac{\Omega_5-\Omega_3}{\Delta}\left(\omega-\sum_{i=1}^{m}\lambda_i I^{\alpha_i+q}h(T)\right) \\ &\quad +\frac{\Omega_1}{\Delta}\left(\sum_{j=1}^{n}\lambda_j I^{\beta_j+q}h(\eta_j)-\sum_{k=1}^{l}\mu_k\bigl(I^{\gamma_k+q}h(T)-I^{\gamma_k+q}h(\xi_k)\bigr)\right). \end{aligned}$$

Substituting constants $c_1$ and $c_2$ into (2.7), we obtain (2.6), as required. □

## 3 Main results

Let $\mathcal{C}=C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ to $\mathbb{R}$ endowed with the norm defined by $\|u\|=\sup_{t\in[0,T]}|u(t)|$. Throughout this paper, for convenience, the expression $I^x f(s,u(s))(y)$ means

$$I^x f\bigl(s,u(s)\bigr)(y)=\frac{1}{\Gamma(x)}\int_0^y (t-s)^{x-1}f\bigl(s,u(s)\bigr)\,ds \quad \text{for } t\in[0,T],$$

where $x\in\{q,\alpha_i+q,\beta_j+q,\gamma_k+q\}$ and $y\in\{t,T,\eta_j,\xi_k\}$, $i=1,2,\dots,m$, $j=1,2,\dots,n$, $k=1,2,\dots,l$.

As in Lemma 2.2, we define an operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ by

$$\begin{aligned} (\mathcal{F}u)(t) &= I^q f\bigl(s,u(s)\bigr)(t) \\ &\quad +\frac{(\Omega_6-\Omega_4)t-(\Omega_5-\Omega_3)}{\Delta}\left(\omega-\sum_{i=1}^{m}\theta_i I^{\alpha_i+q}f\bigl(s,u(s)\bigr)(T)\right) \\ &\quad +\frac{\Omega_1-\Omega_2 t}{\Delta}\left(\sum_{j=1}^{n}\lambda_j I^{\beta_j+q}f\bigl(s,u(s)\bigr)(\eta_j)\right. \\ &\quad \left.-\sum_{k=1}^{l}\mu_k\bigl(I^{\gamma_k+q}f\bigl(s,u(s)\bigr)(T)-I^{\gamma_k+q}f\bigl(s,u(s)\bigr)(\xi_k)\bigr)\right). \end{aligned} \tag{3.1}$$

It should be noticed that problem (1.1) has solutions if and only if the operator $\mathcal{F}$ has fixed points.

In the following subsections we prove existence, as well as existence and uniqueness results, for the boundary value problem (1.1) by using a variety of fixed point theorems.

We set

$$\begin{aligned} \Lambda &= \frac{T^q}{\Gamma(q+1)}+\frac{(|\Omega_3|+|\Omega_5|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|\frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &\quad +\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\left(\sum_{j=1}^{n}|\lambda_j|\frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}+\sum_{k=1}^{l}|\mu_k|\left(\frac{T^{\gamma_k+q}+\xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)}\right)\right) \end{aligned} \tag{3.2}$$

and

$$\Phi = \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|}|\omega|. \tag{3.3}$$

### 3.1 Existence and uniqueness result via Banach's fixed point theorem

**Theorem 3.1** *Assume that*

($H_1$) *there exists a constant $L > 0$ such that $|f(t,u) - f(t,v)| \leq L|u - v|$, for each $t \in [0,T]$ and $u, v \in \mathbb{R}$.*

*If*

$$L\Lambda < 1, \tag{3.4}$$

*where $\Lambda$ is defined by* (3.2), *then the boundary value problem* (1.1) *has a unique solution on* $[0,T]$.

*Proof* We transform the problem (1.1) into a fixed point problem, $u = \mathcal{F}u$, where the operator $\mathcal{F}$ is defined as in (3.1). Observe that the fixed points of the operator $\mathcal{F}$ are solutions of problem (1.1). Applying Banach's contraction mapping principle, we shall show that $\mathcal{F}$ has a unique fixed point.

We let $\sup_{t\in[0,T]} |f(t,0)| = M < \infty$ and choose

$$r \geq \frac{\Lambda M + \Phi}{1 - L\Lambda},$$

where a constant $\Phi$ is defined by (3.3).

Now, we show that $\mathcal{F}B_r \subset B_r$, where $B_r = \{u \in \mathcal{C} : \|u\| \leq r\}$. For any $u \in B_r$, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}u)(t)\bigr| \\
&\quad\leq \sup_{t\in[0,T]} \Biggl\{ I^q \bigl|f\bigl(s,u(s)\bigr)\bigr|(t) + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)t}{|\Delta|}|\omega| \\
&\qquad + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)t}{|\Delta|} \sum_{i=1}^{m} |\theta_i| I^{\alpha_i+q} \bigl|f\bigl(s,u(s)\bigr)\bigr|(T) \\
&\qquad + \frac{|\Omega_1| + |\Omega_2|t}{|\Delta|} \Biggl( \sum_{j=1}^{n} |\lambda_j| I^{\beta_j+q} \bigl|f\bigl(s,u(s)\bigr)\bigr|(\eta_j) \\
&\qquad + \sum_{k=1}^{l} |\mu_k| \bigl( I^{\gamma_k+q} \bigl|f\bigl(s,u(s)\bigr)\bigr|(T) + I^{\gamma_k+q} \bigl|f\bigl(s,u(s)\bigr)\bigr|(\xi_k) \bigr) \Biggr) \Biggr\} \\
&\quad\leq I^q \bigl( \bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(T) + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|}|\omega| \\
&\qquad + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| I^{\alpha_i+q} \bigl( \bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(T) \\
&\qquad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \Biggl( \sum_{j=1}^{n} |\lambda_j| I^{\beta_j+q} \bigl( \bigl|f\bigl(s,u(s)\bigr) - f(s,0)\bigr| + \bigl|f(s,0)\bigr| \bigr)(\eta_j)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{l}|\mu_k|\Big(I^{\gamma_k+q}\big(\big|f\big(s,u(s)\big)-f(s,0)\big|+\big|f(s,0)\big|\big)(T)\\
&+I^{\gamma_k+q}\big(\big|f\big(s,u(s)\big)-f(s,0)\big|+\big|f(s,0)\big|\big)(\xi_k)\Big)\Bigg)\\
&\le (Lr+M)I^q(1)(T)+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega|\\
&+(Lr+M)\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}(1)(T)\\
&+(Lr+M)\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}(1)(\eta_j)\\
&+\sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k+q}(1)(T)+I^{\gamma_k+q}(1)(\xi_k)\big)\Bigg)\\
&\le (Lr+M)\frac{T^q}{\Gamma(q+1)}+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega|\\
&+(Lr+M)\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|\frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&+(Lr+M)\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|\frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}+\sum_{k=1}^{l}|\mu_k|\bigg(\frac{T^{\gamma_k+q}+\xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)}\bigg)\Bigg)\\
&=(Lr+M)\Lambda+\Phi\le r,
\end{aligned}
$$

which implies that $\mathcal{F}B_r\subset B_r$.

Next, we let $u,v\in\mathcal{C}$. Then for $t\in[0,T]$, we have

$$
\begin{aligned}
&\big|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\big|\\
&\le I^q\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(t)\\
&+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(T)\\
&+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(\eta_j)\\
&+\sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(T)+I^{\gamma_k+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(\xi_k)\big)\Bigg)\\
&\le L\|u-v\|\frac{T^q}{\Gamma(q+1)}\\
&+L\|u-v\|\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|\frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)}\\
&+L\|u-v\|\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|\frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}+\sum_{k=1}^{l}|\mu_k|\bigg(\frac{T^{\gamma_k+q}+\xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)}\bigg)\Bigg)\\
&=L\Lambda\|u-v\|,
\end{aligned}
$$

which implies that $\|\mathcal{F}u - \mathcal{F}v\| \leq L\Lambda\|u - v\|$. As $L\Lambda < 1$, $\mathcal{F}$ is a contraction. Therefore, we deduce, by Banach's contraction mapping principle, that $\mathcal{F}$ has a fixed point which is the unique solution of problem (1.1). The proof is completed. □

### 3.2 Existence and uniqueness result via Banach's fixed point theorem and Hölder inequality

**Theorem 3.2** *Suppose that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following assumption*:

($H_2$) $|f(t,u) - f(t,v)| \leq \delta(t)|u - v|$, *for* $t \in [0,T]$, $u, v \in \mathbb{R}$ *and* $\delta \in L^{\frac{1}{\sigma}}([0,T],\mathbb{R}^+)$, $\sigma \in (0,1)$.

*Denote* $\|\delta\| = (\int_0^T |\delta(s)|^{\frac{1}{\sigma}}\,ds)^\sigma$. *If*

$$\|\delta\|\Biggl\{\frac{T^{q-\sigma}}{\Gamma(q)}\biggl(\frac{1-\sigma}{q-\sigma}\biggr)^{1-\sigma} + \frac{(|\Omega_6| + |\Omega_4|)T + (|\Omega_5| + |\Omega_3|)}{|\Delta|} \times \sum_{i=1}^m \frac{|\theta_i| T^{\alpha_i+q-\sigma}}{\Gamma(\alpha_i+q)}\biggl(\frac{1-\sigma}{\alpha_i+q-\sigma}\biggr)^{1-\sigma} + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n \frac{|\lambda_j|\eta^{\beta_j+q-\sigma}}{\Gamma(\beta_j+q)}\biggl(\frac{1-\sigma}{\beta_j+q-\sigma}\biggr)^{1-\sigma} + \sum_{k=1}^l \frac{|\mu_k|}{\Gamma(\beta_j+q)}\bigl(T^{\gamma_k+q-\sigma} + \xi_k^{\gamma_k+q-\sigma}\bigr)\biggl(\frac{1-\sigma}{\gamma_k+q-\sigma}\biggr)^{1-\sigma}\Biggr)\Biggr\} < 1,$$

*then the boundary value problem* (1.1) *has a unique solution*.

*Proof* For $u, v \in C([0,T],\mathbb{R})$ and for each $t \in [0,T]$, by Hölder's inequality, we have

$$\begin{aligned}
&\bigl|(\mathcal{F}u)(t) - (\mathcal{F}v)(t)\bigr| \\
&\quad\leq I^q\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|(t) \\
&\qquad + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|}\sum_{i=1}^m |\theta_i| I^{\alpha_i+q}\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|(T) \\
&\qquad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n |\lambda_j| I^{\beta_j+q}\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|(\eta_j) \\
&\qquad + \sum_{k=1}^l |\mu_k|\bigl(I^{\gamma_k+q}\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|(T) + I^{\gamma_k+q}\bigl|f\bigl(s,u(s)\bigr) - f\bigl(s,v(s)\bigr)\bigr|(\xi_k)\bigr)\Biggr) \\
&\quad\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\delta(s)\bigl|u(s) - v(s)\bigr|\,ds \\
&\qquad + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|}\sum_{i=1}^m \frac{|\theta_i|}{\Gamma(\alpha_i+q)}\int_0^T (T-s)^{\alpha_i+q-1}\delta(s)\bigl|u(s) - v(s)\bigr|\,ds \\
&\qquad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n \frac{|\lambda_j|}{\Gamma(\eta_j+q)}\int_0^{\eta_j} (\eta_j-s)^{\eta_j+q-1}\delta(s)\bigl|u(s) - v(s)\bigr|\,ds \\
&\qquad + \sum_{k=1}^l \frac{|\mu_k|}{\Gamma(\gamma_k+q)}\biggl(\int_0^T (T-s)^{\gamma_k+q-1}\delta(s)\bigl|u(s) - v(s)\bigr|\,ds \\
&\qquad + \int_0^{\xi_k} (\xi_k-s)^{\gamma_k+q-1}\delta(s)\bigl|u(s) - v(s)\bigr|\,ds\biggr)\Biggr)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(q)}\left(\int_0^t \bigl((t-s)^{q-1}\bigr)^{\frac{1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^t \bigl(\delta(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}\|u-v\| \\
&\quad+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|} \\
&\quad\times\sum_{i=1}^{m}\frac{|\theta_i|}{\Gamma(\alpha_i+q)}\left(\int_0^T \bigl((T-s)^{\alpha_i+q-1}\bigr)^{\frac{1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^T \bigl(\delta(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}\|u-v\| \\
&\quad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\left(\sum_{j=1}^{n}\frac{|\lambda_j|}{\Gamma(\eta_j+q)}\left(\int_0^{\eta_j} \bigl((\eta_j-s)^{\beta_j+q-1}\bigr)^{\frac{1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^{\eta_j} \bigl(\delta(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}\right. \\
&\quad+\sum_{k=1}^{l}\frac{|\mu_k|}{\Gamma(\gamma_k+q)}\left(\left(\int_0^T \bigl((T-s)^{\gamma_k+q-1}\bigr)^{\frac{1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^T \bigl(\delta(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}\right. \\
&\quad\left.\left.+\left(\int_0^{\xi_k} \bigl((\xi_k-s)^{\gamma_k+q-1}\bigr)^{\frac{1}{1-\sigma}}\,ds\right)^{1-\sigma}\left(\int_0^{\xi_k} \bigl(\delta(s)\bigr)^{\frac{1}{\sigma}}\,ds\right)^{\sigma}\right)\right)\|u-v\| \\
&\leq \|\delta\|\left[\frac{T^{q-\sigma}}{\Gamma(q)}\left(\frac{1-\sigma}{q-\sigma}\right)^{1-\sigma}+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}\frac{|\theta_i|T^{\alpha_i+q-\sigma}}{\Gamma(\alpha_i+q)}\right. \\
&\quad\times\left(\frac{1-\sigma}{\alpha_i+q-\sigma}\right)^{1-\sigma}+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\left(\sum_{j=1}^{n}\frac{|\lambda_j|\eta_j^{\beta_j+q-\sigma}}{\Gamma(\eta_j+q)}\left(\frac{1-\sigma}{\beta_j+q-\sigma}\right)^{1-\sigma}\right. \\
&\quad\left.\left.+\sum_{k=1}^{l}\frac{|\mu_k|}{\Gamma(\gamma_k+q)}\bigl(T^{\gamma_k+q-\sigma}+\xi_k^{\gamma_k+q-\sigma}\bigr)\left(\frac{1-\sigma}{\gamma_k+q-\sigma}\right)^{1-\sigma}\right)\right]\|u-v\|.
\end{aligned}
$$

It follows that $\mathcal{F}$ is contraction mapping. Hence Banach's fixed point theorem implies that $\mathcal{F}$ has a unique fixed point, which is the unique solution of the problem (1.1). The proof is completed. □

### 3.3 Existence and uniqueness result via nonlinear contractions

**Definition 3.1** Let $E$ be a Banach space and let $F: E \to E$ be a mapping. $F$ is said to be a nonlinear contraction if there exists a continuous nondecreasing function $\Psi: \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\varepsilon) < \varepsilon$ for all $\varepsilon > 0$ with the property

$$\|Fu - Fv\| \leq \Psi\bigl(\|u-v\|\bigr), \quad \forall u, v \in E.$$

**Lemma 3.1** (Boyd and Wong [22]) *Let $E$ be a Banach space and let $F: E \to E$ be a nonlinear contraction. Then $F$ has a unique fixed point in $E$.*

**Theorem 3.3** *Let $f: [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the assumption*

($H_3$) $|f(t,u) - f(t,v)| \leq h(t)\frac{|u-v|}{H^*+|u-v|}$, $t \in [0,T]$, $u, v \geq 0$, *where $h: [0,T] \to \mathbb{R}^+$ is continuous and a constant $H^*$ defined by*

$$
\begin{aligned}
H^* = I^q h(T) &+ \frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}h(T) + \frac{|\Omega_1|+|\Omega_2|T}{|\Delta|} \\
&\times\left(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}h(\eta_j) + \sum_{k=1}^{l}|\mu_k|\bigl(I^{\gamma_k+q}h(T) + I^{\gamma_k+q}h(\xi_k)\bigr)\right).
\end{aligned}
\tag{3.5}
$$

*Then the boundary value problem* (1.1) *has a unique solution.*

*Proof* We define the operator $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ as (3.1) and a continuous nondecreasing function $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ by

$$\Psi(\varepsilon)=\frac{H^*\varepsilon}{H^*+\varepsilon},\quad \forall\varepsilon\geq 0.$$

Note that the function $\Psi$ satisfies $\Psi(0)=0$ and $\Psi(\varepsilon)<\varepsilon$ for all $\varepsilon>0$.

For any $u,v\in\mathcal{C}$ and for each $t\in[0,T]$, we have

$$\begin{aligned}
&\big|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)\big|\\
&\quad\leq I^q\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(t)\\
&\qquad+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(T)\\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(\eta_j)\\
&\qquad+\sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(T)+I^{\gamma_k+q}\big|f\big(s,u(s)\big)-f\big(s,v(s)\big)\big|(\xi_k)\big)\Bigg)\\
&\quad\leq I^q\bigg(h(s)\frac{|u-v|}{H^*+|u-v|}\bigg)(T)\\
&\qquad+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}\bigg(h(s)\frac{|u-v|}{H^*+|u-v|}\bigg)(T)\\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}\bigg(h(s)\frac{|u-v|}{H^*+|u-v|}\bigg)(\eta_j)\\
&\qquad+\sum_{k=1}^{l}|\mu_k|\bigg(I^{\gamma_k+q}\bigg(h(s)\frac{|u-v|}{H^*+|u-v|}\bigg)(T)+I^{\gamma_k+q}\bigg(h(s)\frac{|u-v|}{H^*+|u-v|}\bigg)(\xi_k)\bigg)\Bigg)\\
&\quad\leq\frac{\Psi\|u-v\|}{H^*}\Bigg(I^qh(T)+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^{m}|\theta_i|I^{\alpha_i+q}h(T)\\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Bigg(\sum_{j=1}^{n}|\lambda_j|I^{\beta_j+q}h(\eta_j)+\sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k+q}h(T)+I^{\gamma_k+q}h(\xi_k)\big)\Bigg)\Bigg)\\
&\quad\leq\Psi\big(\|u-v\|\big).
\end{aligned}$$

This implies that $\|\mathcal{F}u-\mathcal{F}v\|\leq\Psi(\|u-v\|)$. Therefore $\mathcal{F}$ is a nonlinear contraction. Hence, by Lemma 3.1 the operator $\mathcal{F}$ has a unique fixed point which is the unique solution of the boundary value problem (1.1). This completes the proof. □

### 3.4 Existence result via Krasnoselskii's fixed point theorem

**Lemma 3.2** (Krasnoselskii's fixed point theorem [23]) *Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that* (a) $Ax+By\in M$ *whenever* $x,y\in M$; (b) *A is compact and continuous*; (c) *B is a contraction mapping. Then there exists* $z\in M$ *such that* $z=Az+Bz$.

**Theorem 3.4** *Let $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* $(\mathrm{H}_1)$. *In addition we assume that*

$(\mathrm{H}_4)$ $|f(t,u)| \le \phi(t)$, $\forall (t,u) \in [0,T] \times \mathbb{R}$, *and* $\phi \in C([0,T],\mathbb{R}^+)$.

*Then the boundary value problem* (1.1) *has at least one solution on* $[0,T]$ *provided*

$$\frac{(|\Omega_3| + |\Omega_5|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| \frac{T^{\alpha_i+q}}{\Gamma(\alpha_i + q + 1)} + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| \frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j + q + 1)} + \sum_{k=1}^{l} |\mu_k| \left( \frac{T^{\gamma_k+q} + \xi_k^{\gamma_k+q}}{\Gamma(\gamma_k + q + 1)} \right) \right) < 1. \tag{3.6}$$

*Proof* Setting $\sup_{t\in[0,T]} |\varphi(t)| = \|\varphi\|$ and choosing

$$\rho \ge \|\varphi\|\Lambda + \Phi \tag{3.7}$$

(where $\Lambda$ and $\Phi$ are defined by (3.2) and (3.3), respectively), we consider $B_\rho = \{u \in \mathcal{C}([0,T],\mathbb{R}) : \|u\| \le \rho\}$. We define the operators $\mathcal{F}_1$ and $\mathcal{F}_2$ on $B_\rho$ by

$$\begin{aligned} \mathcal{F}_1 u(t) &= I^q f\big(s,u(s)\big)(t), \quad t \in [0,T], \\ \mathcal{F}_2 u(t) &= \frac{(\Omega_6 - \Omega_4)t - (\Omega_5 - \Omega_3)}{\Delta} \left( \omega_1 - \sum_{i=1}^{m} \theta_i I^{\alpha_i+q} f\big(s,u(s)\big)(T) \right) \\ &\quad + \frac{\Omega_1 - \Omega_2 t}{\Delta} \left( \sum_{j=1}^{n} \lambda_j I^{\beta_j+q} f\big(s,u(s)\big)(\eta_j) \right. \\ &\quad \left. - \sum_{k=1}^{l} \mu_k \big( I^{\gamma_k+q} f\big(s,u(s)\big)(T) - I^{\gamma_k+q} f\big(s,u(s)\big)(\xi_k) \big) \right), \quad t \in [0,T]. \end{aligned}$$

For any $x, y \in B_\rho$, we have

$$\begin{aligned} &\big|\mathcal{F}_1 u(t) + \mathcal{F}_2 v(t)\big| \\ &\quad \le \sup_{t\in[0,T]} \left\{ I^q \big|f\big(s,u(s)\big)\big|(t) + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)t}{|\Delta|} |\omega| \right. \\ &\qquad + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)t}{|\Delta|} \sum_{i=1}^{m} |\theta_i| I^{\alpha_i+q} \big|f\big(s,v(s)\big)\big|(T) \\ &\qquad + \frac{|\Omega_1| + |\Omega_2|t}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| I^{\beta_j+q} \big|f\big(s,v(s)\big)\big|(\eta_j) \right. \\ &\qquad \left. \left. + \sum_{k=1}^{l} |\mu_k| \big( I^{\gamma_k+q} \big|f\big(s,v(s)\big)\big|(T) + I^{\gamma_k+q} \big|f\big(s,v(s)\big)\big|(\xi_k) \big) \right) \right\} \\ &\quad \le \|\varphi\| \left[ \frac{T^q}{\Gamma(q+1)} + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| \frac{T^{\alpha_i+q}}{\Gamma(\alpha_i + q + 1)} \right. \\ &\qquad \left. + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| \frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j + q + 1)} + \sum_{k=1}^{l} |\mu_k| \left( \frac{T^{\gamma_k+q} + \xi_k^{\gamma_k+q}}{\Gamma(\gamma_k + q + 1)} \right) \right) \right] \end{aligned}$$

$$\begin{aligned}&+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega|\\&=\|\varphi\|\Lambda+\Phi\le\rho.\end{aligned}$$

This shows that $\mathcal{F}_1u+\mathcal{F}_2v\in B_\rho$. It is easy to see using (3.6) that $\mathcal{F}_2$ is a contraction mapping.

Continuity of $f$ implies that the operator $\mathcal{F}_1$ is continuous. Also, $\mathcal{F}_1$ is uniformly bounded on $B_\rho$ as

$$\|\mathcal{F}_1u\|\le\frac{T^q}{\Gamma(q+1)}\|\phi\|.$$

Now we prove the compactness of the operator $\mathcal{F}_1$.

We define $\sup_{(t,u)\in[0,T]\times B_\rho}|f(t,u)|=\overline{f}<\infty$, and consequently we have

$$\begin{aligned}\big|(\mathcal{F}_1u)(t_2)-(\mathcal{F}_1u)(t_1)\big| &= \frac{1}{\Gamma(q)}\bigg|\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]f\big(s,u(s)\big)\,ds\\&\quad+\int_{t_1}^{t_2}(t_2-s)^{q-1}f\big(s,u(s)\big)\,ds\bigg|\\&\le\frac{\overline{f}}{\Gamma(q+1)}\big|t_1^q-t_2^q\big|,\end{aligned}$$

which is independent of $u$ and tends to zero as $t_2-t_2\to0$. Thus, $\mathcal{F}_1$ is equicontinuous. So $\mathcal{F}_1$ is relatively compact on $B_\rho$. Hence, by the Arzelá-Ascoli theorem, $\mathcal{F}_1$ is compact on $B_\rho$. Thus all the assumptions of Lemma 3.2 are satisfied. So the conclusion of Lemma 3.2 implies that the boundary value problem (1.1) has at least one solution on $[0,T]$. □

### 3.5 Existence result via Leray-Schauder's nonlinear alternative

**Theorem 3.5** (Nonlinear alternative for single valued maps [24]) *Let $E$ be a Banach space, $C$ a closed, convex subset of $E$, $U$ an open subset of $C$, and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact* (*that is, $F(\overline{U})$ is a relatively compact subset of $C$*) *map. Then either*

(i) *$F$ has a fixed point in $\overline{U}$, or*
(ii) *there is a $u\in\partial U$* (*the boundary of $U$ in $C$*) *and $\lambda\in(0,1)$ with $u=\lambda F(u)$.*

**Theorem 3.6** *Assume that*

($H_5$) *there exist a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a function $p\in C([0,T],\mathbb{R}^+)$ such that*

$$\big|f(t,u)\big|\le p(t)\psi\big(|u|\big)\quad\textit{for each }(t,u)\in[0,T]\times\mathbb{R};$$

($H_6$) *there exists a constant $M>0$ such that*

$$\frac{M}{\psi(M)\|p\|\Lambda+\Phi}>1,$$

*where $\Lambda$ and $\Phi$ are defined by* (3.2) *and* (3.3), *respectively.*

*Then the boundary value problem* (1.1) *has at least one solution on* $[0,T]$.

*Proof* Let the operator $\mathcal{F}$ be defined by (3.1). Firstly, we shall show that $\mathcal{F}$ *maps bounded sets* (*balls*) *into bounded sets in* $C([0,T],\mathbb{R})$. For a number $r>0$, let $B_r=\{u\in C([0,T],\mathbb{R}):\|u\|\le r\}$ be a bounded ball in $C([0,T],\mathbb{R})$. Then for $t\in[0,T]$ we have

$$\begin{aligned}
&\bigl|(\mathcal{F}u)(t)\bigr| \\
&\quad\le \sup_{t\in[0,T]}\Biggl\{I^q\bigl|f\bigl(s,u(s)\bigr)\bigr|(t)+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)t}{|\Delta|}|\omega| \\
&\qquad+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^m|\theta_i|I^{\alpha_i+q}\bigl|f\bigl(s,u(s)\bigr)\bigr|(T) \\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n|\lambda_j|I^{\beta_j+q}\bigl|f\bigl(s,u(s)\bigr)\bigr|(\eta_j) \\
&\qquad+\sum_{k=1}^l|\mu_k|\bigl(I^{\gamma_k+q}\bigl|f\bigl(s,u(s)\bigr)\bigr|(T)+I^{\gamma_k+q}\bigl|f\bigl(s,u(s)\bigr)\bigr|(\xi_k)\bigr)\Biggr)\Biggr\} \\
&\quad\le\psi\bigl(\|u\|\bigr)I^qp(s)(T)+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega| \\
&\qquad+\psi\bigl(\|u\|\bigr)\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^m|\theta_i|I^{\alpha_i+q}p(s)(T) \\
&\qquad+\psi\bigl(\|u\|\bigr)\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n|\lambda_j|I^{\beta_j+q}p(s)(\eta_j) \\
&\qquad+\sum_{k=1}^l|\mu_k|\bigl(I^{\gamma_k+q}p(s)(T)+I^{\gamma_k+q}p(s)(\xi_k)\bigr)\Biggr) \\
&\quad\le\psi\bigl(\|u\|\bigr)\|p\|\frac{T^q}{\Gamma(q+1)}+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega| \\
&\qquad+\psi\bigl(\|u\|\bigr)\|p\|\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^m|\theta_i|\frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\qquad+\psi\bigl(\|u\|\bigr)\|p\|\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|} \\
&\qquad\times\Biggl(\sum_{j=1}^n|\lambda_j|\frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}+\sum_{k=1}^l|\mu_k|\biggl(\frac{T^{\gamma_k+q}+\xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)}\biggr)\Biggr) \\
&\quad\le\psi\bigl(\|u\|\bigr)\|p\|\Biggl[\frac{T^q}{\Gamma(q+1)}+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}\sum_{i=1}^m|\theta_i|\frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\
&\qquad+\frac{|\Omega_1|+|\Omega_2|T}{|\Delta|}\Biggl(\sum_{j=1}^n|\lambda_j|\frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)}+\sum_{k=1}^l|\mu_k|\biggl(\frac{T^{\gamma_k+q}+\xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)}\biggr)\Biggr)\Biggr] \\
&\qquad+\frac{(|\Omega_5|+|\Omega_3|)+(|\Omega_6|+|\Omega_4|)T}{|\Delta|}|\omega|,
\end{aligned}$$

and consequently

$$\|\mathcal{F}u\|\le\psi(r)\|p\|\Lambda+\Phi.$$

Next we will show that $\mathcal{F}$ *maps bounded sets into equicontinuous sets of* $C([0,T],\mathbb{R})$. Let $\tau_1, \tau_2 \in [0,T]$ with $\tau_1 < \tau_2$ and $u \in B_r$. Then we have

$$\begin{aligned}
&\big|(\mathcal{F}u)(\tau_2) - (\mathcal{F}u)(\tau_1)\big| \\
&\quad \leq \frac{1}{\Gamma(q)}\left|\int_0^{\tau_1}\big[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\big] f\big(s,u(s)\big)\,ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1} f\big(s,u(s)\big)\,ds\right| \\
&\qquad + \frac{(|\Omega_6| + |\Omega_4|)|\omega_1|}{|\Delta|}(\tau_2 - \tau_1) + \frac{|\Omega_6| + |\Omega_4|}{|\Delta|}\sum_{i=1}^{m}|\theta_i| I^{\alpha_i + q}\big|f\big(s,u(s)\big)\big|(T)(\tau_2 - \tau_1) \\
&\qquad + \frac{|\Omega_2|}{|\Delta|}(\tau_2 - \tau_1)\left(\sum_{j=1}^{n}|\lambda_j| I^{\beta_j + q}\big|f\big(s,u(s)\big)\big|(\eta_j)\right. \\
&\qquad \left. + \sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k + q}\big|f\big(s,u(s)\big)\big|(T) + I^{\gamma_k + q}\big|f\big(s,u(s)\big)\big|(\xi_k)\big)\right) \\
&\quad \leq \frac{\psi(r)}{\Gamma(q)}\left|\int_0^{\tau_1}\big[(\tau_2 - s)^{q-1} - (\tau_1 - s)^{q-1}\big] p(s)\,ds + \int_{\tau_1}^{\tau_2}(\tau_2 - s)^{q-1} p(s)\,ds\right| \\
&\qquad + \frac{(|\Omega_6| + |\Omega_4|)|\omega_1|}{|\Delta|}(\tau_2 - \tau_1) + \frac{|\Omega_6| + |\Omega_4|}{|\Delta|}\psi(r)\sum_{i=1}^{m}|\theta_i| I^{\alpha_i + q} p(s)(T)(\tau_2 - \tau_1) \\
&\qquad + \frac{|\Omega_2|}{|\Delta|}\psi(r)(\tau_2 - \tau_1)\left(\sum_{j=1}^{n}|\lambda_j| I^{\beta_j + q} p(s)(\eta_j)\right. \\
&\qquad \left. + \sum_{k=1}^{l}|\mu_k|\big(I^{\gamma_k + q} p(s)(T) + I^{\gamma_k + q} p(s)(\xi_k)\big)\right).
\end{aligned}$$

As $\tau_2 - \tau_1 \to 0$, the right-hand side of the above inequality tends to zero independently of $u \in B_r$. Therefore by the Arzelá-Ascoli theorem the operator $\mathcal{F} : C([0,T],\mathbb{R}) \to C([0,T],\mathbb{R})$ is completely continuous.

Let $u$ be a solution. Then, for $t \in [0,T]$, and following similar computations to those in the first step, we have

$$\big|u(t)\big| \leq \psi\big(\|u\|\big)\|p\|\Lambda + \Phi,$$

which leads to

$$\frac{\|u\|}{\psi(\|u\|)\|p\|\Lambda + \Phi} \leq 1.$$

In view of ($H_6$), there exists $M$ such that $\|u\| \neq M$. Let us set

$$U = \big\{u \in C\big([0,T],\mathbb{R}\big) : \|u\| < M\big\}.$$

We see that the operator $\mathcal{F} : \overline{U} \to C([0,T],\mathbb{R})$ is continuous and completely continuous. From the choice of $U$, there is no $u \in \partial U$ such that $u = \nu\mathcal{F}u$ for some $\nu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type, we deduce that $\mathcal{F}$ has a fixed point $u \in \overline{U}$ which is a solution of the boundary value problem (1.1). This completes the proof. □

## 4 Examples

In this section, we present some examples to illustrate our results.

**Example 4.1** Consider the following fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \frac{\sin^2(2\pi t)}{(3+t)^3} \cdot \frac{|u(t)|}{1+|u(t)|} + \frac{1}{2}, \quad 0 < t < 2, \\ \sqrt{2}I^{3/2}u(2) - \frac{1}{2}I^{\sqrt{3}}u(2) = 1, \\ 3I^{\pi/3}u(2/3) - \frac{7}{4}I^{\sqrt{2}/2}u(4/3) = \frac{1}{2}(I^{\sqrt{\pi}}u(2) - I^{\sqrt{\pi}}u(5/3)). \end{cases} \tag{4.1}$$

Here $q = 3/2$, $T = 2$, $m = 2$, $n = 2$, $l = 1$, $\omega = 1$, $\alpha_1 = 3/2$, $\alpha_2 = \sqrt{3}$, $\beta_1 = \pi/3$, $\beta_2 = \sqrt{2}/2$, $\gamma_1 = \sqrt{\pi}$, $\theta_1 = \sqrt{2}$, $\theta_2 = -1/2$, $\lambda_1 = 3$, $\lambda_2 = -7/4$, $\mu_1 = 1/2$, $\eta_1 = 2/3$, $\eta_2 = 4/3$, $\xi_1 = 5/3$, and $f(t,u) = (\sin^2(2\pi t)|u|)/((3+t)^3(1+|u|)) + (1/2)$. Since $|f(t,u) - f(t,v)| \leq (1/27)|u-v|$, $(H_1)$ is satisfied with $L = 1/27$. We can show that

$$\begin{aligned} \Lambda &= \frac{T^q}{\Gamma(q+1)} + \frac{(|\Omega_3| + |\Omega_5|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| \frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &\quad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| \frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} + \sum_{k=1}^{l} |\mu_k| \left( \frac{T^{\gamma_k+q} + \xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)} \right) \right) \\ &\approx 14.13402398. \end{aligned}$$

Thus $L\Lambda \approx 0.52348237 < 1$. Hence, by Theorem 3.1, the boundary value problem (4.1) has a unique solution on $[0,2]$.

**Example 4.2** Consider the following fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \frac{e^t}{(3+t)^2} \cdot \frac{|u(t)|}{1+|u(t)|} + \frac{3}{4}, \quad 0 < t < 1/2, \\ 10I^{5/3}u(1/2) + \frac{1}{64}I^{\sqrt{3}}u(1/2) = 3/2, \\ 81I^{\sqrt{2}}u(1/36) + \frac{1}{49}I^{1/2}u(1/25) = \frac{1}{25}(I^{\sqrt{\pi}}u(1/2) - I^{\sqrt{\pi}}u(1/9)) \\ \qquad + \frac{1}{81}(I^{1/3}u(1/2) - I^{1/3}u(1/8)). \end{cases} \tag{4.2}$$

Here $q = 3/2$, $T = 1/2$, $m = 2$, $n = 2$, $l = 2$, $\omega = 3/2$, $\alpha_1 = 5/3$, $\alpha_2 = \sqrt{3}$, $\beta_1 = \sqrt{2}$, $\beta_2 = 1/2$, $\gamma_1 = \sqrt{\pi}$, $\gamma_2 = 1/3$, $\theta_1 = 10$, $\theta_2 = 1/64$, $\lambda_1 = 81$, $\lambda_2 = 1/49$, $\mu_1 = 1/25$, $\mu_2 = 1/81$, $\eta_1 = 1/36$, $\eta_2 = 1/25$, $\xi_1 = 1/9$, $\xi_2 = 1/8$, and $f(t,u) = (e^t|u|)/((3+t)^2(1+|u|)) + (3/4)$. We choose $h(t) = e^t/9$ and we obtain

$$\begin{aligned} H^* &= I^q h(T) + \frac{(|\Omega_5| + |\Omega_3|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| I^{\alpha_i+q} h(T) \\ &\quad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| I^{\beta_j+q} h(\eta_j) + \sum_{k=1}^{l} |\mu_k| \big( I^{\gamma_k+q} h(T) + I^{\gamma_k+q} h(\xi_k) \big) \right) \\ &\approx 0.89374610. \end{aligned}$$

Clearly,

$$\big|f(t,u) - f(t,v)\big| = \frac{e^t}{(3+t)^2} \left( \frac{|u| - |v|}{1 + |u| + |v| + |u||v|} \right) \leq \frac{e^t}{9} \left( \frac{|u-v|}{0.89374610 + |u-v|} \right).$$

Hence, by Theorem 3.3, the boundary value problem (4.2) has a unique solution on $[0, 1/2]$.

**Example 4.3** Consider the following fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{3}{2}}u(t) = \frac{e^{-t}\sin^2(2\pi t)}{(2+t)^2} \cdot \frac{|u(t)|}{1+|u(t)|} + 3t, \quad 0 < t < 1, \\ 256I^{3/2}u(1) + \frac{1}{100}I^{\sqrt{3}}u(1) = 1, \\ 1{,}024I^{\pi/3}u(1/5) + \frac{1}{1{,}024}I^{1/\sqrt{2}}u(1/3) = \frac{1}{512}(I^{\sqrt{\pi}}u(1) - I^{\sqrt{\pi}}u(1/3)) \\ \qquad + \frac{1}{1{,}024}(I^{\sqrt{\pi}/2}u(1) - I^{\sqrt{\pi}/2}u(1/6)). \end{cases} \tag{4.3}$$

Here $q = 3/2$, $T = 1$, $m = 2$, $n = 2$, $l = 2$, $\omega = 1$, $\alpha_1 = 3/2$, $\alpha_2 = \sqrt{3}$, $\beta_1 = \pi/3$, $\beta_2 = 1/\sqrt{2}$, $\gamma_1 = \sqrt{\pi}$, $\gamma_2 = \sqrt{\pi}/2$, $\theta_1 = 256$, $\theta_2 = 1/100$, $\lambda_1 = 1{,}024$, $\lambda_2 = 1/1{,}024$, $\mu_1 = 1/512$, $\mu_2 = 1/1{,}024$, $\eta_1 = 1/5$, $\eta_2 = 1/3$, $\xi_1 = 1/3$, $\xi_2 = 1/6$, and $f(t,u) = (e^{-t}\sin^2(2\pi t)|u|)/((2+t)^2(1+|u|)) + 3t$. Since $|f(t,u) - f(t,v)| \leq (1/4)|u-v|$, $(H_1)$ is satisfied with $L = 1/4$. We find that

$$\begin{aligned} &\frac{(|\Omega_3| + |\Omega_5|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| \frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &\quad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| \frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} + \sum_{k=1}^{l} |\mu_k| \left( \frac{T^{\gamma_k+q} + \xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)} \right) \right) \\ &\approx 0.92525429 < 1. \end{aligned}$$

Clearly,

$$\left|f(t,u)\right| = \left| \frac{e^{-t}\sin^2(2\pi t)}{(2+t)^2} \cdot \frac{|u(t)|}{1+|u(t)|} + 3t \right| \leq \frac{e^{-t}}{4} + 3t.$$

Hence, by Theorem 3.4, the boundary value problem (4.3) has at least one solution on $[0, 1]$.

**Example 4.4** Consider the following fractional integral boundary value problem:

$$\begin{cases} {}^cD^{\frac{5}{3}}u(t) = \frac{1}{225}(1+t^2)(\frac{u^2}{1+|u|} + \frac{|u|}{1+|u|}), \quad 0 < t < 3, \\ I^{1/2}u(3) - \sqrt{3}I^{3/2}u(3) - \frac{4}{5}I^{\sqrt{2}}u(3) = \pi, \\ \frac{3}{2}I^{\sqrt{3}}u(2/5) = \frac{2}{9}(I^{8/5}u(3) - I^{8/5}u(3/4)) - 10(I^{1/4}u(3) - I^{1/4}u(1/5)). \end{cases} \tag{4.4}$$

Here $q = 5/3$, $T = 3$, $m = 3$, $n = 1$, $l = 2$, $\omega = \pi$, $\alpha_1 = 1/2$, $\alpha_2 = 3/2$, $\alpha_3 = \sqrt{2}$, $\beta_1 = \sqrt{3}$, $\gamma_1 = 8/5$, $\gamma_2 = 1/4$, $\theta_1 = 1$, $\theta_2 = -\sqrt{3}$, $\theta_3 = -4/5$, $\lambda_1 = 3/2$, $\mu_1 = 2/9$, $\mu_2 = -10$, $\eta_1 = 2/5$, $\xi_1 = 3/4$, $\xi_2 = 1/5$, and $f(t,u) = (1/225)(1+t^2)((u^2/(1+|u|)) + (|u|/(1+|u|)))$. Then we get

$$\begin{aligned} \Lambda &= \frac{T^q}{\Gamma(q+1)} + \frac{(|\Omega_3| + |\Omega_5|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|} \sum_{i=1}^{m} |\theta_i| \frac{T^{\alpha_i+q}}{\Gamma(\alpha_i+q+1)} \\ &\quad + \frac{|\Omega_1| + |\Omega_2|T}{|\Delta|} \left( \sum_{j=1}^{n} |\lambda_j| \frac{\eta_j^{\beta_j+q}}{\Gamma(\beta_j+q+1)} + \sum_{k=1}^{l} |\mu_k| \left( \frac{T^{\gamma_k+q} + \xi_k^{\gamma_k+q}}{\Gamma(\gamma_k+q+1)} \right) \right) \\ &\approx 15.19489487 \end{aligned}$$

and

$$\Phi = \frac{(|\Omega_3| + |\Omega_5|) + (|\Omega_6| + |\Omega_4|)T}{|\Delta|}|\omega_1| \approx 0.80515429.$$

Clearly,

$$\left|f(t,u)\right| = \left|\frac{1}{225}\left(1+t^2\right)\left(\frac{u^2}{1+|u|} + \frac{|u|}{1+|u|}\right)\right| \leq \frac{1}{225}\left(1+t^2\right)\left(|u|+1\right).$$

Choosing $p(t) = (1/225)(1+t^2)$ and $\psi(|u|) = |u| + 1$, we can show that

$$\frac{M}{\psi(M)\|p\|\Lambda + \Phi} > 1,$$

which implies that $M > 4.55994347$. Hence, by Theorem 3.6, the boundary value problem (4.4) has at least one solution on $[0,3]$.

**Competing interests**
The authors declare that they have no competing interests.

**Authors' contributions**
All authors contributed equally in this article. They read and approved the final manuscript.

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